PACIFIC JOURNAL OF MATHEMATICS
Vol. 77, No. 2, 1978

ON THE OSCILLATORY AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF FIFTH ORDER SELFADJOINT DIFFERENTIAL EQUATIONS

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In this paper the fifth order selfadjoint differential equation

$$(1) \qquad (z'''' + 2p(x)z)' + 2p(x)z' = 0$$

is considered under the assumption that $p(x)$ is a positive continuous function defined on the half axis $[0, \infty)$. The oscillation and asymptotic properties of certain solutions of (1) will be discussed after which connections between the solutions of (1) and the solutions of the fourth order differential equation

$$(2) \qquad y'''' - p(x)y = 0$$

are investigated. More spcifically, it is shown that (1) is oscillatory if and only if (2) is oscillatory.

While the literature is unusually scanty on the solutions of odd order selfadjoint differential equations, oscillation properties of the selfadjoint third order equation

$$(3) \qquad (y'' + 2b(x)y)' + 2b(x)y' = 0$$

has been studied by several authors, including J. H. Barrett [3], G. D, Jones [4] and S. C. Tefteller [7]. All of these works utilized the fact that if u and v are solutions of

$$(4) \qquad w'' + b(x)w = 0 ,$$

then the functions u^2, uv, and v^2 are solutions of (3).

Recall that a nontrivial solution of (1) or (2) is said to be *oscillatory* if it has arbitrarily large zeros, otherwise it is termed *nonoscillatory*. In case (1) has an oscillatory solution we say that (1) is *oscillatory*. A similar definition holds for equation (2). For convenience, the term "solution" for the remainder of this work will refer to nontrivial solutions unless otherwise noted.

Tefteller in [7] proved that (3) is oscillatory if and only if (4) is oscillatory. While in [4] it was shown that if (3) is oscillatory then the solution space of (3) has a basis with i oscillatory solutions and $3 - i$ nonoscillatory solutions for $i = 0, 1, 2, 3$. It is these observations and the aforementioned connections between the solutions of (1) and (2) that motivates this study. Finally, we refer to the works of Ahmad [1] and Leighton and Nehari [5] on selfadjoint fourth order differential equations.

II. Zero properties of solutions. In order to facilitate our study of (1), the following definitions are needed:

DEFINITION. A solution z of (1) is said to have an (i, j)-*distribution* of zeros if it has a zero of *multiplicity* i followed by a zero of *multiplicity* j. For $t \geqq 0$, $r_{ij}(t)$ is the infimum of the set of numbers $b > t$ such that there is a nontrivial solution of (1) having an (i, j)-distribution of zeros on $[t, b]$. Equation (1) is termed (i, j)-disconjugate if no solution has an (i, j)-distribution of zeros.

DEFINITION. The first conjugate point of t, $\eta_1(t)$, is the smallest number $s > t$ such that there exists a nontrivial solution of (1) which vanishes at t and has five zeros on $[t, s]$.

In his paper [6], T. L. Sherman has established for nth order linear differential equations that

$$\eta_1(t) = \min_{i+j=n} \{r_{ij}(t)\} .$$

Applying this result to (1) and using the selfadjointness of (1) it is clear that

$$\eta_1(t) = \min \{r_{32}(t), r_{41}(t)\} . \tag{5}$$

LEMMA 2.1. *If z is a solution of* (1), *then*

$$J(z) = z(z'''' + 2p(x)z) - z'z''' + 1/2z''^2$$

is a constant determined by the initial values of z. Furthermore, for a solution z of (1), *the functional*

$$F(z) = zz''' - 2z'z''$$

is nonincreasing on $[0, \infty)$ *whenever* $J(z) \leqq 0$.

Proof. Computing $J'(z)$ and making appropriate substitutions from (1), we find that $J'(z) \equiv 0$, from which the first part of the lemma follows. For the proof of the remaining part of the lemma, note that $F'(z) = J(z) - 5/2z''^2 - 2p(x)z^2$.

THEOREM 2.2. *Equation* (1) *is both* (2, 3)-*disconjugate and* (3, 2)-*disconjugate on* $(0, \infty)$.

Proof. Suppose z is a solution of (1) having a zero of multiplicity at least three at $x = a$, $a > 0$. Then $J(z) \equiv 0$ and consequently, $F[z]$ is decreasing. Thus $0 < x < a$ implies $F[z(x)] > 0$ and $a < x$ implies $F[z(x)] < 0$, from which it follows easily that z cannot have a multiple zero to the right nor left of $x = a$.

COROLLARY 2.3. *Let u and v be independent solutions of* (1) *having triple zeros at $x=c$. Then $W(u, v)(x)=u(x)v'(x)-v(x)u'(x)\neq 0$ for $x \neq c$. Consequently, the zeros of u and v separate each other on $[0, c)$ and (c, ∞).*

From (5) and Theorem 2.2 we can now obtain more precise information about how the first conjugate points are determined, namely

THEOREM 2.4. *If $\eta_1(t)$ exists for $t \geqq 0$, then $\eta_1(t) = r_{41}(t)$.*

III. Oscillation properties. Using $J(z)$ and $F(z)$ defined above we now group the solutions of (1) into three distinct classes. A solution z of (1) is termed *Type A* if $J(z) \equiv 0$ and $F[z] > 0$ on $[0, \infty)$, *Type B* if $J(z) \leqq 0$ and $F[z] \leqq 0$ on $[b, \infty)$, for some $b > 0$. Finally, a *Type C* solution is any solution z with $J(z) > 0$.

To obtain results concerning the solutions in these various classes we will need the so-called "double zero" lemma.

LEMMA 3.1. *Suppose $u, v \in C'[a, b]$ and satisfies $u(\alpha) = u(\beta) = 0$, $a < \alpha < \beta < b$, $u(x) \neq 0$ on (α, β) and $v(x) \neq 0$ on $[\alpha, \beta]$. Then some nontrivial linear combination of u and v has a double zero in (α, β).*

The condition $r_{41}(t) < \infty$ for each $t \geqq 0$ is both necessary and sufficient for the oscillation of (1) as we see in our next result.

THEOREM 3.2. *Equation* (1) *is oscillatory if and only if $r_{41}(t) < \infty$ for each t.*

Proof. Suppose (1) is oscillatory and that $r_{41}(t) = \infty$ for some t. Then $r_{41}(s) = \infty$ for all $s > t$ and so (1) is disconjugate on (t, ∞), contradicting the fact that (1) has an oscillatory solution.

For the converse suppose $r_{41}(t) < \infty$ for all t. We will show that Type B solutions are oscillatory.

Suppose there is a nonoscillatory Type B solution. Then there is a number b and a solution $y(x)$ of (1) satisfying $y(x) > 0$ on $[b, \infty)$, $F[y(x)]<0$ on $[b, \infty)$ and $J(y)\leqq 0$. Let $u(x)$ be a solution of (1) having a (4, 1)-distribution of zeros, say at $x = c$ and $x = d$, where $b \leqq$

$c \leqq d$ and $u(x) \neq 0$ on (c, d). We assume without loss of generality that $u''''(c) = 1$. Since $y(x) > 0$ on (c, d) and $u(x) > 0$ on (c, d), there is a positive constant k such that $v(x) = y(x) - ku(x)$ has a double zero in (c, d). Since $F[v(x)]$ is decreasing ($J(v) \leqq 0$) it follows that $F[v(c)] > 0$. But $F[v(c)] = F[y(c)] < 0$, a contradiction. Consequently, it follows that $y(x)$ must be oscillatory. Using the fact that $r_{41}(t) = r_{14}(t)$ and making some easy modifications in the above argument we can show that Type A solutions must oscillate.

COROLLARY. *If* (1) *is oscillatory then every Type A and Type B solution is oscillatory.*

COROLLARY. *If* (1) *is oscillatory then a nonoscillatory solution is Type C.*

While it is clear that Type B and Type C solutions exist, our next result shows that equation (1) always has a Type A solution.

THEOREM 3.4. *There exists a Type A solution of equation* (1).

Proof. Suppose $a \geqq 0$. Let $\{x_n\}_{n=1}^{\infty}$ be an increasing sequence of points such that $a < x_1$ and $\lim_{n\to\infty} x_n = \infty$.

Let $z_1(x)$, $z_2(x)$, $z_3(x)$, $z_4(x)$, $z_5(x)$ be five independent solutions of (1).

Define

$$u_n(x) = c_{1n}z_1(x) + c_{2n}z_2(x) + c_{3n}z_3(x) + c_{4n}z_4(x) + c_{5n}z_5(x)$$

where

$$u_n(a) = 0 \,,$$

$$u_n(x_n) = u_n'(x_n) = u_n''(x_n) = 0$$

and

$$c_{1n}^2 + c_{2n}^2 + c_{3n}^2 + c_{4n}^2 + c_{5n}^2 = 1 \,.$$

Note that $J(u_n(x)) \equiv 0$ and $F[u_n(x)] > 0$ on $[0, x_n)$, for each n.

The sequences $\{c_{in}\}_{n=1}^{\infty}$ for $i = 1, 2, 3, 4, 5$ are bounded and hence we can assume without loss of generality that $\lim_{n\to\infty} c_{in} = c_i$, $n = 1, 2, 3, 4, 5$. Since $c_1^2 + c_2^2 + c_3^2 + c_4^2 + c_5^2 = 1$, it follows that

$$u(x) = c_1z_1(x) + c_2z_2(x) + c_3z_3(x) + c_4z_4(x) + c_5z_5(x)$$

is a nontrivial solution of (1).

We claim that $u(x)$ is Type A. First note that $F[u_n(x)] > 0$ on $[0, x_n)$, consequently, $F[u(x)] \geqq 0$ on $[0, \infty)$.

If $F[u(x_0)] = 0$ for some $x_0 \geqq 0$, then $F[u(x)] \equiv 0$ on $[x_0, \infty)$, but this implies $F'[u(x)] = -5/2u''^2(x) - 2p(x)u^2)x) \equiv 0$, contradicting the fact that $u(x)$ is nontrivial. Therefore we must have $F[u(x)] > 0$ on $[0, \infty)$.

In the previous proof note that the Type A solution u constructed vanished at $x = a$. If $x = b$ is a point such that $u(b) \neq 0$, then a Type A solution which vanishes at $x = b$ can be constructed which is clearly independent of the one which vanished at $x = a$. Thus equation (1) has at *least* two independent Type A solutions.

We now list some properties of Type A solutions.

THEOREM 3.5. *Suppose* (1) *has an oscillatory solution. Let* z *be a Type* A *solution of* (1), *then the following hold*:

(i) $\int^\infty z''^2(x)dx < \infty$,

(ii) $\int^\infty p(x)z^2(x)dx < \infty$, *and*

(iii) z' *is bounded.*

Proof. Since $J(z) \equiv 0$, $F[z(x)]$ is decreasing. By differentiating $F[z(x)]$ and then integrating from 0 to x, we obtain

$$0 < F[z(x)] = F[z(0)] - \frac{5}{2}\int_0^x z''^2(t)dt - \frac{3}{2}\int_0^x p(t)z^2(t)dt$$

from which (i) and (ii) follow.

To prove (iii) note that

$$0 < F[z(x)] = z(x)z'''(x) - 2z'(x)z''(x) = \left[z(x)z''(x) - \frac{3}{2}z'^2(x)\right]' .$$

Clearly $F[z(x)] \to 0$ as $x \to \infty$, for if not, we would have $z(x)z''(x) - 3/2z'^2(x) \to \infty$ as $x \to \infty$ which is obviously impossible since $z(x)$ is oscillatory. Since $z(x)z''(x) - 3/2z'^2(x)$ is negative and increasing, letting $x \to \infty$ along the zeros of $z''(x)$ we see that $z'(x)$ is bounded.

For Type B solutions we offer the following result, but omit the proof.

THEOREM 3.6. *Suppose* (1) *has an oscillatory solution. Let* z *be a Type* B *solution with* $J(z) < 0$, *then*

(i) *Either* $\int^\infty p(x)z^2(x)dx = \infty$ *or* $\int^\infty z''^2(x)dx = \infty$, *and*

(ii) $\limsup_{x\to\infty} z'(x) = \infty$, *i.e.*, $z'(x)$ *is unbounded.*

For the case $p(x) \equiv 1$, the resulting equation for (1) is

$$z'''' + 4z' = 0 . \tag{6}$$

A basis for the solution space of (6) is

$$\{e^{x}\sin x,\ e^{x}\cos x,\ e^{-x}\sin x,\ e^{-x}\cos x,\ 1\}\ .$$

Note that the solutions $e^{-x}\sin x$ and $e^{x}\sin x$ are Type A and Type B solutions of (6), respectively, and satisfy the conclusions of Theorems 3.5 and 3.6.

In the preceding example we note that (6) has a nonoscillatory solution. To show that this is true in general we proceed through equation (2).

THEOREM 3.7. *Let u and v be independent solutions of*

$$y'''' - p(x)y = 0\ .$$

Then $z(x) = u(x)v'(x) - v(x)u'(x)$ is a solution of (1).

It is well known [1] that (2) has a pair of solutions u and v satisfying

$$(7)\qquad u(x) > 0\ ,\qquad u'(x) > 0\ ,\qquad u''(x) > 0\ ,\qquad u'''(x) > 0\ ,$$

and

$$(8)\qquad v(x) > 0\ ,\qquad v'(x) < 0\ ,\qquad v''(x) > 0\ ,\qquad v'''(x) < 0$$

on $[0, \infty)$. Clearly the Wronskian of this pair of solutions does not vanish on $[0, \infty)$ and we have proven

THEOREM 3.8. *Equation* (1) *has a nonvanishing solution.*

Actually (1) always has at least three independent nonoscillatory solutions. Whether or not three is the maximum possible when (1) is oscillatory remains an open question.

Finally, we establish a connection between the oscillation of (1) and the oscillation of (2).

THEOREM 3.9. *Equation* (1) *is oscillatory if and only if* (2) *is oscillatory.*

Proof. Suppose (2) is oscillatory. Let u be an oscillatory solution of (2). Suppose v is a nonoscillatory solution of (1) satisfying either (7) or (8). Then $z = uv' - vu'$ is an oscillatory solution of (1).

For the converse, suppose (2) is nonoscillatory, then (2) is eventually disconjugate, see [5]. So there exists $a \geqq 0$ so that no solution of (2) has more than three zeros on $[a, \infty)$. Let y_1 and y_2 be independent solutions of (2) having double zeros at $x = a$. Then $W(y_1, y_2)(x) \neq 0$ for $x > a$, for if $W(y_1, y_2)(\bar{x}) = 0$ for some $\bar{x} > a$, then some combination of y_1 and y_2 has four zeros on $[a, \bar{x}]$ contradicting the fact that (2) is disconjugate on $[a, \infty)$. But $z = W(y_1, y_2)$ is a solution of (1) having a zero of order four at $x = a$. Thus $\eta_1(a) = r_{41}(a) = \infty$ since z does not vanish on (a, ∞). Consequently, (1) is nonoscillatory, in fact, disconjugate on (a, ∞).

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Received October 3, 1977.

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