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ONLY TRIVIAL BOREL MEASURES ON S_∞ ARE QUASI-INVARIANT UNDER AUTOMORPHISMS

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Let S_∞ be the group of all permutations of the integers. Then the only σ-finite Borel measures on S_∞ which are quasi-invariant under automorphisms are supported on the finite permutations.

1. Introduction. S_∞ is a complete separable metrizable group with the topology of pointwise convergence on the integers. S_∞ is not locally compact with this topology, and hence there is no σ-finite Borel measure on S_∞ which is invariant under left translations. For if there were such a measure, then there would be a locally compact group topology with a countable basis on S_∞ whose Borel structure coincides with the usual Borel structure (Theorem 7.1, Mackey [7]). This is a contradiction since the Borel structure of a complete separable metric group uniquely determines its topology. In fact, Mackey's result shows that there is no σ-finite Borel measure on S_∞ which is quasi-invariant under left translations. (Recall that a Borel measure μ on a Borel space X is said to be quasi-invariant under a group of Borel automorphisms G if μ and each of its translates under elements of G have precisely the same null sets.) However, even if G is a complete separable metric group which is not locally compact, then there may well be many Borel measures on G which are quasi-invariant under inner automorphisms. For example, let G be any Banach Space. Since G is abelian, any measure on G is invariant under inner automorphisms. The purpose of this paper is to prove the following theorem. It answers a generalization of a question posed by S. M. Ulam, and shows that the above phenomena cannot occur for S_∞. It roughly states that the inner automorphism action on S_∞ is so rich that some natural structures are precluded. This is a common occurence for S_∞. For example, Schreier and Ulam [8] have shown that every automorphism of S_∞ is inner, and Kallman [3] noted that S_∞ has a unique topology in which it is a complete separable metric group.

THEOREM 1.1. *The only σ-finite Borel measures on S_∞ which are quasi-invariant under automorphisms are supported by the finite permutations.*

A result of A. Lieberman [4] will be the main tool used to prove

Theorem 1.1. If μ is a σ-finite Borel measure on S_∞ which is quasi-invariant under automorphisms, then there is, in a natural manner, a unitary representation of S_∞ on $L^2(S_\infty, \mu)$. It is not a priori obvious that this representation is continuous. To show that this representation is continuous, some new theorems on Radon-Nikodym derivatives of measures are proved in §2. These results might be of independent interest. In §3 we show that the quotient of S_∞ by S_∞, acting as inner automorphisms, is countably separated—i.e., there exists a countable set of invariant Borel sets in S_∞ which separate orbits. Finally, the results of §2 are used in §4 to show that the natural unitary representation of S_∞ on $L^2(S_\infty, \mu)$ is continuous, and then Theorem 1.1 is proved using A. Lieberman's result and §3. See Mackey [7] for the basic definitions, theorems, and further references on Borel spaces used in this paper.

2. A result on Radon-Nikodym derivatives. Consider the following setup. Let X be a complete separable metric space, T a Borel space, and for each t in T, let ν_t and μ_t be two positive finite Borel measures on X. Suppose that the mappings $t \rightarrow \nu_t$ and $t \rightarrow \mu_t$ are Borel mappings in the sense that $t \rightarrow \nu_t(E)$ and $t \rightarrow \mu_t(E)$ are realvalued Borel mappings on T for every Borel subset E of X.

PROPOSITION 2.1. *Suppose that for every t in T, ν_t is absolutely continuous with respect to μ_t. Then there is a Borel function $d(t, x)$ on $T \times X$ so that for each t in T, $d(t, \cdot)$ is a Radon-Nikodym derivative of ν_t with respect to μ_t.*

Proposition 2.1 is quite reminiscent of Lemma 3.1 of Mackey [5], but the two results and their methods of proof are disjoint.

LEMMA 2.2. *It may be assumed that $\nu_t \leqq \mu_t$ for every t in T.*

Proof. For each t in T, let $\lambda_t = \mu_t + \nu_t$. Then λ_t is a finite Borel measure on X, $t \rightarrow \lambda_t(E)$ is a Borel mapping on T for each Borel subset E of X, and $\nu_t \leqq \lambda_t$. There exists a Borel function $d'(t, x)$ on $T \times X$ so that for each t in T, $d'(t, \cdot)$ is a Radon-Nikodym derivative of ν_t with respect to λ_t. Define $d''(t, x)$ by setting $d''(t, x) = d'(t, x)$ if $0 \leqq d'(t, x) < 1$, and by setting $d''(t, x) = 0$ if $d'(t, x) < 0$ or $1 \leqq d'(t, x)$. $d''(t, x)$ is then a Borel function on $T \times X$, and $d''(t, \cdot)$ is a Radon-Nikodym derivative of ν_t with respect to λ_t for every t in T. Let $d(t, x) = d''(t, x)/(1 - d''(t, x))$. Then $d(t, x)$ is a Borel function on $T \times X$, and $d(t, \cdot)$ is a Radon-Nikodym derivative of ν_t with respect to λ_t for every t in T.

LEMMA 2.3. *Let $f(t, x)$ be a real-valued Borel function on $T \times X$ so that for each t in T, $f(t, \cdot)$ is a bounded function on X. Then the mapping $t \to \int_X f(t, x)d\mu_t(x)$ is a Borel function.*

Proof. It suffices to prove the lemma in case f is nonnegative. If f is the characteristic function of a Borel rectangle in $T \times X$, the lemma is true since the mapping $t \to \mu_t(E)$ is a Borel function for each Borel subset E of X. Consider the set $\underline{S}$ of all Borel subsets B of $T \times X$ such that the lemma holds for the characteristic function of B. $\underline{S}$ contains all rectangles and is closed under complements since $t \to \mu_t(X)$ is a Borel mapping. $\underline{S}$ is closed under countable increasing unions by the monotone convergence theorem and the fact that a pointwise limit of a sequence of Borel functions is a Borel function. Hence, $\underline{S}$ contains all Borel subsets of $T \times X$. Therefore, the lemma is true for characteristic functions of Borel subsets of $T \times X$, and hence is true for Borel step functions. Choose a sequence of nonnegative Borel step functions $f_n(t, x)$ on $T \times X$ so that for each (t, x) in $T \times X$, $[f_n(t, x) | n \geqq 1]$ is a monotone increasing sequence which converges to $f(t, x)$. Such a sequence exists by standard arguments since f is nonnegative. The lemma then holds for f by again appealing to the monotone convergence theorem and the fact that a pointwise limit of a sequence of Borel functions is again a Borel function.

LEMMA 2.4. *There exists a sequence $[f_n(t, x) | n \geqq 1]$ of Borel functions on $T \times X$ so that $f_n(t, \cdot)$ is bounded for each $n \geqq 1$ and each t in T, and so that the nonzero members of $[f_n(t, \cdot) | n \geqq 1]$ form a basis for $L^2(X, \mu_t)$.*

Proof. Let $V_m (m \geqq 1)$ be a basis for the topology of X, and let $g_n(x)$ be the sequence of characteristic functions for the V_m and the $X - V_m (m \geqq 1)$ in some order. For each t in T, the g_n's are in $L^2(X, \mu_t)$, and there is no element of $L^2(X, \mu_t)$ which is orthogonal to them all. The idea is now to apply a minor variant of the Gram-Schmidt process to the g_n's. Define $f_1(t, x) = 0$ if $\int_X g_1(y)^2 d\mu_t(y) = 0$, $f_1(t, x) = g_1(x)\left(\int_X g_1(y)^2 d\mu_t(y)\right)^{-(1/2)}$ otherwise. Having defined $f_1, \cdots, f_{k-1}$, define $h_k(t, x) = g_k(x) - \sum_{1 \leqq j \leqq k-1} f_j(t, x)\left(\int_X f_j(t, y) g_k(y) d\mu_t(y)\right)$ and set $f_k(t, x) = 0$ if $\int_X h_k(t, y)^2 d\mu_t(y) = 0$, $f_k(t, x) = h_k(t, x)\left(\int_X h_k(t, y)^2 d\mu_t(y)\right)^{-(1/2)}$ otherwise. It is an easy induction using Lemma 2.3 that each $f_n(t, x)$ is a Borel function on $T \times X$, and that $f_n(t, \cdot)$ is bounded for each $n \geqq 1$ and each t in T. Furthermore, if $f_n(t, \cdot)$ is a null function with respect to μ_t, then $f_n(t, \cdot) = 0$, and so the nonzero members of

$[f_n(t, \cdot) | n \geqq 1]$ form a basis for $L^2(X, \mu_t)$, since the span of $f_1(t, \cdot), \cdots, f_k(t, \cdot)$ is the same as the span of $g_1, \cdots, g_k$ in $L^2(X, \mu_t)$.

Let $f_n(t, x)(n \geqq 1)$ be as Lemma 2.4. Define

$$h_n(t, x) = \sum_{1 \leqq j \leqq n} \nu_t(f_j(t, \cdot))f_j(t, x) .$$

Each h_n is a Borel function on $T \times X$ and $h_n(t, \cdot)$ is a bounded function on X for all t in T and $n \geqq 1$. Furthermore, as is well known, the sequence $[h_n(t, \cdot) | n \geqq 1]$ converges in $L^2(X, \mu_t)$, and therefore in $L^1(X, \mu_t)$, to a Radon-Nikodym derivative of ν_t with respect to μ_t. This is true since $\nu_t \leqq \mu_t$ for all t in T.

Proof of Proposition 2.1. Define

$$S(m, n) = \left[t \text{ in } T \middle| \int_X \left| h_m(t, y) - h_j(t, y) \right| d\mu_t(y) \leqq 2^{-n} \text{ for all } j \geqq m \right] .$$

Each $S(m, n)$ is a Borel subset of T by Lemma 2.3, and $\bigcup_{m \geqq 1} S(m, n) = T$ for each $n \geqq 1$. Define $g_n(t, x)$ as follows. Set $g_n(t, x) = h_1(t, x)$ if t is in $S(1, n), \cdots$, and set $g_n(t, x) = h_k(t, x)$ if t is in $S(k, n) - \bigcup_{1 \leqq j \leqq k-1} S(j, n)$. Then g_n is a Borel function on $T \times X$, and

$$[g_n(t, \cdot) | n \geqq 1]$$

converges μ_t-almost everywhere to a Radon-Nikodym derivative of ν_t with respect to μ_t. Define $d(t, x) = \lim_{n \to \infty} g_n(t, x)$ if this limit exists, and set $d(t, x) = 0$ otherwise. Then $d(t, x)$ is a Borel function on $T \times X$, and $d(t, \cdot)$ is a Radon-Nikodym derivative of ν_t with respect to μ_t for all t in T.

3. A countable separability result. S_∞ acts on itself by inner automorphisms, giving rise to an equivalence relation $\equiv$ on S_∞. Recall that a Borel space is standard if it is Borel isomorphic to a Borel subset of $[0, 1]$.

PROPOSITION 3.1. *The quotient space $S_\infty / \equiv$ is standard.*

Proof. It suffices to show that there is a Borel subset B of S_∞ such that every element of S_∞ is conjugate to one and only one element of B. To see this, let C be any Borel subset of B. Then $[aCa^{-1} | a \text{ is in } S_\infty]$ and $[a(B - C)a^{-1} | a \text{ is in } S_\infty]$ are disjoint analytic sets whose union is S_∞. Hence, these two sets are both Borel sets. Therefore, if B_n $(n \geqq 1)$ is a sequence of Borel subsets of B which separate the points of B, then the $C_n = [aB_na^{-1} | a \text{ is in } S_\infty]$ form a sequence of invariant Borel subsets of S_∞ which separate orbits.

Hence, the quotient space $S_\infty/\equiv$ is countably separated. The natural mapping of $B \to S_\infty/\equiv$ is Borel and one-to-one onto its range. Hence, Souslin's theorem now shows that the quotient space $S_\infty/\equiv$ is standard.

One may easily check that two elements of S_∞ are conjugate under inner automorphisms if and only if they have the same number of cycles of length k, for every positive integer k, and the same number of infinite cycles.

For each of the symbols $k = 1, 2, 3, \cdots, \infty$, let $N_k = \{0, 1, 2, \cdots, \infty\}$, considered as a topological space with the discrete topology. If N is a product of certain of the N_k's, then N is a complete separable metric space. If $f: N \to S_\infty$ is continuous and injective, then $f(N)$ is a Borel subset of S_∞ by Souslin's theorem. B will be a finite union of sets of the form $f(N)$, for certain choices of f and N.

First of all, let $N = \prod_{1 \leq k < \infty} N_k$. There is a continuous injective mapping $f: N \to S_\infty$ onto a transversal for the permutations which contain only finite cycles. Identify the integers with the positive integers. If $a = (a_1, a_2, \cdots)$ is an element of N, think of a_k as representing the number of cycles of length k. Define $f(a)$ by the obvious Cantor diagonal process, starting from 1 and moving right. For example, $f((3, 2, 0, 3, 1, \cdots)) = (1)(2)(3, 4)(5, 6)(7)(8, 9, 10, 11)(12, 13, 14, 15, 16)(17, 18, 19, 20) \cdots$. Check easily that f is continuous and one-to-one onto a transversal for the permutations which contain only finite cycles. Let $B_1 = f(N)$.

There are only countably many conjugacy classes of infinite cycles permutations which contain only finitely many finite cycles. Let B_2 be a countable set which is a transversal for these conjugacy classes.

The only permutations which remain to be considered are those which contain an infinite number of finite cycles and at least one infinite cycle. Let B_2' be those elements of B_2 which contain no finite permutations. Identify the integers with the even integers, and then with the odd integers. Let B_3 be those permutations which on the even integers are an element of B_1, and which on the odd integers are an element of B_2'. B_3 is a Borel set which is a transversal for those permutations which contain an infinite number of finite cycles and at least one infinite cycle.

Let B be the union of B_1, B_2, and B_3. B is a Borel set which is a transversal for $S_\infty/\equiv$.

Note that one cannot use the results of Effros [1] for Proposition 3.1, as there are infinitely many conjugacy classes which are dense in S_∞. As this is the case, one cannot conclude that S_∞/G is countably separated, where G is an open subgroup of S_∞ which acts by inner automorphisms. This will cause a slight technical complication in the next section.

A computation shows that if a is an element of S_∞ which is not a finite permutation, then the conjugacy class of a has power of the continuum. Hence, if μ is a σ-finite Borel measure on S_∞ which is quasi-invariant under inner automorphisms, then the only point masses of μ must lie in the finite permutations.

Let G be an open subgroup of S_∞ and let a be an element of S_∞ which is not a finite permutation. Then $[bab^{-1} | b \text{ is in } G]$ is not compact. Indeed, simple computations show that $[bab^{-1} | b \text{ is in } G]$ is not even bounded. In this computation we use the fact that any open subgroup of S_∞ contains an open subgroup of the form G_B, where B is a finite set of integers, and G_B is the subgroup of S_∞ which leaves B pointwise fixed.

LEMMA 3.2. *Let a be an element of S_∞ which is not a finite permutation. Let B be a nonempty subset of the integers, and let B' be a much larger subset. For b in G_B, let $C_b = [bcac^{-1}b^{-1} | c \text{ is in } G_{B'}]$. Then for sufficiently large B', infinitely many C_b's are disjoint.*

Proof. Choose B'' so large that a does not act as the identity on B'' and $a(B'')$ is not contained in B. This is possible since a is not a finite permutation. Let B' be the union of B'' and $a(B'')$. Then $cac^{-1} | B'' = a | B''$ for all c in $G_{B'}$. Choose p in B'' so that $a(p) = q$, $p \neq q$, and p and q are not in B. For a fixed large r and each integer n, let $b_n = (p, r)(q, n)$. Each b_n is an element of G_B for all large n. A computation shows that $b_n a b_n^{-1}(r) = n = b_n cac^{-1} b_n^{-1}(r)$ for all c in $G_{B'}$, and for all large n. Hence, for all large n, the C_{b_n}'s are disjoint.

4. Proof of Theorem 1.1.

The proof of Theorem 1.1 is largely carried out through a sequence of lemmas. Let μ be a σ-finite Borel measure on S_∞ which is quasi-invariant under automorphisms and which is not supported by the finite permutations. It may then be supposed that $\mu(\{x\}) = 0$ for all x in S_∞. For each t in S_∞, let $\mu_t(E) = \mu(t^{-1}Et)$ for all Borel subsets E of S_∞. By assumption, each μ_t is absolutely continuous with respect to μ. We may assume that $\mu(S_\infty)$ is finite.

LEMMA 4.1. *The mapping $t \to \mu_t(E)$ is a Borel mapping on S_∞ for each Borel subset E of S_∞.*

Proof. If E is an open subset of S_∞, the mapping $t \to \mu_t(E)$ is upper semicontinuous by Fato's Lemma. Let $\underline{S} = [E | E \text{ is a Borel subset of } S_\infty, \text{ and the mapping } t \to \mu_t(E) \text{ is a Borel function on } S_\infty]$. $\underline{S}$ contains the open sets, is closed under complements since $\mu(S_\infty)$

is finite, and is closed under countable unions by the monotone convergence theorem and the fact that a limit of a sequence of Borel functions is again a Borel function. Hence, $t \to \mu_t(E)$ is a Borel function for all Borel sets E.

Proposition 2.1 now shows that there is a Borel function $d(t, x)$ on $S_\infty \times S_\infty$ so that $d(t, \cdot)$ is a Radon-Nikodym derivative of μ_t with respect to μ. If f is in $L^2(S_\infty, \mu)$, define $(U(t)f)(x) = f(t^{-1}xt)(d(t, x))^{1/2}$. One can compute that $t \to U(t)$ is a homomorphism of S_∞ into $U(L^2(S_\infty, \mu))$.

LEMMA 4.2. *The mapping $t \to U(t)$ is continuous in the strong operator topology.*

Proof. It is an easy consequence of Theorem B, p. 168, Halmos [2], that $L^2(S_\infty, \mu)$ is separable. Hence, $U(L^2(S_\infty, \mu))$ is a Polish group. In order to show that $t \to U(t)$ is continuous, it suffices, by a well known theorem of Banach, to show that $t \to U(t)$ is a Borel mapping. To do this, simple approximations show that it suffices to prove that $t \to \int (U(t)\chi_E)(x)\chi_F(x)d\mu(x) = \int \chi_E(t^{-1}xt)\chi_F(x)d(t, x)^{1/2}d\mu(x)$ is a Borel mapping, for every pair of Borel subsets E and F of S_∞. This, in turn, will be true, if the following holds. Let $f(t, x)$ be a nonnegative Borel function on $S_\infty \times S_\infty$ so that each $f(t, \cdot)$ is in $L^1(S_\infty, \mu)$. Then the mapping $t \to \int f(t, x)d\mu(x)$ is a Borel mapping. This statement is clearly true if f is the characteristic function of a Borel rectangle in $S_\infty \times S_\infty$. Let $\underline{S} = [B|B$ is a Borel subset of $S_\infty \times S_\infty$, and the mapping $t \to \int_{\chi_{B(t,x)}} d\mu(x)$ is a Borel mapping. $\underline{S}$ contains all Borel rectangles, is closed under complements since $\mu(S_\infty)$ is finite, and is closed under countable unions by the monotone convergence theorem and the fact that the limit of a sequence of Borel functions is again a Borel function. Hence, $\underline{S}$ contains all Borel subsets of $S_\infty \times S_\infty$. Therefore, the above statement holds for all nonnegative Borel step functions. Choose an increasing sequence of nonnegative step functions $f_n(t, x)$ on $S_\infty \times S_\infty$ which converge to $f(t, x)$. Such a sequence exists by standard arguments since f is nonnegative. The statement now follows in general, again by the monotone convergence theorem and the fact that the limit of a sequence of Borel functions is a Borel function.

LEMMA 4.3. *There is an open subgroup G of S_∞ and a finite G-invariant Borel measure ν on S_∞ which is absolutely continuous with respect to μ.*

Proof. By Theorem 3 of Lieberman [4], there is an open subgroup G of S_∞ so that $U|G$ contains the trivial representation as a direct summand. Hence, there exists a unit vector $f(x)$ in $L^2(S_\infty, \mu)$ so that $f(t^{-1}xt)(d(t, x))^{1/2} = f(x)$ μ-almost everywhere, for every t in G. Hence, $|f(t^{-1}xt)|^2 d(t, x) = |f(x)|^2$ μ-almost everywhere. Define $\nu(E) = \int |f(x)|^2 \chi_E(x) d\mu(x)$. A simple computation completes the proof.

Proof of Theorem 1.1. Let C be the support of ν. C is G-invariant. Let $q\colon S_\infty \to S_\infty/\equiv\, = Y$ be the natural quotient mapping. $q(C)$ is an analytic subset of the standard Borel space Y. Define $\tilde{\nu}(E) = \nu(q^{-1}(E))$ for all Borel subsets E of Y. For each y in Y, there is a Borel measure λ_y supported on $q^{-1}(y) \cap C$, so that if f is a positive Borel function on C, then $y \to \int f d\lambda_y$ is a Borel mapping, and $\int f d\nu = \int\left(\int f d\lambda_y\right) d\tilde{\nu}(y)$. Furthermore, if $y \to \lambda'_y$ is another such choice of measures on Y, then $\lambda'_y = \lambda_y$ for $\tilde{\nu}$-most all y (see Lemma 11.1, Mackey [6], and the references cited there). Since $\nu(C)$ is finite, $\lambda_y(C)$ must be finite for $\tilde{\nu}$-almost all y. By altering the λ_y's to be the zero measure on a $\tilde{\nu}$-null Borel set, it may be supposed that $\lambda_y(C)$ is finite for all y in Y. Let $[a_n | n \geqq 1]$ be a dense sequence in G. As ν is G-invariant and q is G-equivariant, an argument analogous to the preceding one shows it may be assumed that each λ_y is invariant under each a_n. Now suppose that λ is a finite Borel measure on C which is invariant under each a_n. Then if U is open in C, $\lambda(aUa^{-1}) = \lambda(U)$ for all open sets U, by two uses of Fatou's Lemma. A standard argument now shows that λ is invariant under G. Hence, it may be assumed that each λ_y is G-invariant. Choose an a in C so that a is not a finite permutation and $\lambda_{q(a)}$ is not the zero measure. Such an a must exist. As G is open in S_∞, there are only countably many G-orbits in $q^{-1}(q(a)) \cap C$. At least one of these orbits must have been positive $\lambda_{q(a)}$-measure. It may be supposed that the G-conjugacy class of a is this orbit. Let G_a be the centralizer of a in G. There is a natural continuous, bijective, G-equivariant mapping $\psi\colon G/G_a \to$ (G-conjugacy class of a). Use ψ to transfer $\lambda_{q(a)}$ to a finite G-invariant Borel measure λ on G/G_a. By Lemma 3.2 there is an open subgroup H of G and a sequence of elements $b_n (n \geqq 1)$ in G so that the sets $b_n H G_a$ are disjoint in G/G_a. As H is open and λ is G-invariant, $\lambda(b_n H G_a) = \lambda(H G_a) > 0$. Hence, $\lambda(G/G_a) = \infty$. This is a contradiction. Thus, there is no σ-finite Borel measure on S_∞ which is quasi-invariant under automorphisms and is not supported on the finite permutations.

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