PACIFIC JOURNAL OF MATHEMATICS
Vol. 77, No. 2, 1978

THE STRONG BIDUAL OF $\Gamma(K)$

D. C. TAYLOR

Let A be a C^*-algebra, K the Pedersen ideal of A, and $\Gamma(K)$ the two-sided multipliers of K under the κ-topology. In this paper a study is made of the strong bidual of $\Gamma(K)$, which we denote by $\Gamma(K)''$. Here it is shown that the Arens products in $\Gamma(K)''$ are well defined and coincide, and therefore make $\Gamma(K)''$ a $*$-algebra; moreover, if A is a PCS-algebra or has a σ-compact spectrum, it is shown that $\Gamma(K)''$ is a metrizable b^*-algebra which is isometrically $*$-isomorphic to the Cartesian product of W^*-algebras.

Now suppose A is just a Banach algebra. In [3] Arens defined two natural extensions of the product of A to the strong bidual A''. If it is assumed that A is also a $*$-algebra, then it is well known and easy to verify that a natural extension of the involution of A can be defined for A'' whenever the two products coincide. When A is a C^*-algebra, it is also well known that the two Arens products for A'' coincide and the resulting $*$-algebra is a C^*-algebra. For proofs of these facts, we refer the reader to [7], [20], and [23].

The fact that A'' is a C^*-algebra whenever A is a C^*-algebra has been very useful. For example, by focusing on certain elements from A'', noncommutative analogues of the bounded Baire, Borel, and semicontinuous functions on a locally compact Hausdorff space have been developed. All of this suggests that the strong bidual of other locally convex topological algebras may be of equal importance. It is the purpose of this paper to study the strong bidual of an important class of topological algebras which we will shortly define.

Let A be a C^*-algebra, K the Pedersen ideal of A, $\Gamma(K)$ the two-sided multipliers of K under the κ-topology. When A is commutative, $\Gamma(K)$ is the $*$-algebra of all complex valued continuous functions defined on the spectrum of A under the compact open topology. In the general case, $\Gamma(K)$ can be viewed as a $*$-algebra of unbounded operators on a pre-Hilbert space that has many of the nice properties of C^*-algebras. In this paper we shall study the strong bidual of $\Gamma(K)$, which we denote by $\Gamma(K)''$. We show that the Arens products in $\Gamma(K)''$ are well defined and coincide, and therefore make $\Gamma(K)''$ a $*$-algebra. Moreover, if A is a PCS-algebra or has a σ-compact spectrum, we show that $\Gamma(K)''$ is a metrizable b^*-algebra which is isometrically $*$-isomorphic to the Cartesian product of W^*-algebras. It is our opinion that noncommutative analogues of unbounded Baire, Borel, and semicontinuous functions on a locally compact Hausdorff space can be developed in this setting. However, in this paper we study only $\Gamma(K)'$, the strong dual of $\Gamma(K)$, and $\Gamma(K)''$, leaving these other topics for study at a later time.

Another study of this type has already been made. Gulick in his two fine papers [8] and [9] investigated the algebraic properties of the strong bidual of a locally multiplicatively-convex topological algebra. However, multiplication in $\Gamma(K)$ is not jointly continuous, or even hypo-continuous, so Gulick's work can not be applied in our setting. Moreover, due to the special nature of $\Gamma(K)$, our results are of a special nature.

In § 2 we characterize the κ-bounded subsets of $\Gamma(K)$ and we study the strong dual of $\Gamma(K)$. In § 3 we study the strong bidual of $\Gamma(K)$. In both of these studies PCS-algebras and PCS-representations play a fundamental role. In fact the main result is obtained by regarding the C^*-algebra A with σ-compact spectrum as a sort of inductive limit of PCS-algebras. The reader is referred to [10], [17], [18] for basic definitions and concepts of topological vector spaces and to [5], [12], [14], [15], [16] for basic concepts and definitions of multipliers and Pedersen's ideal.

2. The strong dual of $\Gamma(K)$. Throughout this section A will denote a C^*-algebras and A' the dual of A under the norm topology. Let K_A denote the Pedersen ideal of A (or K when A is understood), $\Gamma(K)$ the two-sided multipliers of K, and $\Delta(K)$ the bounded multipliers of K. Let $\Gamma(K)^*$ denote the algebraic dual of $\Gamma(K)$ and let $\Gamma(K)'$ denote the space of κ-continuous linear functionals defined on $\Gamma(K)$ under the $s(\Gamma(K)', \Gamma(K))$ topology, that is, the topology of uniform convergence of κ-bounded subsets of $\Gamma(K)$. The $s(\Gamma(K)', \Gamma(K))$ topology is called the strong topology for $\Gamma(K)'$. In this section we will study the locally convex topological linear space $\Gamma(K)'$ and its topological completion $\tilde{\Gamma}(K)'$.

The C^*-algebra A is called a PCS-algebra if $\Gamma(K_A) = \Delta(K_A)$. A representation φ of A is called a PCS-representation if $A/\ker \varphi$ is a PCS-algebra (see [12, 10.2, p. 94]), and we denote the set of all PCS-representations of A by PCS(A). For each $a \in K$ the canonical representation of A generated by a [12, 5.3, p. 18] is a PCS-representation [12, 5.4, p. 19]. Any irreducible representation is a PCS-representaion [12, 10.4, p. 94]. For any representation φ of A there exists a unique κ-continuous linear map $\bar{\varphi}: \Gamma(K_A) \to \Gamma(K_{\varphi(A)})$ that extends φ [12, 3.11, p. 10]. We will also let φ denote the extension $\bar{\varphi}$ when no confusion will arise.

2.1. LEMMA. *Let $\{e_\lambda\}_{\lambda\in\Lambda}$ be a positive increasing approximate identity for A contained in K. Then for a subset Q of $\Gamma(K)$ the following statements are equivalent:*

(i) *For each $\varphi\in PCS(A)$, $\{\varphi(x): x\in Q\}$ is a uniformly bounded subset of $\Delta(K_{\varphi(A)})$;*

(ii) *For each $a\in K$, $\{\varphi_a(x): x\in Q\}$ is a uniformly bounded sebset of $\Delta(K_{\varphi_a(A)})$, where φ_a is the canonical representation of A generated by a;*

(iii) *For each $\lambda\in\Lambda$, $\{\varphi_{e_\lambda}(x): x\in Q\}$ is a uniformly bounded subset of $\Delta(K_{\varphi_{e_\lambda}(A)})$, where φ_{e_λ} is the canonical representation of A generated by e_λ;*

(iv) *Q is a κ-bounded subset of $\Gamma(K)$.*

Proof. Clearly, (i) implies (ii) and (ii) implies (iii). Suppose (iii) holds and $a\in K$. Recall that for $x\in\Gamma(K)$, $\|\varphi_a(x)\|=\|L_x^{(a)}\|$, where $L_x^{(a)}:\mathscr{L}_a\to\mathscr{L}_a$ and $\mathscr{L}_a$ is the closed left ideal generated by a [12, 5.2, p. 17]. We know there exists a $\lambda\in\Lambda$, unitary elements $u_1, u_2, \cdots, u_n$ in $\tilde{A}$ (the C^*-algebra formed by adjoining the identity to A), and positive numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ such that $aa^*\leqq\sum_{i=1}^n\alpha_i u_i e_\lambda u_i{}^*$ [12, p. 20]. Since $e_\lambda^{1/2}\in\mathscr{L}_{e_\lambda}$, it now follows that for $x\in Q$

$$\|xa\|^2\leqq\sum_{i=1}^n\alpha_i\|xu_ie_\lambda u_i^*x^*\|\leqq\sum_{i=1}^n\alpha_i\|\varphi_{e_\lambda}(x)\|^2 .$$

Since $\{\varphi_{e_\lambda}(x): x\in Q\}$ is uniformly bounded, $\sup\{\|xa\|: x\in Q\}<\infty$. Similarly, $\sup\{\|ax\|: x\in Q\}<\infty$, so Q is κ-bounded and (iv) holds.

Finally assume (iv) holds and $\varphi\in\mathrm{PCS}(A)$. Due to the fact that $\varphi(K_A)=K_{\varphi(A)}$ [16, Corollary 6, p. 268], $\varphi(Q)$ is a κ-bounded subset of $\Gamma(K_{\varphi(A)})=\Delta(K_{\varphi(A)})$. Suppose that $\varphi(Q)$ is not uniformly bounded. Then we can choose a sequence $\{z_n\}$ of elements in Q such that $\|\varphi(z_n)\|\geqq 16^n$. Set

$$x_n=(z_n^*z_n)/\|\bar{\varphi}(z_n)\|^{3/2} ,$$

and note $x_n\geqq 0$ and $\|x_n\|\geqq 4^n$. Let $a\in K_{\varphi(A)}{}^+$. Since $\varphi(Q)$ is κ-bounded, there is a $\delta>0$ such that $\|z_na\|<\delta$ for each integer n. It follows that

$$\begin{aligned}\|(\sum_{k=m}^n x_k)a\| &\leqq\sum_{k=m}^n\|\varphi(z_k^*)\varphi(z_k)a\|/\|\varphi(z_k)\|^{3/2}\\ &\leqq\sum_{k=m}^n\delta/\|\varphi(z_k)\|^{1/2}\leqq\delta\sum_{k=m}^n 1/4^k\leqq\delta/4^{m-1} .\end{aligned}$$

Consequently, the sequence of partial sums $\{\sum_{k=1}^n x_k\}$ is κ-Cauchy. Since $\Gamma(K_{\phi(A)})$ is complete, the sequence of partial sums $\{\sum_{k=1}^n x_k\}$ converges in the κ-topology to some $x\in\Delta(K_{\phi(A)})$. Since $x\geqq x_n\geqq 0$ for each positive integer n, $\|x\|\geqq\|x_n\|\geqq 4^n\to\infty$, which is a contradiction. Hence our proof is complete.

2.2. COROLLARY. *If* Q_1 *and* Q_2 *are* κ*-bounded subsets of* $\Gamma(K)$, *then so is* $Q_1 \cdot Q_2 \equiv \{xy: x \in Q_1, y \in Q_2\}$.

2.3. COROLLARY. *A subset* Q *of* $\Gamma(K)$ *is* κ*-bounded if and only if* $\{|x|: x \in Q\}$ *is* κ*-bounded.*

Let $\pi: A \to B(H_\pi)$ be an irreducible representation of A. Recall that π has a natural extension $\tilde{\pi}: \Gamma(K_A) \to B(H_\pi)$, which we still denote by π (cf. [12, 4.5, p. 15]).

2.4. PROPOSITION. *A subset* Q *of* $\Gamma(K_A)$ *is* κ*-bounded if and only if*

$$\sup_{x \in Q} \sup_{\pi \in F} \|\pi(x)\| < \infty$$

for each compact subset F *of the spectrum* $\hat{A}$.

Proof. Suppose Q is a κ-bounded subset of $\Gamma(K)$ and $a_1, a_2, \cdots, a_n$ are elements of K_A. Let $\mathscr{L}$ be the closed left ideal of A generated by $\{a_i\}_{i=1}^n$. From [12, Chapter 3] we see $\mathscr{L} \subseteqq K_A$ and $L_x: \mathscr{L} \to \mathscr{L}$ (left multiplication by x) is a bounded linear operator for each $x \in Q$. Since Q is κ-bounded, we have by the principle of uniform boundedness that $\sup_{x \in Q} \|L_x\| < \infty$. Now the assertion follows exactly as in [12, 7.2, p. 48]. The converse is a direct consequence of [12, 5.39, p. 35].

2.5. PROPOSITION. *If* $\varphi_1, \varphi_2 \in PCS(A)$, *then there is a* $\varphi \in PCS(A)$ *such that* $\ker \varphi_1 \cap \ker \varphi_2 = \ker \varphi$.

Proof. Let $J_i = \ker \varphi_i$, $i = 1, 2$, and set $J = J_1 \cap J_2$. Clearly all we need to show is that A/J is a PCS-algebra. Let Q be a κ-bounded subset of A/J and $\psi_i: A/J \to A/J_i$ the map defined by $\psi_i(x+J) = x + J_i$. By virtue of [16, Corollary 6, p. 268] and [12, 10.1, p. 89] the set $\psi_i(Q)$ is a uniformly bounded subset of A/J_i. Let M be an upper bound of $\{\|\psi_i(x + J)\|: x + J \in Q, i = 1, 2\}$. It follows from [6, 3.2] and [6, 3.3] that $\|x + J\| = \sup \{\|\pi(x)\|: \pi \in \hat{A}_J\} = \sup\{\|\pi(x)\|: \pi \in \hat{A}_{J_1} \cup \hat{A}_{J_2}\} \leqq M$ for all $x + J$ in Q. Hence by [12, 10.1, p. 89] A/J is a PCS-algebra and our proof is complete.

We are now ready to begin our study of $\Gamma(K)'$ and its topological completion $\tilde{\Gamma}(K)'$. The topological linear space $\tilde{\Gamma}(K)'$ will be viewed as the space of all linear functionals on $\Gamma(K)$ that are

κ-continuous on κ-bounded subsets of $\Gamma(K)$ [17, Theorem 2, p. 103]. Now let φ be a *-homomorphism of A onto the C^*-algebra B and $\bar{\varphi}: \Gamma(K_A) \to \Gamma(K_B)$ the unique κ-continuous *-homomorphism that extends φ. Note that if $\psi: \Gamma(K_A) \to \Gamma(K_B)$ is any *-homomorphism with $\psi(x) = \varphi(x)$, $x \in A$, then ψ is κ-continuous so must be equal to $\bar{\varphi}$ everywhere. Let $\bar{\varphi}': \Gamma(K_B)' \to \Gamma(K_A)'$ be defined by $\bar{\varphi}'(f)(x) = f(\bar{\varphi}(x))$, $x \in \Gamma(K_A)$. This mapping is known as the adjoint of $\bar{\varphi}$. For $f \in \tilde{\Gamma}(K_A)'$, define f^* on $\Gamma(K_A)$ by $f^*(x) = \overline{f(x^*)}$, which clearly belongs to $\tilde{\Gamma}(K_A)'$. Also note that if $f \in \Gamma(K_A)'$, then $f^* \in \Gamma(K_A)'$.

2.6. PROPOSITON. *Let φ be a *-homomorphism of $\Gamma(K_A)$ into $\Gamma(K_B)$ with $\varphi(A) = B$. Then for the adjoint map φ' the following hold*: (i) $\varphi'(\Gamma(K_B)') \subseteq \Gamma(K_A)'$; (ii) *$\varphi'$ is continuous with respect to the $s(\Gamma(K_B)', \Gamma(K_B))$ and $s(\Gamma(K_A)', \Gamma(K_A))$ topologies*; (iii) *φ' has a unique extension $\tilde{\varphi}'$ which is a continuous linear mapping of $\tilde{\Gamma}(K_B)'$ into $\tilde{\Gamma}(K_A)'$*; (iv) $\tilde{\varphi}'(f^*) = \tilde{\varphi}'(f)^*$, $f \in \tilde{\Gamma}(K_B)'$.

Proof. The first assertion follows from the fact that φ is κ-continuous. The second and third follow from [16, Corollary 6, p. 268] and [17, Theorem 6, p. 107], respectively. The last assertion is due to the fact that the map $f \to f^*$ is strongly continuous.

2.7. PROPOSITION. *Let $\psi: f \to f \mid A$ be the restriction mapping of $\tilde{\Gamma}(K_A)'$ into A'. Then ψ is one-to-one and $\psi(f^*) = \psi(f)^*$.*

Proof. Let $f \in \tilde{\Gamma}(K_A)'$ be such that $\psi(f) = 0$. Let $x \in \Gamma(K_A)^+$ and $\{e_\lambda\}$ a positive increasing approximate identity for A contained in K_A. By virtue of [12, Chapter 5], $\{x^{1/2}e_\lambda x^{1/2}\}$ is a κ-bounded subset of $\Gamma(K)^+$; moreover, by [12, 3.4, p. 8] $x^{1/2}e_\lambda x^{1/2} \to x$ in the κ-topology. Thus $f(x) = 0$, since f is κ-continuous on κ-bounded sets. It follows $f(x) = 0$ for all $x \in \Gamma(K)$ by virtue of [12, 5.25, p. 29], so ψ is one-to-one. The other assertion is trivial to prove.

Let I be a closed two-sided ideal of A and J the κ-closure of I. It is known that J is a self adjoint two-sided ideal of $\Gamma(K_A)$ [12, Chapter 8]. Now define

$$J^\perp \equiv \{f \in \tilde{\Gamma}(K_A)': f \mid J = \{0\}\} \quad \text{and} \quad I^\perp \equiv \{f \in A': f \mid I = \{0\}\} .$$

2.8. PROPOSITION. *Let $\psi; J^\perp \to I^\perp$ be defined by $\psi(f) = f \mid A$. If A/I is a PCS-algebra, then the map ψ is a bicontinuous mapping of $J^\perp$ onto $I^\perp$ under the strong and uniform topologies.*

Proof. Let $g \in I^\perp$ and $\varphi: A \to A/I$ the natural quotient map. It is well known that there is a unique $h_0 \in (A/I)'$, $||h_0|| = ||g||$, such that $g(x) = h_0(\varphi(x))$ for each $x \in A$. By virtue of [21, Theorem 2.1, p. 634] there exist elements u, v in A/I and a uniformly continuous linear function h on $\Delta(K_{A/I})$ such that $\|u \cdot h \cdot v\| = \|h_0\|$ and $h_0(x) = u \cdot h \cdot v(x)$ for every $x \in A/I$. Set $f = u \cdot h \cdot v$. Since A/I is a PCS-algebra, f is defined everywhere on $\Gamma(K_{A/I})$; moreover, by 2.1, the κ-bounded subsets of $\Gamma(K_{A/I})$ are uniformly bounded, so it is easy to verify that $f \in \tilde{\Gamma}(K_{A/I})'$. Now let φ also denote the κ-continuous map that extends φ to $\Gamma(K_A)$, and let $\tilde{\varphi}'$ be the unique extension of the adjoint map to $\tilde{\Gamma}(K_{A/I})'$. By 2.6, $\tilde{\varphi}'(f) \in \tilde{\Gamma}(K_A)'$. It follows that for each $x \in A$,

$$\begin{aligned}\tilde{\varphi}'(f)(x) &= \lim \tilde{\varphi}'(f_\alpha)(x) = \lim f_\alpha(\varphi(x)) \\ &= f(\varphi(x)) = h_0(\varphi(x)) = g(x) ,\end{aligned}$$

where $\{f_\alpha\}$ is a net in $\Gamma(K_{A/I})'$ that converges to f strongly. So $\psi(\tilde{\varphi}'(f)) = g$.

It is clear that ψ is a continuous mapping of $J^\perp$ onto $I^\perp$. Now let g_α be a net in $I^\perp$ that converges to 0 uniformly and f_α the linear functional $\Gamma(\tilde{K}_{A/I})'$ constructed as above. Since $\|f_\alpha\| = \|g_\alpha\|$, $f_\alpha \to 0$ uniformly. Hence $f_\alpha \to 0$ strongly by virtue of 2.1. Thus by 2.6, $\tilde{\varphi}'(f_\alpha) \to 0$ strongly. But $\tilde{\varphi}'(f_\alpha) | A = g_\alpha$, $\psi^{-1}(g_\alpha) = \tilde{\varphi}'(f_\alpha)$ and our proof is complete.

Let $\varphi \in \mathrm{PCS}(A)$. As noted earlier, φ has a natural extension to $\Gamma(K_A)$ into $\Gamma(K_{\varphi(A)}) = \Delta(K_{\varphi(A)})$ which we still denote by φ. Let J_φ be the κ-closed two-sided ideal of $\Gamma(K_A)$ given by $J_\varphi = \ker \varphi$.

2.9. COROLLARY. *For each $\varphi \in PCS(A)$, $\tilde{\Gamma}(K_{\varphi(A)})'$ is the linear span of its positive part, $J_\varphi^\perp = \tilde{\varphi}'(\Gamma(\tilde{K}_{\varphi(A)})')$ and*

$$\|f\| = \|f | \varphi(A)\| = \|\tilde{\varphi}'(f) | A\|$$

for each f in $\tilde{\Gamma}(K_{\varphi(A)})'$.

2.10. PROPOSITION. *The space*

$$\Gamma(K_A)' = \bigcup_{a \in K_A} \varphi_a'(\Gamma(K_{\varphi(A)})') \subseteqq \bigcup_{a \in K_A} J_{\varphi_a}^\perp \subseteqq \bigcup_{\varphi \in PCS(A)} J_\varphi^\perp \subseteqq \Gamma(\tilde{K}_A)'$$

where φ_a is the canonical representation of A generated by a.

Proof. The proof is straightforward, by virtue of [12, 6.1, p. 42] and given that $K_{\varphi_a(A)} = \varphi_a(K_A)$.

2.11. COROLLARY. *The linear span of the positive, κ-continuous, linear functionals defined on $\Gamma(K_A)$ is strongly dense in $\tilde{\Gamma}(K_A)'$.*

Proof. Let $\varphi \in \mathrm{PCS}(A)$ and $f \in J_\varphi^\perp$. By virtue of 2.9, we may assume $f \geqq 0$; moreover, we may assume that there is a $g \in \tilde{\Gamma}(K_{\varphi(A)})'$ and an $a \in A^+$ such that $\tilde{\varphi}'(\varphi(a)\cdot g\cdot\varphi(a)) = a\cdot\tilde{\varphi}'(g)\cdot a = f$ [21, Theorem 2.1, p. 634]. Let Q be a κ-bounded subset of $\Gamma(K_A)$ and $\{e_\lambda\}$ an approximate identity for A contained in K. Then for $x \in Q$

$$\begin{aligned}|e_\lambda\cdot f\cdot e_\lambda(x) - f(x)| &= |g(\varphi(e_\lambda a)\varphi(x)\varphi(ae_\lambda) - \varphi(a)\varphi(x)\varphi(a))| \\ &\leqq ||\varphi(e_\lambda a) - \varphi(a)||\, ||\varphi(xae_\lambda)|| + ||\varphi(ae_\lambda) - \varphi(a)||\, ||\varphi(ax)||\ ,\end{aligned}$$

so $e_\lambda\cdot f\cdot e_\lambda \to f$ strongly by virtue of 2.1. Since $e_\lambda\cdot f\cdot e_\lambda \in \Gamma(K_A)'^+$, the assertion follows from 2.10.

2.12. LEMMA. *Let $f \in \Gamma(K_A)'^+$. Then the linear maps $x \to f\cdot x$ and $x \to x\cdot f$ of $\Gamma(K_A)$ into $\Gamma(K_A)'$ are continuous under the κ and strong topologies.*

Proof. The proof follows immediately from [12, 6.5, p. 46].

Let $F \in \Gamma(K)'^*$ and define V to be the linear span of $\Gamma(K_A)'^+$. For each $f \in V$ define the linear functionals $\lambda_F(f)$ and $\rho_F(f)$ on $\Gamma(K_A)$ by the formulas:

$$\lambda_F(f)(x) = F(f\cdot x)\ , \qquad \rho_F(f)(x) = F(x\cdot f)\ .$$

It follows from 2.12 that $\lambda_F(f)$ and $\rho_F(f)$ are well defined.

2.13. LEMMA. *If F is strongly continuous, then λ_F, ρ_F are strongly continuous linear mappings of V into $\Gamma(K_A)'$.*

Proof. For each $f \in V$, it follows immediately from 2.12 that $\lambda_F(f)$ and $\rho_F(f)$ belong to $\Gamma(K_A)'$. It is obvious the maps are linear, so it remains to be shown that they are strongly continuous. Suppose $\{f_\alpha\}$ is a net in V that converges strongly to 0. Let $\varepsilon > 0$. Since F is strongly continuous there exists a κ-bounded set $Q_1 \subseteqq \Gamma(K_A)$ such that $|F(g)| < \varepsilon$ for all $g \in \Gamma(K_A)'$ with the property $|g(x)| \leqq 1$ whenever $x \in Q_1$. Now let Q_2 be any κ-bounded subset of $\Gamma(K_A)$. From 2.2, it follows that $Q_1\cdot Q_2$ is κ-bounded. Choose α_0 so that for $\alpha > \alpha_0$ $\sup\{|f_\alpha(x)|: x \in Q_1\cdot Q_2\} < 1$. This implies, for $\alpha > \alpha_0$ and $z \in Q_2$, $|z\cdot f_\alpha(x)| \leqq 1$ for each $x \in Q_1$. Hence

$$|F(z\cdot f_\alpha)| = |\rho_F(f_\alpha)(z)| < \varepsilon$$

for all $z \in Q_2$. This means $\rho_F(f_\alpha) \to 0$ strongly and therefore ρ_F is strongly continuous. Similarly, λ_F is strongly continuous and our proof is complete.

2.14. DEFINITION. Let $\{I_n\}_{n=1}^\infty$ be a decreasing sequence of closed two-sided ideals of A and let I_n^0 denote the closed two-sided ideal of A (not necessarily proper) given by $I_n^0 \equiv \{x \in A: xI_n = \{0\}\}$. If, for each positive integer n, A/I_n is a PCS-algebra, and if $\bigcup_{n=1}^{\infty} I_n^0$ is dense in A, then $\{I_n\}_{n=1}^{\infty}$ is called a PCS-sequence of closed two-sided ideals. Note $K_A \subseteqq \bigcup_{n=1}^{\infty} I_n^0$ (cf. [11, 2]).

2.15. PROPOSITION. *Suppose A has a σ-compact spectrum $\hat{A}$. Then A contains a PCS-sequence of closed two-sided ideals.*

Proof. Since $\hat{A}$ is σ-compact, one can show by using [6, 3.3.2, p. 63] and [12, 5.39, p. 35] that there is an increasing sequence $\{a_n\}$ in K_A^+ such that

$$\bigcup_{n=1}^{\infty} \{\pi \in \hat{A}: \|\pi(a_n)\| > 0\} = \hat{A} .$$

For each integer n set $I_n = \ker \varphi_{a_n}$, where φ_{a_n} is the canonical representation of A generated by a_n. As noted earlier A/I_n is a PCS-algebra and clearly $\{I_n\}_{n=1}^{\infty}$ is a decreasing sequence, so all that remains to be shown is $U_{n=1}^{\infty} I_n^0$ is dense in A. Set

$$U_n = \{\pi \in \hat{A}: \|\pi(a_n)\| > 0\}$$

and

$$I_n' = \{x \in A: \pi(x) = 0 , \qquad \pi \in \hat{A}/U_n\} .$$

Note that for $\pi \in U_n$ and $x \in I_n$, $\pi(x) = 0$. Hence $I_n' \subseteqq I_n^0$. Let $b \in K_A$. Since $\{U_n\}$ is an open cover of $\hat{A}$, it follows from [12, 5.39, p.35] that $\{\pi \in \hat{A}: \|\pi(b)\| > 0\} \subseteqq U_n$ for some positive integer n. Thus $b \in I_n'$. It now follows that $K_A \subseteqq \bigcup_{n=1}^{\infty} I_n' \subseteqq \bigcup_{n=1}^{\infty} I_n^0$. Hence $\{I_n\}_{n=1}^{\infty}$ is a PCS-sequence of closed two-sided ideals.

2.16. COROLLARY. *If A is a C^*-algebra with a countable approximate identity, then A contains a PCS-sequence of closed-two sided ideals.*

2.17. COROLLARY. *If A is a separable C^*-algebra, then A contains a PCS-sequence of closed two-sided ideals.*

2.18. LEMMA. *Suppose A contains a PCS-sequence of closed two-sided ideals $\{I_n\}_{n=1}^{\infty}$ and let $\varphi_n: A \to A/I_n$ denote the natural quotient mapping. A subset Q of $\Gamma(K_A)$ is κ-bounded if and only if $\{\bar{\varphi}_n(x): x \in Q\}$ is a uniformly bounded subset of $\Delta(K_{\varphi_n(A)})$ for each positive integer n. Here $\bar{\varphi}_n$ denotes the natural extension to $\Gamma(K_A)$.*

Proof. Let $\{e_\lambda\}_{\lambda \in \Lambda}$ be a positive increasing approximate identity for A contained in K_A. Recall that $K_A \subseteqq \cup I_n^0$. Hence for each $\lambda \in \Lambda$ there is a positive integer n for which $e_\lambda \in I_n^0$, so $I_n \subseteqq \ker \varphi_{e_\lambda}$, where φ_{e_λ} is the canonical representation of A generated by e_λ. Set $I'_\lambda = \ker \varphi_{e_\lambda}$. Assume $\{\bar{\varphi}_n(x): x \in Q\}$ is a uniformly bounded subset of $\Delta(K_{\varphi_n(A)})$ for each positive integer n. Thus, for each n,

$$\sup_{x \in Q} \sup_{\pi \in \hat{A}_{Jn}} ||\pi(x)|| < \infty ,$$

where $\hat{A}_{J_n} = \{\pi \in \hat{A}: \pi(I_n) = 0\}$. Now let F be a compact subset of $\hat{A}$. Clearly there is a e_λ such that $\{\pi \in \hat{A}: ||\pi(e_\lambda)|| > 0\} \supseteqq F$. Consequently, $F \subseteqq \hat{A}'_{I_\lambda}$. So there is a positive integer n for which $F \subseteqq \hat{A}_{I_n}$. It follows that Q is κ-bounded by virtue of 2.4. The converse is immediate and our proof is complete.

2.19. PROPOSITION. *Suppose A is a C^*-algebra with a PCS-sequence $\{I_n\}_{n=1}^\infty$ of closed two-sided ideals. Let J_n denote the κ-closure of I_n for each positive integer n. Then a subset Q' of $\Gamma(\tilde{K}_A)'$ is strongly bounded if and only if $Q' \subseteqq J_n^\perp$, for some positive integer n, and*

$$\sup_{f \in Q'} ||f \mid A|| < \infty .$$

Proof. First, recall that a subset Q' of $\Gamma(\tilde{K}_A)'$ is strongly bounded if and only if

$$\sup_{f \in Q'} \sup_{x \in Q} |f(x)| < \infty$$

for each κ-bounded subset Q of $\Gamma(K_A)'$. Now assume $Q' \subseteqq J_n^\perp$, for some n and $\sup\{||f|A||: f \in Q'\} < \infty$. Let $\varphi_n: A \to A/I_n$ be the natural quotient map. Let φ_n also denote the natural extension $\varphi_n: \Gamma(K_A) \to \Gamma(K_{\varphi_n(A)})$ and let $\varphi'_n: \Gamma(\tilde{K}_A)' \to \Gamma(\tilde{K}_A)'$ the extended adjoint map. By virtue of 2.9 there is a subset W of $\Gamma(\tilde{K}_{\varphi_n(A)})'$ such that $\varphi'_n(W) = Q'$ and

$$||g|| = ||\varphi'_n(g) \mid A||$$

for each $g \in W$. It follows that each κ-bounded subset Q of $\Gamma(K_A)$

$$\begin{aligned}\sup_{f \in Q'} \sup_{x \in Q} |f(x)| &= \sup_{g \in W'} \sup_{x \in Q} |g(\varphi_n(x))| \leqq \sup_{g \in W'} ||g|| \sup_{x \in Q} |\varphi_n(x)| \\ &\leqq \sup_{f \in Q'} ||f \mid A|| \sup_{x \in Q} |\varphi_n(x)| < \infty\end{aligned}$$

by virtue of 2.1, so Q' is strongly bounded.

Now assume Q' is a strongly bounded subset. Since the unit ball of A is a κ-bounded subset of $\Gamma(K_A)$, it follows that

$$\sup\{||f \mid A||: f \in Q'\} < \infty ,$$

so all we need to show is $Q' \subseteqq J_n^\perp$ for some positive integer n. Assume not. By virtue of 2.7 there is a sequence $\{(f_n, x_n)\}_{n=1}^\infty$ of ordered pairs such that $f_n \in Q'$, $x_n \in I_n^+$, and $f_n(x_n) = n$. Let $a \in K_A^+$. Since $\{I_n\}_{n=1}^\infty$ is a PCS-sequence of closed two-sided ideals, there is a positive integer N such that $a \in I_n^0$ for all $n \geqq N$. Thus when $n \geqq N \|ax_n\| = \|x_n a\| = 0$. This means that $\{x_n\}$ is a κ-bounded subset of $\Gamma(K_A)$. But $f_n(x_n) = n$, which contradicts the fact Q' is strongly bounded. It follows that $Q' \subseteqq J_n^\perp$ for some positive integer n and our proof is complete.

2.20. COROLLARY. *The space* $\tilde{\Gamma}(K_A)' = \bigcup_{n=1}^\infty J_n^\perp$; *consequently,* $\tilde{\Gamma}(K_A)'$ *is the linear span of its positive part.*

Proof. The first assertion is a direct result of [17, Corollary 1, p. 102] and 2.10. The second assertion follows from 2.9.

2.21. PROPOSITION. *Suppose A is a C^*-algebra with a PCS-sequence $\{I_n\}_{n=1}^\infty$ of closed two-sided ideals. For each $f \in A'$ there is a unique decomposition $f = \sum_{n=1}^\infty f_n$ such that $f_n \in I_n^\perp$ and*

$$\|f_{n+1}\| = \|f_{n+1} | I_n\| .$$

Moreover, the following statements hold:

(i) $\|f\| = \sum_{n=1}^\infty \|f_n\|$;

(ii) *if there is a g in $\tilde{\Gamma}(K_A)'$ such that $g | A = f$, then there is a positive integer N for which $f_n = 0$ whenever $n \geqq N$;*

(iii) *if f can be extended to a κ-continuous linear functional defined on $\Gamma(K_A)$, then so can each f_n.*

Proof. Let us view A under the universal representation as a self adjoint algebra of operators on a Hilbert space H with trivial null space. Let $\mathscr{B}$ be the closure of A in the weak operator topology. Let $f \in A'$. Then there is unique ultrastrongly continuous function $\tilde{f}$ on $\mathscr{B}$ that extends A [6, 12.1.1, p. 235]. Now let P_n be the central projection in $\mathscr{B}$ that is the identity for the weak closure of I_n. Set $Q_1 = E - P_1$, $Q_2 = P_1 - P_2, \cdots, Q_n = P_{n-1} - P_n$. Here E denotes the identity of $\mathscr{B}$. Next set $\tilde{f}_n = Q_n \cdot \tilde{f}$ and $f_n = \tilde{f}_n | A$. It is clear that $f_n \in I_n^\perp$, $\|f_{n+1}\| = \|f_{n+1} | I_n\|$, and $\sum_{n=1}^\infty \|f_n\| \leqq \|f\|$. Since $\bigcup I_n^0$ is dense in A, it is straightforward to show that $P_n \to 0$ in the ultrastrong topology. Hence for $x \in A$

$$\begin{aligned} |f(x) - \sum_{n=1}^N f_n(x)| &= |f(x) - \tilde{f}(\sum_{n=1}^N Q_n x)| \\ &= |\tilde{f}(P_N x)| \longrightarrow 0 \end{aligned}$$

as $N \to \infty$. Thus $f(x) = \sum_{n=1}^\infty f_n(x)$, so (i) holds. The uniqueness of the decomposition $f = \sum_{n=1}^{\infty} f_n$ follows directly from [22, Corollary 2.3 and Corollary 2.7, pp. 162, 164]. So to complete the proof we must show (ii) and (iii) hold.

Statement (ii) is a trivial consequence of 2.20. Now suppose f can be extended to a κ-continuous functional on $\Gamma(K)$. By virtue of [12, 6.1, p. 42], $f = b\cdot g + h\cdot b$ for some $b \in K$ and g, h in A'. Since it is clear that $f_n = b\cdot g_n + h_n\cdot b$, it follows that (iii) holds. Hence our proof is complete.

3. The strong bidual of $\Gamma(K)$.

Throughout this section we will use the same notation as in § 2. Here we will study the strong bidual of $\Gamma(K)$, that is, the space of all $s(\Gamma(K)', \Gamma(K))$ continuous linear functionals on $\Gamma(K)'$, denoted by $\Gamma(K)''$, under the $s(\Gamma(K)'', \Gamma(K)')$ topology. The $s(\Gamma(K)'', \Gamma(K)')$ topology for $\Gamma(K)''$ is the topology of uniform convergence on $s(\Gamma(K)', \Gamma(K))$ bounded subsets of $\Gamma(K)'$ and this topology is called the strong topology for $\Gamma(K)''$.

Now let $F, G \in \Gamma(K)''$ and also let F, G denote their unique extensions to $\tilde{\Gamma}(K)'$. Let V be the linear span of $[\Gamma(K_A)']^+$. By virtue of 2.13 the map $f \to F(\lambda_G(f))$ is a $s(\Gamma(K)', \Gamma(K))$ continuous linear functional defined on V. Since V is strongly dense in $\tilde{\Gamma}(K)'$ (see 2.11), this linear functional has a unique extension to all of $\tilde{\Gamma}(K)'$ which is also $s(\Gamma(K)', \Gamma(K))$ continuous. Denote this extended map by $F\circ_l G$. It follows that $\Gamma(K)''$ is an algebra, where multiplication is defined by $F\circ_l G$. This multiplication is called Arens left multiplication. Now define $F\circ_r G(f) = G(\rho_F(f))$, $f \in V$. In similar fashion, $\Gamma(K)''$ is an algebra under multiplication given by $F\circ_r G$. This is called Arens right multiplication. If $F\circ_l G = F\circ_r G$ for all F, G in $\Gamma(K)''$, then Arens multiplication is denoted by $F\circ G$.

Recall that a locally convex topological algebra $\mathscr{A}$ is called locally m-convex if there exists a family of continuous seminorms $\{p_i\}_{i\in\Lambda}$ on $\mathscr{A}$ that generates the topology of $\mathscr{A}$ and satisfies the condition:

$$(*) \qquad p_i(xy) \leqq p_i(x)p_i(y)$$

for each x, y in $\mathscr{A}$ and $i \in \Lambda$. A locally convex topological algebra $\mathscr{A}$ is called a b^*-algebra if it is a complete, locally m-convex $*$-algebra, each of whose defining seminorms p_i satisfies $(*)$ and the condition:

$$p_i(x^*x) = [p_i(x)]^2 .$$

The concept of locally m-convex algebras was introduced by Arens [2] and Michael [13]. The concept of b^*-algebras was introduced by Allan [1]. Now if $\mathscr{A}$ and $\mathscr{A}'$ are b^*-algebras with defining seminorms $\{p_i\}_{i\in\Lambda}$ and $\{p'_i\}_{i\in\Lambda'}$, respectively, then $\mathscr{A}$ and $\mathscr{A}'$ are said to be b^*-isomorphic if there is a *-isomorphism τ of $\mathscr{A}$ onto $\mathscr{A}'$ such that for each $i\in\Lambda$ there is a $j\in\Lambda'$ so that $p'_j(\tau(x)) = p_i(x)$ for each $x\in\mathscr{A}$ and similarly for each $j\in\Lambda'$ there is a $i\in\Lambda$ so that $p_i(\tau^{-1}(y)) = p_j(y)$ for each $y\in\mathscr{A}'$.

The next theorem is the main result of this paper.

3.1. THEOREM. *Let A be a C^*-algebra. Then the following statements hold:*

(i) *The Arens products in $\Gamma(K_A)''$ are well defined and coincide; consequently, under Arens multiplication $\Gamma(K_A)''$ is a *-algebra, where involution $F\to F^*$ is given by $F^*(f) = \overline{F(f^*)}$.*

(ii) *The natural embedding $x\to F_x$ of $\Gamma(K_A)$ into $\Gamma(K_A)''$ is an algebraic *-isomorphism of $\Gamma(K_A)$ onto a $s(\Gamma(K_A)'', \Gamma(K_A)')$ closed *-subalgebra of $\Gamma(K_A)''$.*

(iii) *If $\psi: A''\to\Gamma(K_A)'^*$ is the map defined by $\psi(F)(f) = F(f\mid A)$, then ψ is an algebraic *-isomorphism of A'' into $\Gamma(K_A)''$.*

(iv) *If A possesses a PCS-sequence of closed two-sided ideals $\{I_n\}$ (in particular, if A is a PCS-algebra or has a σ-compact spectrum), then $\Gamma(K_A)''$ is a metrizable b^*-algebra that is b^*-isomorphic to a countable Cartesian product of W^*-algebras.*

Before we can proceed with the proof of 3.1 we must first prove a series of lemmas.

3.2. LEMMA. *Let A and B be C^*-algebras and φ a *-homomorphism of $\Gamma(K_A)$ into $\Gamma(K_B)$ with $\varphi(A) = B$. Then for the second adjoint map φ'' the following hold:*

(i) *$\varphi''(\Gamma(K_A)'')\subseteqq\Gamma(K_B)''$.*

(ii) *φ is continuous with respect to the $s(\Gamma(K_A)'', \Gamma(K_A)')$ and $s(\Gamma(K_B)'', \Gamma(K_B)')$ topologies.*

(iii) *φ'' is a homomorphism with respect to both left and right Arens multiplication.*

Proof. It follows from 2.6 and [17, Proposition 12, p. 38] that (i) holds. Since all topologies of a dual pair have the same bounded sets (ii) is trivial. Statement (iii) follows immediately from 2.11, 2.12, 2.13, and the definitions of Arens multiplications and the adjoint map.

Let A be a C^*-algebra. Let $\lambda\in\mathrm{PCS}(A)$ and also let λ denote its extension to $\Gamma(K_A)$. Throughout the remainder of this section we will let $J_\lambda = \{x\in\Gamma(K_A): \lambda(x) = 0\}$ and $I_\lambda = x\in A: \lambda(x) = 0\}$, while letting $J_\lambda^\perp$ and $I_\lambda^\perp$ be defined as in § 2. By 2.5 we see that $\mathrm{PCS}(A)$ is a directed set, where by $\lambda_1\leqq\lambda_2$ we mean $J_{\lambda_2}\subseteqq J_{\lambda_1}$.

If $Q \subseteqq J_\lambda^\perp$, for some $\lambda \in \mathrm{PCS}(A)$, and $\sup\{\|f|A\|: f \in Q\}$ is finite, then by the argument given for 2.19 we have that Q is a $s(\Gamma(K_A)', \Gamma(K_A))$ bounded subset of $\Gamma(K_A)'$. However, in general it is not known whether or not the converse holds, but it does hold for a large class of C^*-algebras. In fact, if A possesses a PCS-sequence of closed two-sided ideals, then the converse follows from 2.19. Now let $\lambda \in \mathrm{PCS}(A)$ and define $p_\lambda: \Gamma(K_A)'' \to R^+$ by the formula

$$p_\lambda(F) = \sup\{|F(f)|: f \in J_\lambda^\perp\,, \qquad \|f|A\| \leqq 1\}\,.$$

Clearly, p_λ is a $s(\Gamma(K_A)'', \Gamma(K_A)')$ continuous seminorm for $\Gamma(K_A)''$.

3.3. LEMMA. *Let $\lambda \in PCS(A)$ and $F \in \Gamma(K_A)''$. Then the following statements hold*:

(i) $p_\lambda(F) = \sup\{|F(a\cdot f)|: f \in J_\lambda^\perp, a \in K_A, \|a\cdot f|A\| \leqq 1\} = \sup\{|F(f\cdot a)|: f \in J_\lambda^\perp, a \in K_A, \|f\cdot a|A\| \leqq 1\}$.

(ii) *If $\{I_{\lambda_n}\}$ is a PCS-sequence of closed two-sided ideals of A, then $\{p_{\lambda_n}\}$ generates the $s(\Gamma(K_A)'', \Gamma(K_A)')$ topology of $\Gamma(K_A)''$.*

Proof. Statement (i) follows from 2.1, 2.9, [21, Theorem 2.1, p. 634], and the fact every C^*-algebra has an approximate identity. Statement (ii) follows from 2.19.

3.4. LEMMA. *Let A be a C^*-algebra. If $a \in K_A$ and B_a is the hereditary C^*-subalgebra of K_A generated by a, then there is a $\lambda_0 \in PCS(A)$ such that $p_\lambda(F_x) = \|x\|$ for each $x \in B_a$ and $\lambda \geqq \lambda_0$. Here p_λ is defined as above.*

Proof. Assume $a \in K_A$ and B_a is the hereditary C^*-algebra of K_A generated by a. Let λ_a be the canonical representation of A generated by a. Clearly, $a \in I_{\lambda_a}^0$, and, therefore, $B_a \subseteqq I_{\lambda_a}^0$. It follows that $\{b\cdot g: b \in B_a, g \in A'\} \subseteqq I_{\lambda_a}^\perp$. Thus for each $x \in B_a$ and $\lambda \geqq \lambda_a$

$$\begin{aligned}
\|x\| &= \sup\{|f(x)|: f \in B_a, \|f\| < 1\} \\
&= \sup\{|b\cdot g(x)|: b \in B_a\,, \qquad \|b\| \leqq 1, g \in A', \|g\| \leqq 1\} \\
&\leqq \sup\{|f(x)|: f \in I_{\lambda_a}^\perp\,, \qquad \|f\| \leqq 1\} \\
&= \delta_{\lambda_a}(F_x) \leqq \delta_\lambda(F_x) \leqq \|x\|
\end{aligned}$$

by virtue of [21, Theorem 2.1, p. 634] and [19, Theorem 2.1, p. 142] Hence (iii) holds and our proof is complete.

3.5. LEMMA. *Suppose A is a PCS-algebra. Then the Arens products in $\Gamma(K_A)''$ are well defined and coincide; consequently, under Arens multiplication $\Gamma(K_A)''$ is a W^*-algebra, where involution $F \to F^*$ is given by $F^*(f) = \overline{F(f^*)}$. Moreover, the map ψ: $A'' \to$*

$\Gamma(K_A)'^*$ *given by* 3.1 (iii) *is an isometric *-isomorphism of* A'' *onto* $\Gamma(K_A)''$.

Proof. We have already observed that Arens products are well defined. By virtue of 2.1, 3.3, and the fact the unit ball of $\{f|A: f \in \Gamma(K_A)'\}$ is dense in the unit ball of A', we can easily see that $\Gamma(K_A)''$ is a Banach space and the map $\psi: A'' \to \Gamma(K_A)'^*$ given above is an isometric isomorphism of the Banach space A'' onto the Banach space $\Gamma(K_A)''$. Moreover, it is clear that ψ preserves both left and right Arens products. But Arens products in A'' coincide, hence they coincide in $\Gamma(K_A)''$. Now it is easy to show that $F \to F^*$ given above defines an involution for $\Gamma(K_A)''$. So $\Gamma(K_A)''$ is a Banach *-algebra and $\psi: A'' \to \Gamma(K_A)''$ is an isometric *-isomorphism of the W^*-algebra A'' onto $\Gamma(K_A)''$. Thus $\Gamma(K_A)''$ is a W^*-algebra and our proof is complete.

Proof of 3.1. We have already observed that Arens products are well defined, so statement (i) follows immediately from 2.9, 3.2, and 3.5. We shall assume throughout the remainder of the proof that $\Gamma(K_A)''$ is a *-algebra, where the multiplication is Arens multiplication and the involution is defined as in statement (i).

It is routine to show that $\Gamma(K_A)$ can be viewed under the natural embedding as a $s(\Gamma(K_A)'', \Gamma(K_A)')$ closed *-subalgebra of $\Gamma(K_A)''$. Furthermore, it is routine to show that the map $\psi: A'' \to \Gamma(K_A)'^*$ defined by the formula

$$\psi(F)(f) = F(f\,|\,A)$$

is a *-isomorphism of A'' onto the *-subalgebra $\mathscr{A}$ of $\Gamma(K_A)''$, where $\mathscr{A} = \{F \in \Gamma(K_A)'' : \sup\{p_\lambda(F): \lambda \in PCS(A)\} < \infty\}$. Also note that the adjoint mapping φ'' considered in 3.3 preserves involution.

Next let $\{I_n\}$ be a PCS-sequence of closed two-sided ideals in A and let $\{p_{\lambda_n}\}$ denote the corresponding sequence of seminorms defined above 3.3. By 3.3 the seminorms $\{p_{\lambda_n}\}$ generate the $s(\Gamma(K_A)''$, $\Gamma(K_A)')$ topology, so $\Gamma(K_A)''$ is a metric space. Note that from 2.9 and 3.3 we can deduce $p_{\lambda_n}(F) = ||\lambda_n''(F)||$ for all $F \in \Gamma(K_A)''$ and positive integer n. It follows that $p_{\lambda_n}(F \circ G) \leqq p_{\lambda_n}(F) p_{\lambda_n}(G)$ and $p_{\lambda_n}(F^* F) = [p_{\lambda_n}(F)]^2$, $F, G \in \Gamma(K_A)''$. So we will have shown $\Gamma(K_A)''$ is a b^*-algebra once we prove $\Gamma(K_A)''$ is complete.

Let F be a linear functional defined on $\Gamma(K_A)'$ that is $s(\Gamma(K)'$, $\Gamma(K))$ continuous on each $s(\Gamma(K_A)', \Gamma(K_A))$ bounded subset of $\Gamma(K_A)'$. Note that if Q is a $s(\Gamma(K_A)', \Gamma(K_A))$ bounded subset of $\Gamma(K_A)'$, then $F(Q)$ is bounded [17, Corollary 1, p. 102]. Let

$$r_n = \sup\{|F(f)| : f \in J_n^\perp \cap \Gamma(K_A)' , \quad ||f\,|\,A|| \leqq 1\} ,$$

which is finite, and observe $r_n \leqq r_{n+1}$ for each positive integer n. Now let $Q_n = \{x \in \Gamma(K_A): ||\varphi_n(x)|| \leqq 2^n r_n\}$ and set $Q = \cap Q_n$. By 2.18, Q is a κ-bounded subset of $\Gamma(K_A)$. Let Q^p denote the polar of Q taken in $\Gamma(K_A)'$ and Q^{pp} the polar of Q^p taken in $\Gamma(K_A)'^*$. Let $f \in Q^p$. Then, by 2.21, $f = \sum_{i=1}^n f_i$, where $f_i \in J_i^\perp \cap \Gamma(K_A)'$ and

$$||f_{i+1}|A|| = ||f_{i+1}|I_i|| , \qquad i = 1, 2, \cdots, n-1 .$$

Set $I_0 = A$ and then let $M_i = \{x \in I_{i-1}: 2^i r_i > ||x||\}$ for $i = 1, 2, \cdots, n$. Since $I_0 \supseteqq I_1 \supseteqq I_2, \cdots, M_i \subseteqq Q$ for $i = 1, 2, \cdots, n$. It follows that

$$\begin{aligned} 1 \geqq \sup\{|f(x)|: x \in M_i\} &= 2^i r_i ||f\,|I_{i-1}|| \\ &= 2^i r_i \sum_{j=1}^n ||f_j|A|| \geqq 2^i r_i ||f_i|A|| , \end{aligned}$$

so $||f_i|A|| \leqq 1/2^i r_i$. Therefore

$$\begin{aligned} |F(f)| &\leqq \sum_{i=1}^n ||f_i|A||\,|F((1/||f_i|A||)f_i)| \\ &\leqq \sum_{i=1}^n (1/2^i r_i)\,|F((1/||f_i|A||)f_i)| < 1 . \end{aligned}$$

So $F \in Q^{pp}$ which means $F \in \Gamma(K_A)''$ by [17, Proposition 2, p. 47]. But this implies $\Gamma(K_A)''$ is complete by virtue of [17, Theorem 2, p. 103] and therefore $\Gamma(K_A)''$ is a metrizable b^*-algebra.

Finally, we will show that $\Gamma(K_A)''$ is b^*-isomorphic to the Cartesian product of W^*-algebras. Let P_n be the central projection in A'' that is the identity for the $\sigma(A'', A')$ closure of I_n. Note that $(I - P_n)\circ F = F\circ(I - P_n) \in A''$ for each $F \in \Gamma(K_A)''$. Set $Q_1 = I - P_1$, $Q_2 = P_1 - P_2, \cdots, Q_n = P_{n-1} - P_n$ and let B_n denote the W^*-algebra $Q_n\Gamma(K_A)''$ for each positive integer n. It follows easily that $\Gamma(K_A)''$ is b^*-isomorphic to the Cartesian product $\pi_{n=1}^\infty B_n$. Hence (i), (ii), (iii), and (iv) hold and our proof is complete.

REFERENCES

1. G. R. Allan, *On a class of locally convex algebras*, Proc. London Math. Soc., (3) **17** (1967), 91–114.

2. R. F. Arens, *The space L^ω and convex topological rings*, Bull. Amer. Math. Soc', **52** (1946), 931–935.

3. ———, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc., **2** (1951), 839–848.

4. N. Bourbaki, *Espaces vectoriels topologiques*, Act. Sc. Ind. 1229, Hermann, Paris, 1964.

5. H. S. Collins, *Strict, weighted, and mixed topologies and applications*, Advances in Mathematics, **19** (1976), 207–237.

6. J. Dixmier, *Les C*-algebres et leur representations*, Gauthier-Villars, Paris, 1964.

7. A. Grothendiek, *Un résultat sul le dual d'une C*-algébra*. J. Math. Pures Appl., (9) **36** (1957), 97–108.

8. S. L. Gulick, *The bidual of a locally multiplicatively convex algebra*, Pacific J. Math., **17** (1966), 71-96.
9. ———, *Commutativity and ideals in the biduals of topological algebras*, Pacific J. Math., **18** (1966), 121-137.
10. J. Kelley and I. Namioka, *Linear Topological Spaces*, D. Van Nostrand Company, New York, 1966.
11. K. B. Laursen and A. M. Sinclair, *Lifting matrix units in C*-Algebras II*, Math. Scand., **37** (1975), 167-172.
12. A. J. Lazar and D. C. Taylor, *Multipliers of Pedersen's ideal*, Memoirs Amer. Math. Soc., **169** (1976).
13. E. Michael, *Locally multiplicatively-convex topological algebras*, Memoirs Amer. Math. Soc., **11** (1952).
14. G. K. Pedersen, *Measure theory for C*-algebras*, Math. Scand., **19** (1966), 131-145.
15. ———, *Measure theory for C*-algebras II*, Math. Scand., **22** (1968), 63-74.
16. ———, *A decomposition theorem for C*-algebras*, Math. Scand., **22** (1968), 266-268.
17. A. Robertson and W. Robertson, *Topological Vector Spaces*, Cambridge Univ. Press, New York, 1964.
18. H. Schaefer, *Topological Vector Spaces*, The MacMillan Company, New York, 1966.
19. F. D. Sentilles and D. C. Taylor, *Factorization in Banach algebras and the general strict topology*, Trans. Amer. Math. Soc., **142** (1969), 141-152.
20. Z. Takeda, *Conjugate spaces of operator algebras*, Proc. Japan Acad., **30** (1952), 90-95.
21. D. C. Taylor, *The strict topology for double centralizer algebras*, Trans. Amer. Math. Soc., **150** (1970), 633-643.
22. ———, *Interpolation in algebras of operator fields*, J. Functional Analysis, **10** (1972), 159-190.
23. M. Tomita, *The second dual of a C*-algebra*, Mem. Fac. Sci. Kyushu Univ. Ser. A **22** (1967), 185-193.

Received July 26, 1977.

MONTANA STATE UNIVERSITY
BOZEMAN, MT 59717