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SMOOTH G-MANIFOLDS AS COLLECTIONS OF FIBER BUNDLES

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This paper is about the general theory of differentiable actions of compact Lie groups. Let G be a compact Lie group acting smoothly on a manifold M. Both M and M/G have natural stratifications, and M/G inherits a "smooth structure" from M. The map $M \to M/G$ exhibits many of the properties of a smooth fiber bundle. For example, it is proved that a smooth G-manifold can be pulled back via a "weakly stratified" map of orbit spaces. Also, it is well-known (and obvious) that a smooth G-manifold is determined by a certain collection of fiber bundles together with some attaching data. Several precise formulations of this observation are given.

Introduction. We develop some elementary ideas in what might be termed "the bundle theoretical aspect" of compact transformation groups. Suppose that a compact Lie group G acts smoothly on a manifold M. First consider the case where this action has only one type of orbit, that is, where all the isotropy groups are conjugate. In this case, it follows from the Differentiable Slice Theorem that M/G is a smooth manifold and that M is a smooth fiber bundle over M/G. When the action has more than one type of orbit; this is no longer true; however, there are two related points of view.

The first of these is to regard the smooth G-manifold M together with the natural projection $\pi: M \to M/G$ as a prototypical example of a "singular fiber bundle." As such, one might expect smooth G-manifolds to have many of the formal properties of ordinary fiber bundles. In an appropriate context, this is true (as we shall see in Chapter III). One of the main purposes of this paper is to describe this context.

The second point of view is to regard M as a certain collection of fiber bundles. Here the basic idea is to consider all those points in M of a given orbit type (that is, all those points with isotropy groups conjugate to a given subgroup of G). It follows from the Differentiable Slice Theorem, again, that the union of such points is an invariant submanifold of M and, therefore, a smooth fiber bundle over its image in M/G. Thus, M is a union of various fiber bundles.

Our actual approach is a slight modification of this. In Chapter I, we define a notion of "normal orbit type," which is better suited to the study of smooth actions than is the notion of orbit type. The normal orbit type of a point takes into account the slice representation as well as the isotropy group at the point. M is also stratified by the invariant submanifolds consisting of those points of a given normal orbit type. The normal bundle in M of such a stratum can be regarded as a fiber bundle over the corresponding stratum of M/G. The associated principal bundle is called a "normal orbit bundle." Given the equivariant normal bundles of the strata and information which describes how these normal bundles fit together, one can clearly recover M. So, in some sense, a smooth G-manifold is nothing more or less than a certain collection of principal bundles together with some attaching data. This simple observation is one of the most fundamental ideas in the study of smooth G-actions (see, for example, [2] and [6]). One of the first people to isolate this basic intuition and to try to formulate it as a theorem was K. Jänich in his paper on $O(n)$-manifolds [7].

We prove a similar, but more general, version of this theorem in Chapter IV (Theorem 4.3). It states that there is a functor from the category of smooth G-manifolds and (equivariant) stratified maps to the category of "$\mathscr{G}$-normal systems," and that this functor defines a bijection between equivariant diffeomorphism classes of G-manifolds and isomorphism classes of "$\mathscr{G}$-normal systems." By a "normal system" we roughly mean a collection of bundles together with some attaching data. By a "stratified map" of G-manifolds, we mean a smooth equivariant map which preserves the stratification and which maps the normal bundle of each stratum transversely. A similar result (Theorem 4.4 in Chapter IV) is true for orbit spaces (or rather for "local G-orbit spaces"). In order to state this result and in order to describe the context in which G-manifolds behave like fiber bundles, it is necessary to take a close look at the local structure of orbit spaces and to carefully consider what should be meant by a "stratified map" of orbit spaces. This is done in Chapter II.

We should first point out that there is a natural "smooth structure" (i.e., functional structure) on M/G. This is essentially obtained by defining a function $f: M/G \to \boldsymbol{R}$ to be *smooth* if $f \circ \pi: M \to \boldsymbol{R}$ is smooth. Secondly, as we pointed out above, M/G can be stratified by normal orbit types. In Chapter II, we give two possible definitions for a "stratified map" of orbit spaces. These two definitions are distinguished by the use of the terms "weakly stratified" and "stratified." In both definitions we consider smooth, strata preserving maps which "map the normal bundle of each stratum transversely" (however, there are essentially two possible interpretations of this last phrase). In Theorem 4.5 of Chapter II, we show that an (equivariant) stratified of G-manifolds induces a stratified map of their orbit spaces. Also, stratified maps of orbit spaces are weakly stratified (Theorem 4.8 in Chapter II). The converse is an interesting open question.

In Chapter III, we prove that smooth G-manifolds can be pulled back by a weakly stratified map of orbit spaces (Theorem 1.1), a result suggested in [3]. In the second section of that chapter we discuss the Covering Homotopy Theorem of G. Schwarz [10]. These two results are the major justification for the assertion that smooth G-manifolds exhibit the same formal properties as do smooth fiber bundles.

The reason for introducing the stronger concept of a stratified map of orbit spaces is that the theory of normal systems for orbit spaces works only with this stronger definition.

There are many concrete applications of the above ideas, but we do not discuss them in this paper. It should be mentioned, however, that the pullback construction and the theory of normal systems play key roles in the classification of regular $O(n)$, $U(n)$ and $Sp(n)$-manifolds in [4] and [5] (also see [3]).

Some of the work in this paper was done in my thesis (in the special case of regular $O(n)$, $U(n)$ and $Sp(n)$-actions). I would like to thank my thesis advisor W. C. Hsiang for his help while I was writing my thesis and for his continuing support and encouragement. I would also like to thank G. Bredon and G. Schwarz for several illuminating conversations. Finally, I want to point out that Bredon's book [2] is an excellent introduction to this material and that references are made to it throughout this paper.

I. Stratification by Normal Orbit Type

1. Normal orbit types. This section contains some preliminary material. The theorems are well-known; however, some of the definitions are not.

Let G be a compact Lie group. Suppose that H is a closed subgroup and that V is an H-module. Define an action of H on $G \times V$ by $h \cdot (g, v) = (gh^{-1}, hv)$. Let $G \times_H V$ denote the orbit space and let $[g, v]$ denote the image of (g, v) in $G \times_H V$. Then $G \times_H V$ has the structure of a G-vector bundle over G/H with projection map defined by $[g, v] \to gH$. In fact, any G-vector bundle over G/H must be of this form. For if E is such a bundle, then the fiber of E at the identity coset is an H-module. If we denote this H-module by V, then the map $[g, v] \to gv$ defines an isomorphism $G \times_H V \cong E$.

Now suppose that G acts smoothly on a manifold M. Given $x \in M$, let G_x denote the isotropy group at x and let $G(x)$ denote the orbit passing through x. $G(x)$ can be identified with G/G_x via the equivariant embedding $gG_x \to gx$ (see page 302 in [2]). Hence, the normal bundle of $G(x)$ in M can be identified with a G-vector bundle over G/G_x. The fiber of this normal bundle at x is the G_x-module

$$S_x = T_xM/T_xG(x) .$$

S_x is called the *slice representation at* x. By our initial remarks, the normal bundle of $G(x)$ is isomorphic to $G \times_{G_x} S_x$. By the Equivariant Tubular Neighborhood Theorem, there is an equivariant diffeomorphism from the normal bundle of $G(x)$ onto a G-invariant neighborhood of $G(x)$ (see [2]). Taken together these two observations constitute the following well-known theorem of Koszul.

THEOREM 1.1. (*The Differentiable Slice Theorem*). *Each point* $x \in M$ *has a* G*-invariant neighborhood of the form* $G \times_H S$, *where* $H = G_x$ *and* $S = S_x$.

Next, let F_x be the subspace of S_x on which G_x acts trivially, i.e., let $F_x = (S_x)^{G_x}$. Define a G_x-module V_x, called the *normal representation at* x, by

$$V_x = S_x/F_x .$$

So to each $x \in M$ we have associated a closed subgroup G_x and a G_x-module V_x.

This situation can be abstracted as follows. Consider all pairs (H, V) where H is a closed subgroup of G and where V is a H-module with $V^H = \{0\}$. Two such pairs (H, V) and (H', V') are *equivalent* if there is an element $a \in G$ and a linear isomorphism $L\colon V \to V'$ such that $aHa^{-1} = H'$ and such that the following diagram commutes

$$\begin{array}{ccc} H & \xrightarrow{\phi} & \mathrm{Aut}\,(V) \\ {\scriptstyle i_a}\downarrow & & \downarrow{\scriptstyle i_L} \\ H' & \xrightarrow{\phi'} & \mathrm{Aut}\,(V') . \end{array}$$

Here ϕ and ϕ' are the associated representations, $i_a(h) = aha^{-1}$ and $i_L(f) = LfL^{-1}$. An equivalence class of such pairs will be called a *normal* G*-orbit type*. Let $[H, V]$ denote the class of (H, V).

If G acts smoothly on M and $x \in M$, then $[G_x, V_x]$ is called the *normal orbit type of* $G(x)$. For this definition to make sense we must have that (G_x, V_x) and (G_{gx}, V_{gx}) are equivalent. This is indeed the case; for, the required equivalence is (g, L), where $L\colon V_x \to V_{gx}$ is the map induced by dg.

REMARK 1.2. Traditionally, one speaks of an "orbit type" by which one means a conjugacy class of an (isotropy) subgroup of G. The notion of normal orbit type should be regarded as a slight refinement of this, appropriate to the study of smooth actions.

PROPOSITION 1.3. *Let (a, L) be an equivalence from (H, V) to (H', V'). Then the map $\theta_{(a,L)}\colon G \times_H V \to G \times_{H'} V'$ defined by $\theta_{(a,L)}([g, v]) = [ga^{-1}, Lv]$ is a well-defined isomorphism of G-vector bundles. Conversely, any isomorphism from $G \times_H V$ to $G \times_{H'} V'$ must be of this form.*

The proof of this is a routine matter and is left to the reader.

Thus, the normal orbit type $[H, V]$ may also be regarded as the isomorphism class of the G-vector bundle $G \times_H V$. This is completely analogous to the fact that the conjugacy class of H can be regarded as the G-diffeomorphism class of G/H. (Compare § I.4 in [2].)

Next, we consider the group of automorphisms of $G \times_H V$. By an "automorphism" we simply mean an invertible equivariant bundle map, that is, a bundle map which covers some equivariant diffeomorphism of G/H (not necessarily the identity). There is an embedding $H \to G \times \mathrm{Aut}(V)$ defined by $h \to (h, \phi(h))$, where ϕ is the representation associated to the H-module V. To simplify notation we shall identify H with its image under this embedding. Let $N_H(G \times \mathrm{Aut}(V))$ be the normalizer of H in $G \times \mathrm{Aut}(V)$. Then

$$N_H(G \times \mathrm{Aut}(V)) = \{(a, L) \mid aHa^{-1} = H \text{ and } \phi(aha^{-1}) = L\phi(h)L^{-1} \text{ for all } h \in H\}\,.$$

Thus, $N_H(G \times \mathrm{Aut}(V))$ is just the group of self-equivalences of (H, V). Also notice that $\theta_{(a,L)}$ is the identity if and only if $a \in H$ and $L = \phi(a)$. Therefore, (1.3) has the following

COROLLARY 1.4. *Let S be the group of automorphisms of $G \times_H V$. Then the map $(a, L) \to \theta_{(a,L)}$ defines an isomorphism of Lie groups $N_H(G \times \mathrm{Aut}(V))/H \cong S$.*

Returning to our G-action on M, suppose that α is a normal orbit type. The *α-stratum* M_α is the union of orbits of type α, i.e.,

$$M_\alpha = \{x \in M \mid [G_x, V_x] = \alpha\}\,.$$

Let B be the orbit space of M and let $\pi\colon M \to B$ be the natural projection (π is called the *orbit map*). Set $B_\alpha = \pi(M_\alpha)$ and $\pi_\alpha = \pi|_{M_\alpha}$. B_α is called the *α-stratum of B*.

PROPOSITION 1.5. *Both M_α and B_α are smooth manifolds and*

$\pi_\alpha: M_\alpha \to B_\alpha$ is the projection map of a smooth fiber bundle.

Proof. Ssuppose that α is represented by (H, V). Let $x \in M_\alpha$. By the Slice Theorem, x has an invariant neighborhood $U \cong G \times_H S$. Clearly, $(G \times_H S)_\alpha = G \times_H F \cong G/H \times F$, where F is the subspace fixed by H. Hence,

$$(*) \qquad U_\alpha \cong G/H \times F .$$

Thus, M_α is a smooth manifold. It follows from $(*)$ that $\pi_\alpha(U_\alpha) \cong F$. Therefore, B_α is also a smooth manifold (since F is isomorphic to euclidean space). Finally, notice that $(*)$ provides a local trivialization for π_α, and so M_α is a smooth fiber bundle over B_α.

This proposition shows that both M and B are "stratified spaces" in the sense that they both are the union of smooth manifolds.

Next, we consider $I(G)$, the set of normal G-orbit types. There is a natural partial ordering on $I(G)$, namely: $[H, V] \leqq [K, W]$ if the G-manifold $G \times_H V$ contains an orbit of type $[K, W]$. Notice that maximal elements of $I(G)$ are of the form $[H, O]$ where O denotes the zero dimensional H-module.

The following two results are essentially classical.

THEOREM 1.6. *Let $\alpha \in I(G)$. Then*

$$\overline{M_\alpha} = \bigcup_{\beta \leqq \alpha} M_\beta .$$

THEOREM 1.7. (*The Principal Orbit Theorem*). *Suppose that B is connected. Then there is a maximum normal orbit type γ for G on M called the "principal orbit type." M_γ is open and dense in M, and B_γ is open and dense in B. Moreover, B_γ is connected.*

Both theorems follow fairly easily from the Differentiable Slice Theorem (see pages 179 and 182 in [2]). The Principal Orbit Theorem is due to Montgomery, Samelson and Yang.

Next, suppose that $F: M \to M'$ is a smooth equivariant map (where G acts smoothly on M and M'). It follows that for each $x \in M$, $G_x \subset G_{F(x)}$ and that the differential of F induces a linear map $F_*: V_x \to V_{F(x)}$, which is G_x-equivariant.

DEFINITION 1.8. A smooth equivariant map $F: M \to M'$ is *stratified* if for each $x \in M$, the following two conditions hold:

(i) $G_x = G_{F(x)}$

(ii) $F_*: V_x \to V_{F(x)}$ is an isomorphism.

Thus, stratified maps preserve normal orbit types. In this paper, *all maps of G-manifolds will be stratified.*

2. Normal orbit bundles. In this section we investigate the structure of the equivariant normal bundle of a stratum.

Fix a normal G-orbit type α and let (H, V) be a representative for α. Let S_α be the group of automorphisms of $G \times_H V$, i.e., let $S_\alpha = N_H(G \times \mathrm{Aut}(V))/H$.

As usual, suppose that G acts smoothly on M with orbit map $\pi: M \to B$. Let $\nu_\alpha(M)$ be the total space of the normal bundle of M_α in M. When there is no ambiguity we will write simply ν_α instead of $\nu_\alpha(M)$. Let $q_\alpha: \nu_\alpha \to M_\alpha$ be the projection map and let $r_\alpha = \pi_\alpha \circ q_\alpha$.

PROPOSITION 2.1. *The map $r_\alpha: \nu_\alpha(M) \to B_\alpha$ is the projection map of a smooth fiber bundle with fiber $G \times_H V$ and with structure group S_α.*

Proof. Let $x \in M_\alpha$. By the Slice Theorem, x has a neighborhood in M_α of the form $G(x) \times D$ where D is a disk in B_α. By the definition of the α-stratum, $\nu_\alpha|_{G(x)} \cong G \times_H V$. Therefore, we clearly have that $\nu_\alpha|_{G(x)\times D} \cong \nu_\alpha|_{G(x)} \times D \cong (G \times_H V) \times D$. The composition of these two isomorphisms provides a local trivialization of $r_\alpha^{-1}(D)$. Thus, $\nu_\alpha \to B_\alpha$ is a fiber bundle with fiber $G \times_H V$. Since local trivializations such as the above are isomorphisms of G-vector bundles, it follows that the structure group is S_α. This completes the proof.

Let $P_\alpha(M) \to B_\alpha$ be the principal S_α-bundle associated to the bundle $\nu_\alpha(M) \to B_\alpha$. $P_\alpha(M)$ is called the *α-normal orbit bundle of M* (and is denoted by P_α when there is no ambiguity). Explicitly, P_α is the fiber bundle with base space B_α and with the fiber over b consisting of all isomorphisms of G-vector bundles $(\nu_\alpha)_b \to G \times_H V$.

These normal orbit bundles are the basic building blocks for a smooth G-manifold. In the theory which we are developing they take the place of the more traditional "orbit bundles."

REMARK 2.2. Suppose that $F: M \to M'$ is stratified. Then the differential of F induces a bundle map $F_*: \nu_\alpha(M) \to \nu_\alpha(M')$ which is an isomorphism on the fibers. Hence, F_* induces a map of the associated principal bundles, which we denote by $F_\alpha: P_\alpha(M) \to P_\alpha(M')$.

Next consider the structure group S_α. Recall (1.4) that $S_\alpha = N_H(G \times G')/H$, where $G' = \mathrm{Aut}(V)$ and where H is embedded in $G \times G'$ via $h \to (h, \phi(h))$. (ϕ is the representation associated to the H-module V.) Set $H' = \phi(H)$.

It is clear that the inclusion map $a \to (a, 1) \in G \times G'$ restricts to a map $C_H(G) \hookrightarrow N_H(G \times G')$, where $C_H(G)$ means the centralizer of H in G. Let j denote the composition $C_H(G) \to N_H(G \times G') \to N_H(G \times G')/H = S_\alpha$. In a similar fashion, inclusion of the second factor induces a homomorphism $j': C_{H'}(G') \to S_\alpha$. Also, there are homomorphisms

$$s: N_H(G \times G')/H \longrightarrow N_H(G)/H$$

and

$$s': N_H(G \times G')/H \longrightarrow N_H(G')/H'$$

induced by the projections on the first and second factors, respectively. The proof of the next proposition is immediate.

PROPOSITION 2.3. *The following sequence is exact*

$$1 \longrightarrow C_{H'}(G') \xrightarrow{j'} S_\alpha \xrightarrow{s} N_H(G)/H .$$

Moreover, if the representation $\phi: H \to G'$ is faithful, then the fullowing sequence is also exact

$$1 \longrightarrow C_H(G) \xrightarrow{j} S_\alpha \xrightarrow{s'} N_{H'}(G')/H' .$$

For each α, there are two other important bundles associated to an action. The first is the "orbit bundle," $\pi_\alpha: M_\alpha \to B_\alpha$ which has fiber G/H. The second is the bundle $\bar{r}_\alpha: \nu_\alpha/G \to B_\alpha$, which has fiber $(G \times_H V)/G$. Here $\bar{r}_\alpha$ is the map induced by r_α. We clearly have that

$$M_\alpha \cong G/H \times_{S_\alpha} P_\alpha$$

and that

$$\nu_\alpha/G \cong (G \times_H V)/G \times_{S_\alpha} P_\alpha .$$

Here S_α acts on G/H and on $(G \times_H V)/G$ in an obvious fashion. The problem with the above formulation is that these S_α-actions will generally be ineffective.

First let us consider the action of S_α on G/H. If $a \in C_{H'}(G')$, then $j'(a) \cdot [g, 0] = [g, a \cdot 0] = [g, 0]$. In other words, the image of $C_{H'}(G')$ acts trivially on G/H. So we define a quotient group

$$R_\alpha = S_\alpha/j'(C_{H'}(G')) \cong s(S_\alpha)$$

and a principal R_α-bundle $O_\alpha \to B_\alpha$ by

$$O_\alpha = P_\alpha/C_{H'}(G') .$$

$N_H(G)/H$ acts freely on G/H and the action of R_α may be identified with the action of the subgroup $s(S_\alpha) \subset N_H(G)/H$. Hence R_α acts freely on G/H.

The kernel of the action of S_α on $(G \times_H V)/G$ cannot be described quite so explicitly. But we can still define a group

$$T_\alpha = S_\alpha / Z_\alpha$$

where Z_α is defined as the subgroup of S_α which acts trivially on the orbit space $(G \times_H V)/G$. (Notice that $\ker s' \subset Z_\alpha$.) We can also define a principal T_α-bundle $Q_\alpha \to B_\alpha$ by

$$Q_\alpha = P_\alpha / Z_\alpha \,.$$

The above remarks are collected in the following proposition.

PROPOSITION 2.4. *There are natural isomorphisms*

$$M_\alpha \cong G/H \times_{R_\alpha} O_\alpha$$

and

$$\nu_\alpha / G \cong (G \times_H V)/G \times_{T_\alpha} Q_\alpha \,.$$

II. The Structure of Orbit Spaces

1. Smooth invariant theory. Suppose that G acts smoothly on M and that $\pi\colon M \to M/G$ is the orbit map. Let $\mathscr{S}$ be the smooth structure on M, i.e., let $\mathscr{S}$ be the sheaf of germs of C^∞ functions on M. The induced functional structure $\pi_* \mathscr{S}$ is called the *quotient smooth structure on* M/G. In this spirit, a function $f\colon M/G \to \boldsymbol{R}$ is *smooth* if $f \circ \pi$ is smooth, and a map of such spaces which preserves the functional structure is a *smooth map*. This terminology is not in anyway meant to suggest that M/G is a smooth manifold (see [2] and [8] for further details).

By the Slice Theorem, each orbit in M has an invariant neighborhood of the form $G \times_H S$, where S is the slice representation. It is clear that a smooth function on $G \times_H S$ is G-invariant if and only if its restriction to S is smooth and H-invariant. In other words, the inclusion of the fiber induces a homeomorphism $S/H \cong (G \times_H S)/G$ of spaces and an isomorphism of smooth structures. Hence, determining the local smooth structure of M/G is equivalent to determining the invariant C^∞ functions on the various slice representations.

Classical invariant theory deals with the related problem of computing the invariant polynomials of a representation. One of its fundamental theorems, proved by Weyl (on page 275 of [11]), states that if H is a compact Lie group and if V is an H-module, then the ring of invariant polynomials $\boldsymbol{R}[V]^H$ is finitely generated. The following theorem, proved by G. Schwarz in [8], shows how the invariant smooth functions are determined by the invariant polynomials.

THEOREM 1.1 (Schwarz). *Let H be a compact Lie group and let V be a H-module. Let $\{\theta_1, \cdots, \theta_s\}$ be a set of generators for $\boldsymbol{R}[V]^H$ and let $\theta = (\theta_1, \cdots, \theta_s)$: $V \to \boldsymbol{R}^s$. Then $\theta^* C^\infty(\boldsymbol{R}^s) = C^\infty(V)^H$.*

REMARK 1.2. Since θ is invariant, it induces a map $\bar{\theta}$: $V/H \to \boldsymbol{R}^s$, which is easily seen to be a topological embedding. Give $\theta(V)$ the smooth structure induced by the inclusion $\theta(V) \subset \boldsymbol{R}^s$, i.e., a function on $\theta(V)$ is smooth if and only if it extends to a smooth function on $\boldsymbol{R}^s$. Then Schwarz's theorem is equivalent to the statement that $\bar{\theta}$: $V/H \to \theta(V)$ is an isomorphism. Taken together with our remarks at the beginning of this section, this shows that M/G has a smooth structure locally isomorphic to that of certain semi-algebraic subsets of euclidean space. This result is one of the technical underpinnings of the theory developed in this paper.

If H acts trivially on $\boldsymbol{R}^k$, then it follows from (1.1) that $(V \times \boldsymbol{R}^k)/H$ is smoothly isomorphic to $V/H \times \boldsymbol{R}^k$, where the smooth structure on $V/H \times \boldsymbol{R}^k$ is induced by the embedding $\bar{\theta} \times id$: $V/H \times \boldsymbol{R}^k \to \boldsymbol{R}^s \times \boldsymbol{R}^k$ (in fact, this is a key lemma in Schwarz's proof). From this observation we see that it suffices to study the invariant polynomials of the normal representations.

2. The tangent space of M/G. Let $y \in M/G$. The stalk of $\pi_* \mathscr{S}$ at y is a local ring with maximal ideal $\mathscr{M}_y$, the germs of G-invariant smooth functions which vanish at y. The (Zariski) *cotangent space* at y is defined as

$$T_y^*(M/G) = \mathscr{M}_y / \mathscr{M}_y^2 \; .$$

Its dual $T_y(M/G)$ is the *tangent space at y.*

Let us now consider the tangent space of V/H, where V is an H-module (or, equivalently, the tangent space of $((G \times_H V)/G)$. The symbol 0 will be used to denote both the origin in V and its image in V/H. Let $\mathscr{M}_0$ be the ring of germs of H-invariant smooth functions which vanish at 0. There is a corresponding algebraic object $\hat{\mathscr{M}}_0 \subset \boldsymbol{R}[V]^H$. This is the ideal of invariant polynomials with no constant terms.

LEMMA 2.1. *$\mathscr{M}_0/\mathscr{M}_0^2 \cong \hat{\mathscr{M}}_0/\hat{\mathscr{M}}_0^2$. Furthermore, if $\{\theta_1, \cdots, \theta_s\}$ is a minimal set of generators for $\hat{\mathscr{M}}_0$, then the dimension of $T_0(V/H)$ is s.*

Proof. The inclusion $\hat{\mathscr{M}}_0 \subset \mathscr{M}_0$ induces a linear map λ: $\hat{\mathscr{M}}_0/\hat{\mathscr{M}}_0^2 \to$

$\mathscr{M}_0/\mathscr{M}_0^2$. If $f \in \mathscr{M}_0$, then by (1.1) $f = r \circ q$, where $r\colon \boldsymbol{R}^s \to \boldsymbol{R}$ is smooth. By clearly have that

$$f \equiv r'(0) \cdot (\theta_1, \cdots \theta_s) \qquad (\text{mod } \mathscr{M}_0^2) .$$

But $r'(0) \cdot (\theta_1, \cdots, \theta_s) \in \hat{\mathscr{M}}_0$; hence, Λ is surjective. A similar argument shows that Λ is injective and therefore, an isomorphism. The image of $\{\theta_1, \cdots, \theta_s\}$ in $\hat{\mathscr{M}}_0/\hat{\mathscr{M}}_2^0$ is clearly a basis; so $s = \dim(\hat{\mathscr{M}}_0/\hat{\mathscr{M}}_0^2) = \dim(\mathscr{M}_0/\mathscr{M}_0^2) = \dim(T_0(V/H))$.

REMARK 2.2. Let $\{\theta_1, \cdots, \theta_s\}$ be a minimal set of generators for $\hat{\mathscr{M}}_0$. Then polynomials in $\{\theta_i\}$ generate $\boldsymbol{R}[V]^H$. Let $\theta = (\theta_1, \cdots, \theta_s)\colon V \to \boldsymbol{R}^s$ and $\bar{\theta}\colon V/H \to \boldsymbol{R}^s$ be as in § 1. Then $\bar{\theta}$ induces a linear map $\bar{\theta}_*\colon T_0(V/H) \to T_0(\boldsymbol{R}^s)$. Suppose that dx_i denotes the image of θ_i in $\mathscr{M}_0/\mathscr{M}_0^2$. As we pointed out in the above proof, $\{dx_1, \cdots, dx_s\}$ is a basis for $T_0^*(V/H)$. Let $\{D_1, \cdots, D_s\}$ be the dual basis for $T_0(V/H)$. Then $\bar{\theta}_*$ clearly sends $\{D_1, \cdots, D_s\}$ to the standard basis for $T_0(\boldsymbol{R}^s)$. Hence, $\bar{\theta}_*$ is an isomorphism.

PROPOSITION 2.3. *Let $x \in M$ and let $y = \pi(x) \in M/G$. Then $T_y(M/G) \cong T_0(S_x/G_x)$. In particular, $T_y(M/G)$ is a finite dimensional vector space.*

This is immediate from the Slice Theorem and the above remark.

REMARK 2.4. If $f\colon M/G \to M'/G$ is any smooth map of orbit spaces, then f induces, in an obvious fashion, a linear map $Df\colon T_y(M/G) \to T_{f(y)}(M'/G)$. Df is called the *differential of f at y*.

3. Weakly stratified maps. In this section we define two terms, "the normal bundle" of a stratum of an orbit space and "a weakly stratified map of orbit spaces." We shall show in (3.2) that a stratified map of G-manifolds covers a weakly stratified map of orbit spaces. Then we shall establish some further properties of weakly stratified maps in (3.3) and (3.4).

Set $B = M/G$. Let α be a normal G orbit type represented by (H, V) and let

$$X = G \times_H V .$$

Let

$$E_\alpha = \bigcup_{b \in B_\alpha} T_b(B) .$$

It follows from (2.3) that E_α is a vector bundle over B_α. The ordinary tangent bundle of the stratum $T(B_\alpha)$ is clearly a subbundle of

E_α, so one can define the *normal bundle of* B_α *in* B by

$$N_\alpha(B) = E_\alpha/T(B_\alpha) .$$

We will write simply N_α when there is no confusion. For $b \in B_\alpha$, let N_b denote the fiber of N_α at b. Clearly, $N_b \cong T_0(X/G)$, where 0 denotes the point in X/G which is the image of the zero-section of X.

REMARK 3.1. Let ν_α be the normal bundle of M_α in M. The Equivariant Tubular Neighborhood Theomem states that ν_α is equivariantly diffeomorphic to a neighborhood of M_α in M. Hence, ν_α/G is isomorphic to a neighborhood of B_α in B. Therefore, N_α is isomorphic to the normal bundle of B_α in ν_α/G. In § I.2, it was shown that the bundle $\nu_\alpha/G \to B_\alpha$ has structure group T_α. It follows that the structure group of N_α can also be reduced to T_α. In fact,

$$N_\alpha \cong T_0(X/G) \times_{T_\alpha} Q_\alpha$$

(see I.2.4). More will be said about this action of T_α on $T_0(X/G)$ in the next section.

Next, suppose that $f\colon B \to B'$ is a smooth map of orbit spaces which preserves the stratification, i.e., such that $f(B_\alpha) \subset B'_\alpha$ for each α. For any $b \in B_\alpha$, we have the linear map $Df\colon T_b(B) \to T_{f(b)}(B')$ (see 2.4). Since $f(B_\alpha) \subset B'_\alpha$, Df carries $T_b(B_\alpha)$ into $T_{f(b)}(B'_\alpha)$. Hence, Df induces a map

$$\overline{Df}\colon N_b \longrightarrow N_{f(b)} .$$

The map f is said to be *weakly stratified at* b if $\overline{Df}$ is an isomorphism; f is *weakly stratified* if it is weakly stratified at each point. In other words, a smooth strata preserving map of orbit spaces is weakly stratified if its differential maps the normal bundle of each stratum transversely.

Recall that if $F\colon M \to M'$ is any equivariant map, then there is an induced map of orbit spaces $f\colon B \to B'$ so that the following diagram commutes

$$\begin{array}{ccc} M & \xrightarrow{F} & M' \\ \downarrow & & \downarrow \\ B & \xrightarrow{f} & B' . \end{array}$$

PROPOSITION 3.2. *If* $F\colon M \to M'$ *is stratified, then the induced map of orbit spaces* $f\colon B \to B'$ *is weakly stratified.*

Proof. The map f is clearly smooth and strata preserving. The remainder of the proof can easily be seen to reduce to the following local formulation. Recall that $X = G \times_H V$. Suppose $F: X \to X$ is stratified. Then we must show that the induced map $f: X/G \to X/G$ is weakly stratified at 0. Since the differential of F must be an isomorphism at points in the zero-section of X, it follows from the Inverse Function Theorem that F is an equivariant diffeomorphism on some neighborhood of the zero-section. Hence, f is an isomorphism on a neighborhood of 0. Therefore, $Df: T_0(X/G) \to T_0(X/G)$ must also be an isomorphism, that is, f must be weakly stratified at 0.

We shall need the next result in Chapter III. It is a sort of "Inverse Function Theorem" for weakly stratified maps.

PROPOSITION 3.3. *Let $X = G \times_H V$ and let $f: X/G \to X/G$ be a weakly stratified map. Then f is an isomorphism on some neighborhood of* 0.

Proof. First choose a minimal set of polynomial generators $\{\theta_1, \cdots, \theta_s\}$ for $\boldsymbol{R}[V]^H$ as in (2.2). Each polynomial θ_i extends to a smooth G-invariant function $\tilde{\theta}_i$ on X defined by $\tilde{\theta}_i([g, v]) = \theta_i(v)$. Let $\theta = (\tilde{\theta}_1, \cdots, \tilde{\theta}_s): X \to \boldsymbol{R}^s$ and let $\bar{\theta}: X/G \to \boldsymbol{R}^s$ be the induced embedding. Using Schwarz's theorem (see Remark 1.2), f may be extended to a smooth map $g: \boldsymbol{R}^s \to \boldsymbol{R}^s$. By (2.2), we have the following commutative diagram

$$\begin{array}{ccc} T_0(X/G) & \xrightarrow{Df} & T_0(X/G) \\ \downarrow \cong & & \downarrow \cong \\ T_0(\boldsymbol{R}^s) & \xrightarrow{Dg} & T_0(\boldsymbol{R}^s) \end{array}.$$

Since Df is an isomorphism so is Dg. So by the Inverse Function Theorem, there is a neighborhood U of the origin in $\boldsymbol{R}^s$ such that $g|U: U \to g(U)$ is a diffeomorphism. Set $W = U \cap X/G$. We will be done if we can show that $f(W)$ is open in X/G; for, then the map $g^{-1}|_{f(W)}$ will be a smooth inverse for $f|_W$.

Set $Y = g(U) \cap X/G$. We will show that $f(W) = Y$ and hence, that $f(W)$ is open. Clearly, $f(W) \subset Y$. We may assume (by changing U if necessary) that Y is path connected. Let β be the principal (normal) orbit type for G on X. By the Principal Orbit Theorem, Y_β is path connected (since Y is). Pick a point $x \in f(W_\beta)$. Given an arbitrary point $y \in Y_\beta$, we can find a path $\omega: [0, 1] \to Y_\beta$ from x to y. Let $J = \{t \in [0, 1] | \omega(t) \in f(W_\beta)\} = \omega^{-1}(f(W_\beta))$. Since $g|_U$ is a diffeomorphism, $Df: T_u(W_\beta) \to T_{f(u)}(Y_\beta)$ must be an isomorphism; hence, f maps W_β diffeomorphically onto an open submanifold of Y_β. Thus, $J = \omega^{-1}(f(W_\beta))$ is open. Pull back the path ω to a path $\psi = g^{-1} \circ \omega$: $[0, 1] \to U$ from $g^{-1}(x)$ to $g^{-1}(y)$. Let $C = \omega([0, 1]) \cap W$. Then C is clearly closed in U. Hence, $\psi^{-1}(C) = J$ is closed in $[0, 1]$. Thus, J, being both open and closed, must be all of $[0, 1]$, and so $y = \omega(1) \in f(W_\beta)$. Therefore, $Y_\beta = f(W_\beta)$. According to the Principal Orbit Theorem, Y is the closure of Y_β in Y and W is the closure of W_β in W. Thus, $g^{-1}(Y) = g^{-l}(\bar{Y}_\beta) \subset \overline{g^{-l}(Y_\beta)} = \bar{W}_\beta = W$, and so, $Y \subset f(W)$, as was to be proved.

Here is one further observation which will be needed in Chapter III.

PROPOSITION 3.4. *Let $F: M \to M'$ be a stratified map and let $f: B \to B'$ be the induced map of orbit spaces. Then F is an equivariant diffeomorphism if and only if f is an isomorphism.*

Proof. Obviously if F is an isomorphism then so is f. So suppose that f is an isomorphism. From the facts that f is a strata preserving homeomorphism and that F is equivariant, it follows easily that F is an equivariant homeomorphism (see page 97 in [2]). So it suffices to show that the differential of F is everywhere an isomorphism. The tangent space at $x \in M_\alpha$ splits as a direct sum of G_x-modules as $T_xM = T_xG(x) + V_x + T_{\pi(x)}B_\alpha$. Since F is stratified, DF maps $T_xG(x) + V_x$ isomorphically onto $T_{F(x)}G(F(x)) + V_{F(x)}$. Since $Df: TB_\alpha \to TB'_\alpha$ is an isomorphism, it follows that DF maps $T_{\pi(x)}B_\alpha$ isomorphically onto $T_{f(\pi(x))}B'_\alpha$. Thus, DF is an isomorphism.

4. Stratified maps. In this section the concept of a "stratified map of orbit spaces" is defined. The difference between stratified and weakly stratified maps essentially comes from the fact that there are two possible definitions for the "derivative" of a map of orbit spaces. The first of these definitions was given in (2.4). The second one is given below in (4.2). We will show in (4.5) that a stratified map of smooth G-manifolds induces a stratified map of orbit spaces. Furthermore, every stratified map of orbit spaces is weakly stratified. Conversely, one can ask if every weakly stratified map is stratified. The section concludes with examples.

As before, let α be a normal orbit type and let $X = G \times_H V$ be a representative for α. Scalar multiplication on V induces *scalar multiplication on X*. Explicitly, if $x = [g, v] \in X$, then this is defined by $tx = [g, tv]$.

DEFINITION 4.1. For any $x \in X$, let $\bar{x}$ denote its image in X/G. Scalar multiplication on X induces an action of the real numbers on

X/G defined by $t \circ \bar{x} = \overline{tx}$. This is called *scalar multiplication on* X/G.

The normal bundle of G/H in X can be naturally identified with X. So, if $F: X \to X$ is stratified, then its differential induces an equivariant bundle automorphism $F_*: X \to X$. F_* can be defined by

$$F_*(x) = \lim_{t \to 0} t^{-1} F(tx) .$$

Let $f: X/G \to X/G$ and $f_*: X/G \to X/G$ be, respectively, the maps induced by F and F_*, that is, $f(\bar{x}) = \overline{F(x)}$ and $f_*(\bar{x}) = \overline{F_*(x)}$. It follows that

$$f_*(\bar{x}) = \lim_{t \to 0} t^{-1} \circ f(t \circ \bar{x}) .$$

We have that $F_* \in S_\alpha$ and hence, that $f_* \in T_\alpha$. (Recall that S_α is the group of automorphisms of X and that T_α is the quotient of this group by Z_α.) The above considerations motivate the following definition.

DEFINITION 4.2. Let $f: X/G \to X/G$ be a smooth strata preserving map. For each $t \neq 0$, define $f_t: X/G \to X/G$ by $f_t(z) = t^{-1} \circ f(t \circ z)$. Then f_t is also smooth and strata preserving. We will say that f is *stratified at* 0 ($0 \in X/G$), if, as $t \to 0$, the maps f_t converge smoothly to a smooth isomorphism f_*, and if $f_* \in T_\alpha$.

In the following lemma we prove the chain rule for the above definition of "derivative."

LEMMA 4.3. *Suppose that the maps $f: X/G \to X/G$ and $g: X/G \to X/G$ are stratified at* 0. *Let $h = f \circ g$. Then h is stratified at* 0 *and $h_* = f_* \circ g_*$.*

Proof. The proof is formally the same as the proof of the ordinary chain rule. First pick a minimal set of generators $\{\theta_1, \cdots, \theta_s\}$ for $\boldsymbol{R}[V]^H$, as in (2.2). We may suppose that the θ_i's are homogeneous polynomials. As before, we get a map $\bar{\theta}: X/G \to \boldsymbol{R}^s$, which we regard as an inclusion. The action of $\boldsymbol{R}$ on X/G extends to an action on $\boldsymbol{R}^s$ by the formula $t \circ (x_1, \cdots, x_s) = (t^{d(1)}x_1, \cdots, t^{d(s)}x_s)$, where $d(i)$ is the degree of θ_i. We can write $g(x) = g_*(x) + R_g(x)$, where $R_g = g - g_*: X/G \to \boldsymbol{R}^s$. Clearly, $t^{-1} \circ R_g(t \circ x) \to 0$ as $t \to 0$. So the function $P_g(t, x)$ defined by

$$P_g(t, x) = \begin{cases} t^{-1} \circ R_g(t \circ x) \, ; & t \neq 0 \\ 0 \, ; & t = 0 \end{cases}$$

is continuous. We have that

$$(1)\quad g_*(t\circ x) = \lim_{s\to 0} s^{-1}\circ g(st\circ x) = t\circ g_*(x)$$

$$(2)\quad g(t\circ x) = g_*(t\circ x) + R_g(t\circ x) = t\circ[g_*(x) + P_g(t, x)]$$

$$(3)\quad \lim_{t\to 0} t^{-1}\circ R_f(g(t\circ x)) = \lim_{t\to 0} t^{-1}\circ R_f(t\circ[g_*(x) + P_g(t, x)]) = 0$$

$$(4)\quad t^{-1}\circ h(t\circ x) = t^{-1}\circ f(g(t\circ x)) = t^{-1}\circ f_*(g(t\circ x)) + t^{-1}\circ R_f(g(t\circ x)) .$$

Therefore,

$$\begin{aligned} h_*(x) &= \lim t^{-1}\circ h(t\circ x) \\ &= \lim t^{-1}\circ f_*(g(t\circ x)) && \text{, by (3) and (4)} \\ &= \lim t^{-1}\circ f_*(t\circ[g_*(x) + P_g(t, x)]) && \text{, by (2)} \\ &= \lim f_*(g_*(x) + Pg(t, x)) && \text{, by (1)} \\ &= f_*(g_*(x)) , \end{aligned}$$

which proves the lemma.

Suppose, as before, that G acts smoothly on manifolds M and M' and that $\pi: M\to B$ and $\pi': M'\to B'$ are the orbit maps. Let $f: B\to B'$ be a smooth strata preserving map and let $b\in B_\alpha$. By the Slice Theorem, $\pi^{-1}(b)$ has a neighborhood of the form $G\times_H(V\times \boldsymbol{R}^m)\cong X\times \boldsymbol{R}^m$. So, we can choose an equivariant chart $\psi: U\to X\times \boldsymbol{R}^m$, where U is an invariant neighborhood of $\pi^{-1}(b)$. Let $\bar\psi: \pi(U)\to X/G\times \boldsymbol{R}^m$ be the induced map. We may also choose an equivariant chart $\psi': U'\to X\times \boldsymbol{R}^{m'}$ so that $f(\pi(U))\subset \pi'(U')$. Let g be the germ at $\bar\psi(b)$ of the map $X/G\times \boldsymbol{R}^m\to X/G\times \boldsymbol{R}^{m'}$ defined by $\bar\psi'\circ f\circ\bar\psi^{-1}$. Also, let h be the composition

$$X/G \hookrightarrow X/G\times \boldsymbol{R}^m \xrightarrow{g} X/G\times \boldsymbol{R}^{m'} \longrightarrow X/G$$

where the first map is the appropriate inclusion and the third map is projection on the first factor. Clearly, h is smooth and strata preserving.

DEFINITION 4.4. In the above situation, $f: B\to B'$ is said to be *stratified at* b if h is stratified at 0; f is *stratified* if it is stratified at each point.

We must show that this definition is independent of the choice of charts ψ and ψ'. The effect of changing one of these charts is to alter h be composition with a map from X/G to itself which is stratified at 0. So, it follows from (4.3) that the altered h will also be stratified at 0, as claimed.

THEOREM 4.5. *If $F: M \to M'$ is a stratified map of G-manifolds, then the induced map $f: B \to B'$ of orbit spaces is stratified.*

Proof. It clearly suffices to prove this locally, that is, it suffices to consider the case where $M = X \times \boldsymbol{R}^m$ and $M' = X \times \boldsymbol{R}^{m'}$ and to show that $f: X/G \times \boldsymbol{R}^m \to X/G \times \boldsymbol{R}^{m'}$ is stratified at points of the form $(0, z)$. But this case is obvious, as we observed at the beginning of this section.

In order to prove that stratified maps are weakly stratified, we must examine the action of T_α on X/G more closely. Recall that $s': S_\alpha \to N_{H'}(G')/H'$ and that the action of T_α on X/G may be identified with the restriction of the natural action of $N_{H'}(G')/H'$ (made effective) to the subgroup $s'(S_\alpha)$. (See § I.2.) $N_{H'}(G')$ acts linearly on V and the action preserves H-orbits; hence, $N_{H'}(G')/H'$ acts smoothly on $V/H \cong X/G$. The restriction of action to T_α preserves the stratification on X/G (which may be different from the stratification of V/H by normal H-orbit type). Since $N_{H'}(G')$ acts linearly on V, it acts from the right on polynomials. Suppose that ϕ is an H-invariant polynomial, that $h \in H'$, and that $L \in N_{H'}(G')$. Then $\phi \cdot L(hv) = \phi(Lhv) = \phi(h'Lv) = \phi(Lv) = \phi \cdot L(v)$, for some $h' \in H'$. That is, $\boldsymbol{R}[V]^H$ is invariant under the action of $N_{H'}(G')$. Since H' clearly acts trivially, the action factors through $N_{H'}(G')/H'$ (and hence, the subgroup T_α acts on $\boldsymbol{R}[V]^H$). Let $P_m \subset \boldsymbol{R}[V]^H$ denote the subspace consisting of those H-invariant polynomials which are homogeneous of degree m. P_m is clearly invariant under the action of $N_{H'}(G')/H'$.

LEMMA 4.6. *There is a minimal set of generators $\{\theta_1, \cdots, \theta_s\}$ for $\boldsymbol{R}[V]^H$ such that*

(1) *θ_i is homogeneous degree $d(i)$ and $2 \leqq d(1) \leqq d(2) \cdots \leqq d(s)$.*

(2) *If P'_m is the subspace of P_m spanned by $\{\theta_i \mid d(i) = m\}$, then P'_m is invariant under $N_{H'}(G')/H'$.*

Proof. First we show that the linear representation of $N_{H'}(G')$ on P_m is completely reducible. Since H' is compact, it is a (real) algebraic subgroup of $G' = GL(V)$. Hence, $N_{H'}(G')$ is also an algebraic subgroup. In particular, this implies that $\pi_0(N_{H'}(G'))$ is finite. Next, let $C_{H'}(G')$ denote the centralizer of H' in G' and let $C_{H'} = C_{H'}(G') \cap H'$ be the center of H'. Then $C_{H'}(G')$ is a product of general linear groups (over either the real, complex or quaternionic numbers) and the action of $C_{H'}(G')$ on V is via a product of standard representations.

Hence, $C_{H'}(G')$ is a reductive algebraic group. $C_{H'}(G')/C_{H'}$ is a normal subgroup of $N_{H'}(G')/H'$ and both groups clearly have the same Lie algebra. Therefore, the quotient group is discrete. In fact, the quotient is finite since $|\pi_0(N_{H'}(G')/H')| < \infty$. Since the semi-direct product of a reductive group and a finite group is also reductive, we conclude that the action of $N_{H'}(G')/H'$ on P_m is completely reducible.

Now, pick a basis $\{\theta_1, \cdots, \theta_{p(2)}\}$ for P_2 (since $V^H = \{0\}$, $P_1 = \{0\}$). Assume, by induction, that we have chosen a minimal set of generators $\{\theta_1, \cdots, \theta_{p(i)}\}$ for the image of $P_2 + \cdots + P_i$ in $\boldsymbol{R}[V]^H$. Let Q_{i+1} be the image of $P_2 + \cdots + P_i$ in P_{i+1}. Then Q_{i+1} is obviously invariant under $N_{H'}(G')/H'$. Let P'_{i+1} be a complementary invariant subspace for Q_{i+1} and let $\theta_{p(i)+1}, \cdots, \theta_{p(i+1)}$ be a basis for P'_{i+1}. Since $\boldsymbol{R}[V]^H$ is finitely generated this process stops after a finite number of steps. The result is the desired set of generators.

From now on, we will let $\{\theta_1, \cdots, \theta_s\}$ be a minimal set of generators for $\boldsymbol{R}[V]^H$, chosen as in the above lemma, and we will let $\bar{\theta} = (\bar{\theta}_1, \cdots, \bar{\theta}_s)\colon X/G \to \boldsymbol{R}^s$ be the induced embedding. If dx_i denotes the image of θ_i in $\mathscr{M}_0/\mathscr{M}_0^2 = T_0^*(X/G)$, then $\{dx_i\}$ is a basis. Let $\{D_1, \cdots, D_s\}$ be the dual basis for $T_0(X/G)$ and let $\{e_1, \cdots, e_s\}$ be the standard basis for $\boldsymbol{R}^s$. We will also use D_i to stand for the standard basis element of $T_0\boldsymbol{R}^s$. This notation should cause no confusion, for a function $g\colon X/G \to \boldsymbol{R}$ is smooth if and only if it extends to a smooth function $\tilde{g}$ on $\boldsymbol{R}^s$, in which case $D_i g = D_i \tilde{g}$.

Let $a \in N_{H'}(G')/H'$. The action of $N_{H'}(G')/H'$ extends to a linear action on $\boldsymbol{R}^s$ defined by $e_i \cdot a = \Sigma a_{ij} e_j$ where the matrix (a_{ij}) is defined by the formula $\theta_i \cdot a = \Sigma a_{ij}\theta_j$. Here the summations extend over all j with $d(j) = d(i)$.

LEMMA 4.7. *If $f\colon X/G \to X/G$ is stratified at* 0, *then f is also weakly stratified at* 0.

Proof. We must show that $Df\colon T_0(X/G) \to T_0(X/G)$ is an isomorphism. Regarding X/G as a subset of $\boldsymbol{R}^s$, we can write $f = (f_1, \cdots, f_s)\colon X/G \to \boldsymbol{R}^s$ where $f_i = \bar{\theta}_i \circ f$. We also have $f_* = (f_{1*}, \cdots, f_{s*})$ where

$$(1) \qquad f_{i*}(x_1, \cdots, x_s) = \lim_{t\to 0} t^{-d(i)} f_i(t^{d(1)}x_1, \cdots, t^{d(s)}x_s)\,.$$

Since $f_* \in T_\alpha$, it follows from (4.6) and the above remarks that

$$(2) \qquad f_{i*}(x_1, \cdots, x_s) = \Sigma a_{ij} x_j$$

where $a_{ij} = 0$ whenever $d(i) \neq d(j)$. In other words, (a_{ij}) is a matrix of the form

$$(a_{ij}) = \begin{bmatrix} A & & & & 0 \\ & B & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & C \end{bmatrix}$$

where there are nonzero blocks $A, B, \cdots, C$ only where i and j satisfy $d(i) = d(j)$. Consider what it means for the expression

$$\frac{f_i(t^{d(1)}x_1, \cdots, t^{d(s)}x_s)}{t^{d(i)}} \tag{3}$$

to converge as $t \to 0$. Clearly, the first $d(i) - 1$ derivatives of the numerator (with respect to t) must vanish at $t = 0$. Calculation shows that this implies that $D_{j_1}D_{j_2} \cdots D_{j_m}f_i = 0$ whenever $d(j_1) + d(j_2) + \cdots + d(j_m) < d(i)$. (Recall that D_j is partial differentiation at $0 \in \boldsymbol{R}^s$ with respect to the jth coordinate.) In particular, this shows that $D_jf_i = 0$ whenever $d(j) < d(i)$. Next, use l'Hopital's rule to calculate the limit of expression (3) as $t \to 0$. Differentiating the numerator and denominotor $d(i)$ times we obtain

$$f_{i*}(x_1, \cdots, x_s) = \Sigma(\varepsilon_{j_1\cdots j_m}D_{j_1} \cdots D_{j_m}f_i)x_{j_1} \cdots x_{j_m} \tag{4}$$

where $\varepsilon_{j_1\cdots j_m} = d(j_1)! \cdots d(j_m)!/d(i)!$ and where the summation is taken over all $(j_1, \cdots, j_m)$ with $d(j_1) + \cdots + d(j_m) = d(i)$. But, according to (2), f_{i*} is a linear combination of the x_j's. So, in (4), we must have that $D_{j_1} \cdots D_{j_m}f_i = 0$ whenever $m > 1$. It follows that $f_{i*}(x_1, \cdots, x_s) = \Sigma(D_jf_i)\cdot x_j$ and hence, that $D_jf_i = a_{ij}$ whenever $d(i) = d(j)$. In other words, the matrix (D_jf_i) has the form

$$\begin{bmatrix} A & & & & * \\ & B & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & & & C \end{bmatrix}.$$

Since (a_{ij}) is nonsingular so is (D_jf_i). But (D_jf_i) represents the linear transformation $Df\colon T_0(X/G) \to T_0(X/G)$. Hence, f is weakly stratified at 0.

By using local charts, the above lemma immediately yields the following global formulation.

THEOREM 4.8. *Every stratified map of orbit spaces is weakly stratified.*

The converse is an interesting question. This is again a local problem. According to (3.3), a map $f: X/G \to X/G$ which is weakly stratified at 0 is a smooth strata preserving isomorphism in some neighborhood of 0. Thus, this amounts to the following:

Question 4.9. Is every smooth strata preserving isomorphism $f: X/G \to X/G$ stratified at 0?

The Covering Isotopy Theorem (see [1], [10] and § III.2) provides the following partial answer: if $f_t: X/G \to X/G$ is a smooth one-parameter family of (weakly stratified) isomorphisms and if f_0 is stratified at 0, then so is each f_t. G. Schwarz has pointed out that the answer to (4.9) is not always affirmative. For example, let W denote the real 8-dimensional spin representation of $G = \mathrm{Spin}(7)$, and let $X = 4W$. Then X/G embeds in $\boldsymbol{R}^{11}$ and there is a linear automorphism of $\boldsymbol{R}^{11}$ which restricts to a strata preserving automorphism of X/G and which is not in the image of $N_G(GL(X))$. Hence, this automorphism is not stratified. In the next example we consider a case where the answer to (4.9) is affirmative.

EXAMPLE 4.10. Suppose that $G = O(n + m)$, $H = O(n)$ and that V is $M(n, k)$, the space of $n \times k$ matrices (on which $O(n)$ acts by matrix multiplication on the left). Let $H(k)$ be the space of $k \times k$ symmetric matrices and let $H_+(k)$ be the subspace of positive semi-definite ones. Define a polynomial mapping $\theta: M(n, k) \to H(k)$ by $\theta(x) = x^t \cdot x$. According to [11], the entries of θ generate the invariant polynomials on $M(n, k)$. If $n \geqq k$, then the image of θ is $H_+(k)$, so in this case $X/G \cong H_+(k)$. It is easily checked that $S_\alpha = O(m) \times GL(k)$ and that $T_\alpha = GL(k)/\{\pm 1\}$ which acts on $H_+(k)$ by $(g, x) \to g^t x g$. Suppose that $f: H_+(k) \to H_+(k)$ is a strata preserving isomorphism and that Df is the differential at 0. Then $Df(z) = \lim_{t\to 0} t^{-1}f(tz)$ is an isomorphism. It follows that $f_*(z) = \lim_{t\to 0} t^{-2}f(t^2 z)$ also exists and can be identified with the same isomorphism of $H(k)$. The pertinent point is that the entries of θ are homogeneous polynomials of the same degree (namely, of degree two). We claim that $f_* \in GL(k)$. Let $\det: H(k) \to \boldsymbol{R}$ be the determinant. For $t \neq 0$, let $f_t(z) = t^{-2}f(t^2 z)$. Since f_t preserves the stratification, it preserves $\det^{-1}(0) \cap H_+(k)$. It follows that f_* belongs to the subgroup of $\mathrm{Aut}\,(H(k))$ which preserve the hypersurface $\det^{-1}(0)$ and positive semi-definiteness, but this subgroup is precisely $PGL(k)$. This last statement is essentially a well-known theorem of Frobenius (see [4] for details). Thus, $f_* \in T_\alpha$, i.e., f is stratified at 0.

More generally, suppose $O(n)$ acts smoothly on M. We say that M is a *regular* $O(n)$-*manifold* or is *modeled on* $k\rho_n$ if the normal orbit types of $O(n)$ on M are of the form $[O(n - i), M(n - i, k - i)]$,

$0 \leqq i \leqq k$, (these are the normal orbit types of $O(n)$ on $M(n, k)$). It follows from the above analysis that for orbit spaces of regular $O(n)$-manifolds weakly stratified maps are stratified. The same result is true for regular $U(n)$ or $Sp(n)$-manifolds by an essentially identical analysis (see [2] and [4]).

5. The category of local G-orbit spaces. In this section, we shall define "local G-orbit spaces" and "weak local G-orbit spaces." The definition of the previous section will be extended to define "stratified maps of local G-orbit spaces." Roughly speaking, a local G-orbit space will be a space together with a collection of local charts to orbit spaces so that the transition maps are stratified isomorphisms. Also, for each stratum of a local G-orbit space B, we shall define a bundle $C_\alpha(B) \to B_\alpha$, and examine its properties.

Let $I(G)$ be the set of normal G-orbit types. Pick a representative (H^α, V^α) for each $\alpha \in I(G)$, and let $X^\alpha = G \times_{H^\alpha} V^\alpha$.

If B is a space, then an *$I(G)$-chart of type α* is a homeomorphism $\psi\colon U \to X^\alpha/G \times \boldsymbol{R}^m$ from an open set $U \subset B$ onto an open neighborhood of $(0, 0)$ in $X^\alpha/G \times \boldsymbol{R}^m$. Let $\mathscr{A}$ be a collection of $I(G)$-charts which cover B. Suppose that if ψ and ψ' are charts in $\mathscr{A}$ of type α and α', respectively, then the following two conditions hold:

(1) $U \cap U'$ is empty unless $\alpha \leqq \alpha'$ or $\alpha' \leqq \alpha$ (where U and U' are the domains of ψ and ψ', respectively).

(2) $\psi'\psi^{-1}\colon \psi(U \cap U') \to \psi'(U \cap U')$ is a stratified isomorphism of orbit spaces.

Such an $\mathscr{A}$ is called an *$I(G)$-atlas for B*. A *local G-orbit space* is a Hausdorff space B together with a maximal $I(G)$-atlas. We shall usually also require that $m + \dim X^\alpha$ is constant for each chart (this is automatic if B is connected). Local G-orbit spaces are similar to G. Schwarz's "Q-manifolds."

REMARK 5.1. Suppose that we replace condition (2) in the above definition by condition

(2′) $\psi'\psi^{-1}\colon \psi(U \cap U') \to \psi'(U \cap U')$ is a weakly stratified isomorphism.

Then we obtain the notion of a *weak $I(G)$-atlas* and the corresponding notion of a *weak local G-orbit space*. Clearly, any local G-orbit space is a weak local G-orbit space.

Since each X^α/G has a "smooth" structure (see §1), a weak $I(G)$-atlas defines a *smooth structure* on a weak local G-orbit space B. Also there is a natural stratification on B defined as follows. If $\psi\colon U \to X^\alpha/G \times \boldsymbol{R}^m$ is a chart of type α, then let $U_\alpha = \psi^{-1}(\{0\} \times \boldsymbol{R}^m)$. The *$\alpha$-stratum of B* consists of all those points which lie in some

U_α (for a fixed α). Clearly the strata are disjoint smooth manifolds and $B = \bigcup B_\alpha$.

It should be clear how to define a stratified map of local G-orbit spaces. Explicitly, $f: B \to B'$ is *stratified* if for each chart ψ on B and ψ' on B', the map $\psi' \circ f \circ \psi^{-1}$ is stratified (where the composition is defined). Similarly, one can define a *weakly stratified map* of weak local G-orbit spaces.

Let $\mathscr{B}$ be the category with objects local G-orbit spaces and with morphisms stratified maps. $\mathscr{B}$ is, indeed, a category; for, the only axiom which is not obviously satisfied is that the composition of two stratified maps is stratified, but this is immediate from (4.3). Similarly, there is a category $\mathscr{B}_w$ of weak local G-orbit spaces and weakly stratified maps.

THEOREM 5.2. *Suppose that G acts smoothly on M. Then M/G naturally has the structure of a local G-orbit space.*

Proof. The point is that we can choose a collection of equivariant charts on M of the form $\Phi: W \to X^\alpha \times \boldsymbol{R}^m$, where $\{W\}$ is an open cover for M and where the transition maps are equivariant diffeomorphisms. It then follows from (4.5) that the induced charts $\psi: W/G \to X^\alpha/G \times \boldsymbol{R}^m$ will be an $I(G)$-atlas for M/G.

If B is actually the orbit space of a smooth G-action on M, then there is a fiber bundle over B_α which is isomorphic to a neighborhood of B_α in B; namely, $C_\alpha(B) = \nu_\alpha(M)/G$. We want to show how to define $C_\alpha(B)$ without mentioning M, that is, how to define $C_\alpha(B)$ for an arbitrary local G-orbit space B.

Consider a smooth curve $\omega: [0, 1] \to X/G$ such that $\omega(0) = 0$. We say that ω is a *good curve at* 0 if $t^{-1} \circ \omega(t)$ converges to a point in X/G as $t \to 0$, in which case this point is denoted by $\omega_*(0) \in X/G$. Notice that if $f: X/G \to X/G$ is stratified at 0, then $f \circ \omega$ is also a good curve at 0 and $(f \circ \omega)_*(0) = f_*(\omega_*(0))$. Also, notice that if $\theta: [0, 1] \to X$ is a smooth curve and if $\pi: X \to X/G$ is the orbit map, then $\pi \circ \theta$ is a good curve at 0 and $(\pi \circ \theta)_*(0)$ can be identified with the image of $\theta_*(d/dt)$ in X/G. (Here we are identifying X with the normal bundle of G/H in X, as usual.)

This definition can be promoted to an arbitrary local orbit space B, as follows. Let $b \in B_\alpha$. Suppose that $\omega: [0, 1] \to B$ is a smooth curve with $\omega(0) = b$. Let $\psi: U \to X^\alpha/G \times \boldsymbol{R}^m$ be a chart of type α defined on some neighborhood of b. Let $p: X^\alpha/G \times \boldsymbol{R}^m \to X^\alpha/G$ be projection on the first factor. Finally, let $\tilde{\omega} = p \circ \psi \circ \omega: [0, \varepsilon] \to X/G$, where $[0, \varepsilon] \subset \omega^{-1}(U)$. We say that ω is a *good curve at b* if $\tilde{\omega}$ is a good curve at 0. If ω_1 is another good curve at b, then ω is *equivalent* to ω_1 if $\tilde{\omega}_*(0) = \tilde{\omega}_{1*}(0)$. Clearly, the notions of good curve and of equivalence of good curves are independent of the choice of chart. Thus, let $(C_\alpha)_b$ be the set of equivalence classes of good curves at b, and let

$$C_\alpha(B) = \bigcup_{b \in B_\alpha} (C_\alpha)_b \ .$$

The chart ψ provides us with a bijection $\psi_*: C_\alpha(U) \to C_\alpha(X^\alpha/G \times \boldsymbol{R}^m) \cong X^\alpha/G \times \boldsymbol{R}^m$. If for each chart ψ we require that ψ_* be a homeomorphism, then this defines a topology on $C_\alpha(B)$ and gives $C_\alpha(B)$ the structure of a (locally trivial) fiber bundle over B_α with fiber X^α/G. Since the transition functions are stratified, it follows that the structure group of this bundle is T_α. As usual, we shall often write simply C_α instead of $C_\alpha(B)$.

REMARK 5.3. If $B = M/G$, then $C_\alpha(B) \cong \nu_\alpha(M)/G$.

REMARK 5.4. If $f: B \to B'$ is a stratified map of local G-orbit spaces, then for each $\alpha \in I(G)$, f induces a bundle map $f_*: C_\alpha(B) \to C_\alpha(B')$ defined by $\omega \to f \circ \omega$.

We shall need the following "Tubular Neighborhood Theorem" in Chapter IV.

THEOREM 5.5 (*Tubular Neighborhood Theorem*). *Let B be a local G-orbit space and let α be a normal orbit type.*

(Existence). *There is a stratified map $T: C_\alpha \dashrightarrow B$ which maps C_α isomorphically onto some neighborhood of B_α in B that $T|_{B_\alpha}$ is the inclusion and such that $T_*: C_\alpha \to C_\alpha$ is the identity.*

(Uniqueness). *If $T': C_\alpha \to B$ is another such stratified map then there is a stratified isotopy $\Phi: C_\alpha \times [0, 1] \to B$ such that $\Phi_0 = T$ and $\Phi_1 = T'$.*

If B is actually an orbit space, then this theorem follows immediately from the Equivariant Tubular Neighborhood Theorem (and this is really the only case in which we need the above theorem). The proof of existence in the general case will be omitted, since it would take us too far afield. As we shall show below, the proof of uniqueness is virtually identical to the proof of the uniqueness part of the ordinary Tubular Neighborhood Theorem. By altering T by an isotopy we may assume that $T(C_\alpha) \subset T'(C_\alpha)$. Consider $\Lambda = T^{-1} \circ T': C_\alpha \to C_\alpha$. The real numbers act by fiberwise scalar multiplication on C_α as in (4.1). For $t \in (0, 1]$, define $\Lambda_t(x) = t^{-1} \circ \Lambda(t \circ x)$ and set $\Lambda_0 = \Lambda_* = id$. Define $\Phi(x, t) = T(\Lambda_t(x))$. Then $\Phi(x, 0) = T(x)$

and $\Phi(x, 1) = T \circ T^{-1} \circ T'(x) = T'(x)$, as claimed.

6. The category of smooth G-manifolds.

DEFINITION 6.1. A *smooth G-manifold* is a triple (M, B, p) where M is a manifold on which G acts smoothly, B is a local G-orbit space, and $p: M \to B$ is a smooth map which is constant on orbits and which induces a stratified isomorphism $\bar{p}: M/G \to B$. M is called the *total space*, B is the *base space*, and p is the *projection map*.

In the usual fashion, we will sometimes blur the distinction between a G-manifold and its total space and write simply "M is a G-manifold." When we wish to emphasize the base space, we shall say that M is a *G-manifold over B*.

Let $\mathscr{D}$ be the category, the objects of which are smooth G-manifolds and the morphisms of which are stratified maps of the total spaces (see I.1.8). If $F: M \to M'$ is stratified, then there is an induced map $\pi(F): B \to B'$ which makes the following diagram commute

$$\begin{array}{ccc} M & \xrightarrow{F} & M' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ B & \xrightarrow[\pi(F)]{} & B' \end{array} .$$

It follows from (4.5), that $\pi(F)$ is a stratified map of local orbit spaces. We state this as the following proposition.

PROPOSITION 6.2. *There is a functor $\pi: \mathscr{D} \to \mathscr{B}$ which assigns to each G-manifold its base space and to each stratified map F the induced map $\pi(F)$.*

REMARK 6.3. There is also a functor $W: \mathscr{B} \to \mathscr{B}_w$ which associates to each local G-orbit space the underlying weak local G-orbit space and therefore, a functor $\pi_w = W \circ \pi: \mathscr{D} \to \mathscr{B}_w$.

III. Pullbacks

1. The pullback construction. Let M be a smooth G-manifold over B with projection map $p: M \to B$. Suppose that A is a weak local orbit space and that $f: A \to B$ is weakly stratified (see § II.3). Define $f^*(M)$, the *pullback of M via f*, as

$$f^*(M) = \{(x, a) \in M \times A \mid p(x) = f(a)\} .$$

There is a commutative square

$$\begin{CD} f^*(M) @>{\tilde f}>> M \\ @V{q}VV @VV{p}V \\ A @>{f}>> B \end{CD} ,$$

where $\tilde{f}$ and q are the maps induced by projection on the first and second factor, respectively.

THEOREM 1.1. *With the above hypotheses, A naturally has the structure of a local G-orbit space, and $(f^*(M), A, q)$ is a smooth G-manifold over A. Moreover, $\tilde{f}$ is stratified.*

REMARK 1.2. The subspace $f^*(M) \subset M \times A$ could have been defined for any map f; however, in this generality, it definitely would *not* be a manifold. The main point of Theorem 1.1 is that $f^*(M)$ will be a smooth manifold whenever f is weakly stratified. The proof given below was suggested by the proof of a special case in [3]. For a stratified map f, a different proof could be given by using the results of Chapter IV.

Proof of (1.1). Since M is a smooth manifold and since A is a weak local orbit space, $M \times A$ has a natural "smooth" structure (i.e., a functional structure). Let G act trivially on A and via the product action on $M \times A$. Then this action on $M \times A$ is through smooth isomorphisms. The subspace $f^*(M)$ is clearly G-invariant. Moreover, it inherits a smooth structure induced by the inclusion.

We shall show that with this induced smooth structure gives $f^*(M)$ the structure of a smooth manifold. Since the problem is local, by choosing charts, we may assume that $M = X \times \boldsymbol{R}^n$, that $B = X/G \times \boldsymbol{R}^m$ and that $A = X/G \times \boldsymbol{R}^n$, where X $(=X^\alpha)$ is a G-vector bundle representing a normal orbit type α. As usual, we regard X/G as a subset of $\boldsymbol{R}^s$. Since $f\colon A \to B$ is smooth, it extends to a smooth map $g\colon \boldsymbol{R}^s \times \boldsymbol{R}^m \to \boldsymbol{R}^s \times \boldsymbol{R}^n$. For each $z \in \boldsymbol{R}^m$, define $g_z\colon \boldsymbol{R}^s \to \boldsymbol{R}^s$ by $g_z(y) = \mathrm{pr}\,(g(y, z))$, where $\mathrm{pr}\colon \boldsymbol{R}^s \times \boldsymbol{R}^n \to \boldsymbol{R}^s$ is the projection on the first factor. Let $f_z\colon X/G \to X/G$ be the restriction of g_z. Since f is weakly stratified, so is f_z. So, it follows from the proof of Proposition 3.3 in Chapter II, that there is an open neighborhood U of the origin in $\boldsymbol{R}^s$ such that $g_z|_U$ is an embedding and such that $f_z|_{(X/G \cap U)}$ is an isomorphism onto some neighborhood of 0 in X/G. It is then easy to see that we can choose a neighborhood $Y \times W \subset \boldsymbol{R}^s \times \boldsymbol{R}^m$ so that for each $z \in W$, f_z maps $Y \cap X/G$ isomorphically onto a neighborhood of 0 in X/G. Consider the map $\Phi\colon M \times (Y \times W) \to \boldsymbol{R}^s \times \boldsymbol{R}^n$ defined by

$$\Phi(x, y, z) = p(x) - g(y, z) .$$

We assert that Φ is a submersion. Suppose that $z = (r, s) \in X \times \boldsymbol{R}^n$ (recall that $M = X \times \boldsymbol{R}^n$). Clearly, the differential $Dg_{(y,z)}$ maps $T_y(Y)$ isomorphically onto the subspace $T_{g_z(y)}(\boldsymbol{R}^s)$ and Dp_x maps the subspace $T_s(\boldsymbol{R}^n) \subset T_x(M)$ isomorphically onto $T_{p(x)}(\boldsymbol{R}^n)$. Thus, $D\Phi = Dp - Dg$ is everywhere surjective, i.e., Φ is a submersion. So, by the Implicit Function Theorem, $\Phi^{-1}(0, 0)$ is a smooth submanifold of $M \times (Y \times W)$. Clearly, $f^*(M) \cap M \times (Y \times W) \subset \Phi^{-1}(0, 0)$. On the other hand, if $(x, y, z) \in \Phi^{-1}(0, 0)$, then $g(y, z) = p(x)$ and therefore, $g_z(y) \in X/G$. But by the proof of Proposition II.3.3, $\text{Image}(f_z) = \text{Image}(g_z) \cap X/G$; so it follows that $f_z^{-1}(g_z(y)) = y \in X/G$. Thus, $(x, y, z) \in f^*(M)$ and so $f^*(M) \cap M \times (Y \times W) = \Phi^{-1}(0, 0)$, that is, $f^*(M)$ is a smooth manifold.

Next, consider the smooth map $q: f^*(M) \to A$. It induces a smooth map $\bar{q}: f^*(M)/G \to A$ which clearly preserves the stratification. We can identify $f^*(M)/G$ with the graph of f and $\bar{q}$ with the map $(f(a), a) \to a$. Since $a \to (f(a), a)$ is obviously a smooth inverse for $\bar{q}$, it follows that $\bar{q}$ is a (weakly stratified) isomorphism. Since $f^*(M)/G$ is an orbit space, it is a local orbit space. Thus, the isomorphism $\bar{q}$ defines a local orbit space structure on A.

It remains to check that $\tilde{f}: f^*(M) \to M$ is stratified, i.e., that it is smooth and equivariant and that it preserves normal representations (see I.1.8). $\tilde{f}$ is clearly smooth and equivariant. Moreover, $G_{(x,a)} = G_x = G_{\tilde{f}(x,a)}$. Let us compute the normal representation at $(x, a) \in f^*(M)$. We have that

$$T_{(x,a)}f^*(M) = \{(v, w) \in T_xM \times T_xA \mid Dp_x(v) = Df_a(w)\} .$$

In particular, $\ker dp_x \subset T_{(x,a)}f^*(M)$. Let V_x and $V_{(x,a)}$ be the respective normal representations at $x \in M$ and $(x, a) \in f^*(M)$. Since $\ker dp_x = V_x + T_x(G(x))$, it follows that $T_{(x,a)}f^*(M) = V_x + T_x(G(x)) + F$, where F is a trivial G_x-module. Hence, $V_{(x,a)} = V_x$. Since the projection $T_xM \times T_aA \to T_xM$ maps V_x onto itself, the same is true for its restriction to $D\tilde{f}: T_{(x,a)}f^*(M) \to T_xM$. Thus, $\tilde{f}$ is stratified.

The hypothesis of the above theorem is that A is a weak local orbit space and that f is weakly stratified. If we had assumed at the outset that A was a local orbit space and that f was stratified, then one might well ask if the isomorphism $\bar{q}: f^*(M)/G \to A$ is stratified. This is indeed the case. For, $f \circ \bar{q}: f^*(M)/G \to B$ is certainly stratified, since it is covered by the stratified map $\tilde{f}$. Since f is also assumed to be stratified, it follows easily that so is $\bar{q}$. Therefore, Theorem 1.1 has the following corollary.

THEOREM 1.3. *Let M be a smooth G-manifold over B and let $f: A \to B$ be a stratified map (of local orbit spaces). Then $f^*(M)$*

is a smooth G-manifold over A.

To justify the terminology "pullback," we should show that $f^*(M)$ satisfies the universal property of pullbacks. First let us briefly recall the relevant definitions from category theory. Let $\phi: C \to D$ be a functor. For any object Y in D, we can define a category $\phi^{-1}(Y)$ called the *fiber at Y*. The objects of $\phi^{-1}(Y)$ consist of those $X \in ob\ C$ such that $\phi(X) = Y$, and $\mathrm{Hom}_{\phi^{-1}(Y)}(X, X')$ consists of those morphisms $u: X \to X'$ such that $\phi(u) = \mathrm{id}_Y$. An arbitrary morphism $u: X \to X'$ in C is *cartesian* if for any $X'' \in ob\ \phi^{-1}(\phi(X))$, the natural map $\mathrm{Hom}_{\phi^{-1}(\phi(X))}(X'', X) \to \{t \in \mathrm{Hom}_C(X'', X') \mid \phi(t) = \phi(u)\}$ is a bijection.

$$\begin{tikzcd}
X'' \arrow[dr] \arrow[drr] \arrow[ddr] & & \\
& X \arrow[r, "u"] \arrow[d] & X' \arrow[d] \\
& \phi(X) \arrow[r, "\phi(u)"] & \phi(X')
\end{tikzcd}$$

The functor ϕ is *prefibered* if for every morphism $s: Y \to Y'$ in D with $Y' = \phi(X')$, there is a cartesian arrow $u: X \to X'$ lying over it (i.e., with $\phi(u) = s$). ϕ is *fibered* if, in addition, the composition of cartesian arrows is cartesian (the terminology "C is a fibered category over D" is perhaps more common).

Recall that $\mathscr{B}$ and $\mathscr{B}_w$ are the categories of local G-orbit spaces and weak local G-orbit spaces, respectively, and that $\pi: \mathscr{D} \to \mathscr{B}$ and $\pi_w: \mathscr{D} \to \mathscr{B}_w$ are the canonical functors (see II.8).

THEOREM 1.4. *The functors $\pi: \mathscr{D} \to \mathscr{B}$ and $\pi_w: \mathscr{D} \to \mathscr{B}_w$ are fibered. In fact, every morphism in $\mathscr{D}$ is cartesian (with respect to either π or π_w).*

Proof. We prove the theorem for π_w, the proof for π being identical. Let M be a smooth G-manifold over B (i.e., $\pi_w(M) = B$) and let $f: A \to B$ be weakly stratified. First we show that $\tilde{f}: f^*(M) \to M$ is cartesian. Suppose that M' is another smooth G-manifold over A with projection map $r: M' \to A$ and that $t: M' \to M$ is a stratified map covering f. Define $h: M' \to f^*(M)$ by $h(x) = (t(x), r(x))$.

$$\begin{tikzcd}
M' \arrow[dr, "h"] \arrow[drrr, "t"] \arrow[ddr, "r"'] & & & \\
& f^*(M) \arrow[rr, "\tilde{f}"] \arrow[d] & & M \arrow[d] \\
& A \arrow[rr, "f"] & & B
\end{tikzcd}$$

Since r and t are smooth so is h; furthermore, h is stratified since t is. Thus, π_w is prefibered. Since $\pi_w(h)$ is the identity, it follows from Proposition II.3.4 that h is an isomorphism (i.e., that h is an equivariant diffeomorphism). Thus, t is also cartesian. But we could have started with an arbitrary stratified map $t\colon M' \to M$, set $f = \pi_w(t)$ and reached the same conclusion. Thus, every $\mathscr{G}$-morphism is cartesian and therefore, π_w is a fibered functor.

REMARK 1.5. The fiber $\pi^{-1}(B)$ is called the *category* of *G-manifolds over B*. As we pointed out in the above proof, it follows from II.3.4 that the morphisms of $\pi^{-1}(B)$ are equivariant diffeomorphisms which cover the identity on B.

2. The Covering Homotopy Theorem.

THE COVERING ISOTOPY THEOREM 2.1. *Let M be a smooth G-manifold over B. Suppose that $\Phi\colon B \times I \to B$ is a smooth one parameter family of (weakly stratified) isomorphisms with $\Phi_0 = \mathrm{id}_B$ (Φ is an "isotopy"). Then there is an equivariant isotopy $\psi\colon M \times I \to M$ with $\pi_w(\psi) = \Phi$ and with $\psi_0 = \mathrm{id}_M$.*

This was conjectured by G. Bredon. It was proved for finite groups by E. Bierstone [1], for regular $O(n)$, $U(n)$ and $Sp(n)$ actions in the author's thesis, and in full generality by G. Schwarz [10].

The Covering Isotopy Theorem is implied by the statement that a vector field on B which is tangent to each stratum can be lifted to an invariant vector field on M. The proofs of both Bierstone and Schwarz involve first proving this "Vector Field Lifting Theorem." Using the pullback construction, we shall show how this implies the following smooth version of Palais' Covering Homotopy Theorem (see [2], [8]).

THEOREM 2.2 (*The Covering Homotopy Theorem*). *Let M and M' be smooth G-manifolds over B and B' and let $F\colon M \to M'$ be* *stratified. Suppose that $h: B \times I \to B'$ is a weakly stratified homotopy with $h_0 = \pi_w(F)$. Then, there is a stratified homotopy $H: M \times I \to M'$ extending F and covering h (i.e., with $\pi_w(H) = h$).*

Proof. Since $\tilde{h}_0: h_0^*(M') \to M'$ is cartesian, there is an equivariant diffeomorphism $F': M \to h_0^*(M')$ covering the identity on B (see 1.4). The theorem now follows from the assertion that there is an equivariant diffeomorphism $\psi: h_0^*(M') \times I \to h^*(M')$ covering the identity on $B \times I$. For, assuming this, the homotopy H can be defined to be the composition $H = \tilde{h} \circ \psi \circ (F' \times \mathrm{id}): M \times I \to h_0^*(M') \times I \to h^*(M') \to M'$.

So, it remains to produce the equivalence ψ. Consider the vector field d/dt on $B \times I$. By the Covering Isotopy Theorem, d/dt lifts to an invariant vector field X on $h^*(M')$. Let r be the map $h^*(M') \to B \times I \to I$. Let Λ_z be an integral curve for X through $z \in h^*(M')$. Then, $r_*(d\Lambda_z/dt) = 1$. Hence, $r(\Lambda_z(t)) = t + \text{constant}$. So, by reparameterizing, we may assume that $r(\Lambda_z(t)) = t$. Let c denote the composition $h^*(M') \to B \times I \to B$. Since Λ_z stays inside the compact set $c^{-1}(c(z))$, it extends to a maximal integral curve parameterized by $[0, 1]$. So, define $\psi: h_0^*(M') \times I \to h^*(M')$ by $\psi(z, t) = \Lambda_z(t)$. This is the required equivalence.

Let M and M' be smooth G-manifolds over B and B'. Put the coarse C^∞-topology on $\mathrm{Hom}_{\mathscr{G}}(M, M')$ and on $\mathrm{Hom}_{\mathscr{B}}(B, B')$. Let $\mathrm{Diff}_1^G(M) = \mathrm{Hom}_{\pi^{-1}(B)}(M, M)$ be the group of equivariant diffeomorphisms of M which cover the identity on B. The following conjecture would be a strengthened version of 2.2.

Conjecture 2.3. $\pi: \mathrm{Hom}_{\mathscr{G}}(M, M') \to \mathrm{Hom}_{\mathscr{B}}(B, B')$ is a principal fibration with fiber $\mathrm{Diff}_1^G(M)$, (by "fibration" we only mean that π has the Covering Homotopy Property).

IV. Normal Systems

In this chapter we show how a smooth G-manifold can naturally be regarded as a collection of principal fiber bundles over certain manifolds with corners. These collections of bundles are called "normal systems." The main ideas of this construction are explained in § 1, where we describe a functorial association of normal systems to G-manifolds. The principal results are stated in § 4. One of them, Theorem 4.3, states that this functor from $\mathscr{G}$ to the category of normal systems defines a bijection on isomorphism classes. This result was essentially proved by Jänich in [7]. From a certain point of view, this is just the Equivariant Tubular Neighborhood Theorem. This idea can be pushed further. Indeed, the next result (Theorem

4.4) is the analogous theorem for local G-orbit spaces. Then, using the Covering Homotopy Theorem, we are able to prove another similar result (Theorem 4.5) for the category $\pi^{-1}(B)$ of G-manifolds over B. This last result is the one which, perhaps, best isolates the bundle theoretic aspects of smooth G-manifolds.

1. Removing a tubular neighborhood. In this section, we shall describe a certain functorial process of "removing" a tubular neighborhood of a stratum. Actually, rather than removing a tubular neighborhood, this process involves attaching a boundary to the complement of a stratum—essentially, by passing to "fiberwise polar coordinates" on a tubular neighborhood of the stratum. The discussion follows Jänich [7].

In order to focus our ideas, we shall, for the moment, forget about G-manifolds and local orbit spaces and concentrate simply on "removing" a tubular neighborhood of a submanifold. First, we need some preliminary material.

Suppose that E is a smooth vector bundle over a manifold A. Let E_0 denote the complement of the zero-section. The positive real numbers $\boldsymbol{R}_+$ act on E and on E_0 by fiberwise scalar multiplication. There is an associated bundle C_+E, called *the nonnegative cylinder bundle*, which is defined by

$$C_+E = E_0 \times_{\boldsymbol{R}_+} [0, \infty) ,$$

where a positive real number s acts on $(x, t) \in E_0 \times [0, \infty)$ by $s \cdot (x, t) = (xs^{-1}, st)$. Denote the image of (x, t) in C_+E by $[x, t]$. The boundary of C_+E is called the *sphere bundle*, ΣE; it is the subset

$$\Sigma E = E_0 \times_{\boldsymbol{R}_+} \{0\} \cong E_0/\boldsymbol{R}_+ .$$

There is a canonical map $c: C_+E \to E$ defined by $c([x, t]) = tx$, which takes $C_+E - \Sigma E$ diffeomorphically onto E_0 and ΣE onto A via the projection mapping.

REMARK 1.1. If we pick a metric for E, then the map $[x, 0] \to x/|x|$ identifies ΣE with the set of vectors of unit length in E. Notice that C_+E can also be regarded as a bundle over ΣE with fiber $[0, \infty)$ and with projection map $[x, t] \to [x, 0]$. Furthermore, given a metric, there is a bundle trivialization $s: C_+E \to \Sigma E \times [0, \infty)$ defined by $s([x, t]) = (x/|x|, t|x|)$.

The following lemma is the key to our construction.

LEMMA 1.2. *Let E and E' be smooth vector bundles over A and*

A', respectively. Let $\psi\colon E \to E'$ be a smooth map such that $\psi^{-1}(A') = A$ and such that the restriction of $\psi_*\colon E \to E'$ to any fiber is an isomorphism (i.e., *ψ is transverse to A*). *Then there is a unique smooth map*: *$\bar{\psi}\colon C_+E \to C_+E$ making the following diagram commute*

$$\begin{array}{ccc} C_+E & \xrightarrow{\bar{\psi}} & C_+E' \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\ E & \xrightarrow{\psi} & E\,. \end{array}$$

In fact, $\bar{\psi}$ is defined by the formula

$$\bar{\psi}([x, t]) = \begin{cases} [t^{-1}\psi(tx), t]; & t \neq 0 \\ [\psi_*(x), 0]\ ; & t = 0\,. \end{cases}$$

In particular, this formula shows that $\bar{\psi}|_{\Sigma E}$ is a bundle map.

Proof. Since $c|_{C_+E - \Sigma E}$ is a diffeomorphism, it follows that there is exactly one way to define $\bar{\psi}$ on $C_+E - \Sigma E$ so that the diagram will commute; namely, by $\bar{\psi}([x, t]) = [\psi(tx), 1] = [t^{-1}\psi(t(x)), t]$. Since $\lim_{t\to 0}[t^{-1}\psi(tx), t] = [\psi_*(x), 0]$ and since $\bar{\psi}$ must be continuous, we see that $\bar{\psi}$ must be defined by the given formula. Moreover, this formula clearly defines a smooth map.

Now, suppose that M is a smooth manifold and that A is a submanifold and a closed subset. Let N be the normal bundle of A in M. Pick a "tubular map" $T\colon N \to M$. By a *tubular map*, we mean an embedding $T\colon N \to M$ such that

(i) $T|_A$ is the inclusion and

(ii) $T_*\colon N \to N$ is the identity.

(Here T_* denotes the map from the normal bundle of A in N to the normal bundle of A in M induced by the differential.) A smooth manifold with boundary $M \odot A$ can be defined as follows. As a set, $M \odot A$ is the disjoint union of $M - A$ and ΣN. The tubular map T induces a map $\tau\colon C_+N \to M - A \cup \Sigma N$ defined by

$$\tau([x, t]) = \begin{cases} T(tx)\ ; & t \neq 0 \\ [x, 0]\ ; & t = 0\,. \end{cases}$$

As Jänich points out, $M - A \cup \Sigma N$ has exactly one smooth structure as a manifold with boundary which agrees with the original smooth structure on $M - A$ and with respect to which τ is a diffeomorphism onto a neighborhood of ΣN in $M - A \cup \Sigma N$. This manifold with boundary is denoted by $M \odot A$. The map $\tau\colon C_+N \to M \odot A$ may be regarded as a collared neighborhood of the boundary.

In everything that follows, we shall only be concerned with the "germ" of a tubular map or of a collared neighborhood. So let us adopt the following conventions. If E is a smooth vector bundle, then the symbols "$f: E \to X$" will mean only that the domain of f is some neighborhood of the zero-section. Similarly, the domain of $g: C_+E \to X$ will only be required to be some neighborhood of ΣE. The phrase "$f: E \to X$ (respectively, $g: C_+E \to X$) is a diffeomorphism" will only mean that f (respectively, g) is a diffeomorphism from some neighborhood on the zero-section (respectively, ΣE) onto its image.

Let us consider the effect on our construction of altering the choice of tubular maps. We claim that the smooth structure on $M \odot A$ is independent of the choice of T. For, let $T': N \to M$ be another tubular map and let $\tau': C_+N \to M \odot A$ be the induced collared neighborhood. Since the smooth structure on $M \odot A$ was defined by requiring τ to be a diffeomorphism, our claim amounts to the assertion that $\tau^{-1} \circ \tau': C_+N \to C_+N$ is a diffeomorphism. But, clearly $\tau^{-1} \circ \tau' = \bar{\psi}$, where $\psi = T^{-1} \circ T': N \to N$; so this assertion follows from Lemma 1.2. This argument also suggests how to define an equivalence relation on collared neighborhoods so as to make the equivalence class of τ independent of the choice of tubular maps. To be specific, two collared neighborhoods $\tau: C_+N \to M \odot A$ and $\tau': C_+N \to M \odot A$ are *equivalent* if $\tau^{-1} \circ \tau' = \bar{\psi}$, where $\psi: N \to N$ is the germ of some diffeomorphism with $\psi^{-1}(A) = A$ and with $\psi_* = \mathrm{id}_N$.

So, given a manifold M and a submanifold A, we have constructed a 4-tuple $(M \odot A, N, \theta, [\tau])$ consisting of

(1) a manifold with boundary, $M \odot A$,

(2) a vector bundle N over a manifold A

(3) a diffeomorphism $\theta: \Sigma N \to \partial(M \odot A)$, and

(4) an equivalence class $[\tau]$ of a collared neighborhood $\tau: C_+N \to M \odot A$, where $\tau|_{\Sigma N} = \theta$.

Conversely, given any such 4-tuple $(Y, E, \theta, [\tau])$, one can recover a manifold M together with a submanifold A (the base space of E). For, let $\tau: C_+E \to Y$ be a collared neighborhood in the given equivalence class. As a set, M will then be defined as the disjoint union $(Y - \partial Y) \cup A$. There is an obvious map $T: E \to (Y - \partial Y) \cup A$ defined by

$$T(x) = \begin{cases} \tau([x, 1]); & x \in E_0 \\ x \quad ; & x \in A . \end{cases}$$

M can be given the structure of a smooth manifold by requiring that T be a diffeomorphism from E onto some neighborhood of A in M. This structure is well-defined, since our definition of the equivalence relation on collared neighborhoods is obviously rigged so that the smooth structure on M will be independent of the choice of τ within the given equivalence class.

These constructions are clearly inverse to one another. Moreover, they are functorial in a sense which we shall explain below. If $(Y, E, \theta, [\tau])$ and $(Y', E', \theta', [\tau'])$ are 4-tuples satisfying the above conditions, then by a *morphism* $(Y, E, \theta, [\tau]) \to (Y', E', \theta', [\tau'])$, we mean

(i) a smooth map $h: (Y, \partial Y) \to (Y', \partial Y')$, together with

(ii) a bundle map $\lambda: E \to E'$ which is a fiberwise isomorphism, such that the following diagram commutes

(iii)
$$\begin{array}{ccc} \Sigma E & \xrightarrow{\bar{\lambda}} & \Sigma E \\ \theta\downarrow & & \downarrow\theta' \\ \partial Y & \xrightarrow[h|_{\partial Y}]{} & \partial Y' \end{array}.$$

Furthermore, if τ and τ' are choices of collared neighborhoods in the given equivalence classes, then we require that

(iv) There exists a smooth map (germ) $\psi: E \to E'$ transverse to A', such that $\psi^{-1}(A') = A$, $\psi_* = \lambda$, and $\bar{\psi} = \tau'^{-1} \circ h \circ \tau: C_+E \to C_+E$. Notice that if ψ exists it is uniquely determined by τ, τ' and h.

Given two pairs (M, A) and (M', A'), we consider as morphisms smooth maps $F: M \to M'$ which are transverse to A' and which satisfy $F^{-1}(A') = A$. Such an F induces a morphism of 4-tuples $(M \odot A, N, \theta, [\tau]) \to (M' \odot A', N', \theta', [\tau'])$ defined, in the obvious fashion, by $\lambda = F_*$, $h|_{(M\odot A)-\partial(M\odot A)} = F|_{M-A}$, and $h|_{\partial(M\odot A)} = \bar{F}_*|_{\Sigma N}$. It is immediate from Lemma 1.2 that h is smooth and that condition (iv) holds. It is just as easy to see that, conversely, a morphism of 4-tuples induces a morphism of the associated manifold pairs.

The above construction clearly works equivariantly. For, suppose that M is a smooth G-manifold and that A is an invariant submanifold. Then we can choose an equivariant tubular map $T: N \to M$ and define $M \odot A$ as before (actually, it is not necessary for T to be equivariant). Since G acts smoothly on M and since the construction is functorial, it acts smoothly on $M \odot A$.

We are particularly interested in the case where $A = M_\alpha$ is a minimal stratum of M, i.e., where $M_\beta = \phi$ whenever $\beta < \alpha$. In this case M_α is closed (by I.1.6). So we obtain a G-manifold with boundary $M \odot M_\alpha$, which has one less stratum. Notice that a stratified map $M \to M'$ is a morphism of manifold pairs $(M, M_\alpha) \to (M', M'_\alpha)$ in the sense we discussed above. We can continue this process of "removing" tubular neighborhoods of the strata, all the while keeping track of the relevant information (normal bundles, identifications on the boundaries, and collared neighborhoods). Just as we associated a 4-tuple to a manifold pair, we are led to associate an "augmented normal system" to a smooth G-manifold. As one might suspect, this association is an equivalence of the appropriate categories (see 4.9).

For many purposes an "augmented normal system" contains too much information. To see this, let us again consider 4-tuples $(Y, E, \theta, [\tau])$, as defined above. If we forget about the equivalence class $[\tau]$ (condition (4) in the definition), and consider only the triple (Y, E, θ), then we have not lost anything crucial; for, by the Collared Neighborhood Theorem, ∂Y always has a collared neighborhood and any two such collared neighborhoods are isotopic. Morphisms of such triples can be defined as before, except that now we must forget about condition (iv), which no longer makes sense. Forgetting the collared neighborhoods leads to the notion of a "normal system" associated to a G-manifold. The category of normal systems is conceptually much simpler than the category of augmented normal systems (just as the triples are simpler than the 4-tuples). The association of a normal system to a G-manifold is still a functor, but there is no adjoint. However, a normal system still determines a smooth G-manifold, well-defined up to equivariant diffeomorphism (see 4.3).

REMARK 1.3. There is a similar procedure for local G-orbit spaces. Suppose that B_α is a minimal stratum of a local orbit space B. Recall that in § II.5 we showed that there was a bundle $C_\alpha \to B_\alpha$ with fiber X^α/G. The positive real numbers act by fiberwise scalar multiplication on C_α and on $C_\alpha - B_\alpha$ (see II.4.1). So we can define a *nonnegative cylinder bundle*

$$c_+C_\alpha = (C_\alpha - B_\alpha) \times_{\boldsymbol{R}_+} [0, \infty)$$

and a *sphere bundle*

$$\sigma C_\alpha = (C_\alpha - B_\alpha) \times_{\boldsymbol{R}_+} \{0\} \cong (C_\alpha - B_\alpha)/\boldsymbol{R}_+$$

as before. Also, if $\psi\colon C_\alpha \to C'_\alpha$ is a stratified map, then there is a smooth stratified map $\bar{\psi}\colon c_+C_\alpha \to c_+C'_\alpha$ defined by the formula given in Lemma 1.2. (Also, notice that an equivariant version of this lemma is clearly true.) Finally, by Theorem II.4, there is a "tubular map" $T\colon C_\alpha \to B$. So we can construct $B \odot B_\alpha$ in the same manner as we constructed $M \odot M_\alpha$. Moreover, it is clear that if $M/G \cong B$, then $(M \odot M_\alpha)/G \cong B \odot B_\alpha$.

We should point out, however, that there is no such construction for weak local orbit spaces, for two reasons. First of all, for weak local orbit spaces, we do not know that the bundles C_α exist. Secondly, if $\psi\colon C_\alpha \to C'_\alpha$ is only weakly stratified, thet there is no guarantee that $\bar{\psi}: c_+C_\alpha \to c_+C'_\alpha$ exists (essentially because as $t \to 0$ the limit of $t^{-1} \circ \psi(t \circ x)$ may not exist). Thus, there is no analogue of Lemma 1.2 for weakly stratified maps.

2. The closure of a stratum. In this section we attach boundaries to the strata and to the normal orbit bundles.

Recall that a *smooth n-dimensional manifold with corners M* is differentiably modeled on open subsets of $\{x \in \boldsymbol{R}^n \mid x_1 \geqq 0, \cdots, x_n \geqq 0\}$. If $x \in M$ is represented by local coordinates $(x_1, \cdots, x_n)$, then we denote by $c(x)$ the number of zeros in this n-tuple. Following [7], we say that M is a *manifold with faces* if every $x \in M$ belongs to exactly $c(x)$ different connected components of $\{y \in M \mid c(y) = 1\}$. Any disjoint union of the closures of such components is called a *fact of M*. Notice that any face is an $(n-1)$-dimensional manifold with faces (see [7]).

DEFINITION 2.1. Let S be a partially ordered set. A *manifold with S-faces* is a manifold with faces M together with an S-tuple of faces $(\partial_s M)_{s \in S}$ such that

(i) $\partial M = \bigcup_{s \in S} \partial_s M$.

(ii) $\partial_s M$ and $\partial_t M$ are disjoint unless $s \leqq t$ or $t \leqq s$.

(iii) If $s < t$, then $\partial_s M \cap \partial_t M$ is a fact of $\partial_s M$ and of $\partial_t M$.

Similarly, one has the notion of *G-manifold with S-faces*.

Next we define a function $d: I(G) \to \boldsymbol{Z}_+$. If α is a normal G-orbit type, then $d(a)$, *the length of α*, is the maximum length of any chain beginning at α, i.e., $d(a) = \max\{n \mid \alpha = \alpha_1 < \alpha_2 < \cdots < \alpha_n\}$. Suppose that α and β are represented by (H^α, V^α) and (H^β, V^β), respectively. If $\beta > \alpha$, then $\dim V^\beta < \dim V^\alpha$. Consequently, $d(\alpha) < \dim V^\alpha + 1$.

Suppose that M is a smooth G-manifold. Let J be the set of normal orbit types of G on M, i.e., let $J = \{\alpha \in I(G) \mid M_\alpha \neq \phi\}$. By the remark in the above paragraph, every element of J has length bounded by $\dim M + 1$; hence, the integer $m = \max\{d(\alpha) \mid \alpha \in J\}$ is defined. Set

$$J(i) = \{\alpha \in J \mid d(\alpha) > i\}$$

and

$$J_\beta = \{\alpha \in J \mid \alpha < \beta\}\ .$$

Obviously, $J_\beta \subset J(i)$ if $d(\beta) = i$.

For each $i \leqq m$, we shall now define a G-manifold $M(i)$ with $J(i)$-faces. First, set $M(m) = M$. If $d(\beta) = m$, then M_β is a minimal stratum of M, and hence, a closed subset (by I.1.6). So, we can apply the construction of the previous section to define

$$M(m-1) = M \odot \bigcup M_\beta ,$$

where the union is taken over all β with $d(\beta) = m$. $M(m-1)$ is a G-manifold with boundary. Let $\partial_\beta M(m-1)$ denote those boundary components determined by M_β, that is, let $\partial_\beta M(m-1) = \Sigma\nu_\beta(M)$. This gives $M(m-1)$ the structure of a manifold with $J(m-1)$-faces. The general definition is made by induction (downwards). Suppose that we have defined a G-manifold $M(i+1)$ with $J(i+1)$-faces. Further suppose, by induction, that

(a) $M(i+1)_\gamma \neq \phi$ if and only if $d(\gamma) \leqq i+1$, and that

(b) if $d(\beta) = i+1$, then $(\partial_\alpha M(i+1))_\beta \neq \phi$ if and only if $\alpha < \beta$. Then for all β with $d(\beta) = i+1$, $M(i+1)_\beta$ is a minimal stratum of $M(i+1)$. For each such β, we can choose an equivariant tubular map $\nu_\beta(M(i+1)) \to M(i+1)$, which is compatible with the manifold with corner structure on $M(i+1)$ (see [7]). Therefore, we can define

$$M(i) = M(i+1) \odot \bigcup M(i+1)_\beta ,$$

where, as before, the union is taken over all β with $d(\beta) = i+1$. One can easily check that $M(i)$ is again a manifold with faces. If $d(\alpha) > i+1$, then set $\partial_\alpha M(i) = \partial_\alpha M(i+1) \odot \bigcup (\partial_\alpha M(i+1))_\beta$. This is clearly a face of $M(i)$. If $d(\alpha) = i+1$, then set $\partial_\alpha M(i) = \Sigma\nu_\alpha(M(i+1))$, which is clearly also a face of $M(i)$. With these definitions, it follows from the inductive hypothesis (b), that $M(i)$ is a manifold with $J(i)$-faces. It is also easy to check that the inductive hypothesis (a) and (b) hold for $M(i)$.

If B is a local G-orbit space with strata indexed by J, then, in a similar fashion, we can define $B(i)$, a "local G-orbit space with $J(i)$-faces." In particular, suppose that M is a smooth G-manifold over B with projection map $p: M \to B$. Then the interior of $M(i)$ can be identified with the complement in M of the strata of M length greater than i, and $p|_{\mathrm{int}\, M(i)}$ extends to a map $p(i): M(i) \to B(i)$ in an obvious fashion. For simplicity, we shall use the notation $M(\alpha) = M(i)$, $B(\alpha) = B(i)$ and $p(\alpha) = p(i)$, where $i = d(\alpha)$.

DEFINITION 2.2. Suppose that M is a smooth G-manifold over B. Let

$\bar{M}_\alpha = M(\alpha)_\alpha$.

$\bar{B}_\alpha = B(\alpha)_\alpha$.

$\bar{\nu}_\alpha = \bar{\nu}_\alpha(M)$, the normal bundle of $\bar{M}_\alpha$ in $M(\alpha)$.

$\bar{P}_\alpha = \bar{P}_\alpha(M)$, the principal S_α-bundle over $\bar{B}_\alpha$ associated to $\bar{\nu}_\alpha$.

$\bar{C}_\alpha = \bar{C}_\alpha(B)$, the budle over $\bar{B}_\alpha$ with fiber $X^\alpha/G(\bar{C}_\alpha \cong \bar{\nu}_\alpha/G)$.

$\bar{Q}_\alpha = \bar{Q}_\alpha(M)$, the principal T_α-bundle over $\bar{B}_\alpha$ associated to $\bar{C}_\alpha$ ($\bar{Q}_\alpha \cong \bar{P}_\alpha/Z_\alpha$, where Z_α is the kernel of the natural map $S_\alpha \to T_\alpha$).

$\bar{M}_\alpha$ is called the *closed α-stratum of M*, $\bar{B}_\alpha$ is the *closed α-stratum of B* and $\bar{P}_\alpha$ is the *closed α-normal orbit bundle*.

Next, consider $\partial_\alpha M(\beta) \cap \bar{M}_\beta$, where $d(\alpha) > d(\beta)$. This intersection is empty unless $\alpha < \beta$, in which case we define

$$\partial_\alpha \bar{M}_\beta = \partial_\alpha M(\beta) \cap \bar{M}_\beta \ .$$

This gives $\bar{M}_\beta$ the structure of a manifold with J_β-faces (recall that $J_\beta = \{\alpha \in J \mid \alpha < \beta\}$). Similarly, for each $\alpha < \beta$, we can define $\partial_\alpha \bar{B}_\beta = \partial_\alpha B(\beta) \cap \bar{B}_\beta$, $\partial_\alpha \bar{\nu}_\beta = \bar{\nu}_\beta|_{\partial_\alpha \bar{M}_\beta}$, $\partial_\alpha \bar{P}_\beta = \bar{P}_\beta|_{\partial_\alpha \bar{B}_\beta}$, and $\partial_\alpha \bar{Q}_\beta = \bar{Q}_\beta|_{\partial_\alpha \bar{B}_\beta}$. So, $\bar{B}_\beta$, $\bar{\nu}_\beta$, $\bar{P}_\beta$ and $\bar{Q}_\beta$ are also manifolds with J_β-faces.

Let $F: M \to M'$ be a stratified map of G-manifolds and let $f = \pi(F): B \to B'$. It follows from the discussion in the previous section that for each α, $F|_{\operatorname{int} M(\alpha)}$ extends to a stratified map $F(\alpha)$: $M(\alpha) \to M'(\alpha)$ covering $f(\alpha): B(\alpha) \to B'(\alpha)$. The differential of $F(\alpha)$ induces an equivariant linear bundle map $F(\alpha)_*: \bar{\nu}_\alpha(M) \to \bar{\nu}_\alpha(M')$ and therefore, a map of the associated principal bundles $F_\alpha: \bar{P}_\alpha(M) \to \bar{P}_\alpha(M')$. Similarly, $f(\alpha)_*: \bar{C}_\alpha(B) \to \bar{C}_\alpha(B')$ induces $f_\alpha: \bar{Q}_\alpha(B) \to \bar{Q}_\alpha(B')$.

3. **The linear data.** The purpose of this section is to set up some notation. First of all, for the remainder of this chapter, we shall only be interested in the closed strata. For simplicity we shall change our notation as follows.

NOTATION 3.1. From now on, M_α (respectively, B_α) will denote the *closed* α-stratum of M (respectively B). Similarly, ν_α, P_α and Q_α will denote the appropriate bundles over the *closed* α-stratum.

As before, for each normal orbit type $\alpha \in I(G)$, choose a representative (H^α, V^α) and let $X^\alpha = G \times_{H^\alpha} V^\alpha$. We regard X^α as a left G-space and as a right S_α-space, where S_α is defined as in § I.2. Let J^α be the set of normal orbit types of G on X^α, i.e., let $J^\alpha = \{\beta \in I(G) \mid \beta \geqq \alpha\}$. For each β with $\beta \geqq \alpha$, we shall denote by X^α_β the closed β-stratum of X^α. Similarly, we shall denote the closed β-stratum of X^α/G by B^α_β. Also, we can define the bundles ν^α_β, P^α_β and Q^α_β as in the previous section. . All of these spaces are manifolds with J^α_β-faces, where $J^\alpha_\beta = \{\gamma \in I(G) \mid \beta > \gamma \geqq \alpha\}$.

The action of S_α on X^α preserves the stratification by normal G-orbit types. Hence, the normal bundle of each stratum is a $(G \times S_\alpha)$-vector bundle, and consequently, S_α also acts on the normal sphere bundle of each stratum. It follows by the naturality of the construction, that for each $\beta > \alpha$, $X^\alpha(\alpha)$, X^α_β and ν^α_β are all S_α-spaces. Since S_α acts on ν^α_β through bundle maps, it also acts on P^α_β, the total space of the associated principal bundle. In a similar fashion, $B^\alpha(\beta)$, B^α_β and Q^α_β become right T^α-spaces.

For any smooth G-manifold M, we have (by definition) that $\partial_\alpha M_\beta = \Sigma(\nu_\alpha)_\beta$, where $\nu_\alpha = X^\alpha \times_{S_\alpha} P_\alpha$. If ΣX^α denotes the sphere bundle of $X^\alpha \to G/H^\alpha$, then

$$\begin{aligned}(\Sigma\nu_\alpha)_\beta &= (\Sigma X^\alpha \times_{S_\alpha} P_\alpha)_\beta \\ &= (\Sigma X^\alpha)_\beta \times_{S_\alpha} P_\alpha \\ &= \partial_\alpha X^\alpha_\beta \times_{S_\alpha} P_\beta \ .\end{aligned}$$

Hence, for each $\alpha < \beta$,

$$\partial_\alpha M_\beta = \partial_\alpha X^\alpha_\beta \times_{S_\alpha} P_\alpha \ . \tag{3.1}$$

Similarly,

$$\partial_\alpha P_\beta = \partial_\alpha P^\alpha_\beta \times_{S_\alpha} P_\alpha \tag{3.2}$$

$$\partial_\alpha B_\beta = \partial_\alpha B^\alpha_\beta \times_{T_\alpha} Q_\alpha \tag{3.3}$$

$$\partial_\alpha Q_\beta = \partial_\alpha Q^\alpha_\beta \times_{T_\alpha} Q_\alpha \ . \tag{3.4}$$

4. Normal systems. A subset $J \subset I(G)$ is *closed* if for every $\alpha \in J$ and every β with $\beta > \alpha$, we have that $\beta \in J$. If J is any such subset and if $\alpha \in J$, then let $J_\alpha = \{\gamma \in J \mid \gamma < \alpha\}$.

DEFINITION 4.1. An *n-dimensional $\mathscr{G}$-normal system* is the following data:

(i) A closed set J of normal G-orbit types.

(ii) For each $\alpha \in J$, a principal S_α-bundle P_α over a manifold B_α with J_α-faces (where $\dim B_\alpha + \dim X^\alpha = n$).

(iii) For each pair $(\alpha, \beta) \in J \times J$ with $\beta > \alpha$, an isomorphism of S_β-bundles

$$\theta_{\alpha,\beta} \colon \partial_\alpha P^\alpha_\beta \times_{S_\alpha} P_\alpha \longrightarrow \partial_\alpha P_\beta \ ,$$

where by definition $\partial_\alpha P_\beta = P_\beta|_{\partial_\alpha B_\beta}$. Moreover, there is the following compatibility condition.

(iv) For each triple (α, β, γ) with $\gamma > \beta > \alpha$, the following diagram of isomorphisms commutes

$$\begin{tikzcd}[column sep=tiny] & \partial_\beta P^\beta_\gamma \times_{S_\beta} \partial_\alpha P^\alpha_\beta \times_{S_\alpha} P_\alpha \arrow[dl, "\times\theta_{\alpha,\beta}"'] \arrow[dr, "\partial_\beta\theta_{\alpha,\gamma}"] & \\ \partial_\beta P^\beta_\gamma \times_{S_\beta} \partial_\alpha P_\beta \arrow[rr, "\partial_\alpha\theta_{\beta,\gamma}"'] & & \partial_\alpha P_\alpha \cap \partial_\beta P_\gamma \ . \end{tikzcd}$$

Here $\partial_\beta\theta_{\alpha,\gamma}$ denotes the restriction of $\theta_{\alpha,\gamma}$ to $(\partial_\beta P_\alpha \cap \partial_\alpha P^\alpha_\gamma) \times_{S_\alpha} P_\alpha$ and $\partial_\alpha\theta_{\beta,\gamma}$ denotes the restriction of $\theta_{\beta,\gamma}$ to $\partial_\beta P^\beta_\gamma \times_{S_\beta} \partial_\alpha P_\beta$. If ξ is a $\mathscr{G}$-normal system, then we shall sometimes use the notation $J(\xi)$, $B_\alpha(\xi)$, $P_\alpha(\xi)$ and $\theta_{\alpha,\beta}(\xi)$ for the above data.

If ξ and ξ' are $\mathscr{G}$-normal systems with $J(\xi) \subset J(\xi')$, then by a *morphism* $\phi: \xi \to \xi'$ we mean a collection of smooth bundle maps $\phi_\alpha: P_\alpha(\xi) \to P_\alpha(\xi')$, $\alpha \in J(\xi)$, such that the following obvious diagram commutes

$$\begin{tikzcd}
\partial_\alpha P^\alpha_\beta \times_{S_\alpha} P_\alpha(\xi) \arrow[r, "\times \phi_\alpha"] \arrow[d, "\theta_{\alpha,\beta}(\xi)"] & \partial_\alpha P^\alpha_\beta \times_{S_\alpha} P_\alpha(\xi') \arrow[d, "\theta_{\alpha,\beta}(\xi')"] \\
\partial_\alpha P_\beta(\xi) \arrow[r, "\phi_\beta"'] & \partial_\alpha P_\beta(\xi)
\end{tikzcd}$$

for each α, $B \in J$ with $\beta > \alpha$. For technical reasons, we shall also require that the differential of ϕ_β maps the normal bundle of $\partial_\alpha P_\beta(\xi)$ in $P_\beta(\xi)$ transversely to the normal bundle of $\partial_\alpha P_\beta(\xi')$.

The definition of *$\mathscr{B}$-normal systems* and their morphisms is completely similar—in condition (ii) of 4.1, we merely replace the phrase "a principal S_α-bundle P_α" by "a principal T_α-bundle Q_α."

Let $\mathscr{N}$ be the category of $\mathscr{G}$-normal systems and let $\mathscr{N}'$ be the category of $\mathscr{B}$-normal systems. As we indicated in Sections 1 and 2, there is a faithful functor $D: \mathscr{G} \to \mathscr{N}$ which associates to a G-manifold M the normal system with bundles $\{P_\alpha(M)\}$, the set of closed normal orbit bundles. The maps $\theta_{\alpha,\beta}$ are the identity maps, as indicated in (3.2). Condition (iv) of the definition, which asserts that the identifications agree on the intersection of two faces, is clearly satisfied. In a similar fashion, there is a faithful functor $D': \mathscr{B} \to \mathscr{N}'$ which associates to each local G-orbit space B the $\mathscr{B}$-normal system with bundles $\{Q_\alpha(B)\}$.

Also, there is a functor $\tilde{\pi}: \mathscr{N} \to \mathscr{N}'$ which associates to each $\mathscr{G}$-normal system a $\mathscr{B}$-normal system. To be specific, if $\xi = \{J, B_\alpha, P_\alpha, \theta_{\alpha,\beta}\}$ is a $\mathscr{G}$-normal system, then set $\tilde{\pi}(\xi) = \{J, B_\alpha, Q_\alpha, \lambda_{\alpha,\beta}\}$, where $Q_\alpha = P_\alpha/Z_\alpha$ and where $\lambda_{\alpha,\beta}$ is the bundle map covered by $\theta_{\alpha,\beta}$. We summarize the above remarks in the following proposition.

PROPOSITION 4.2. *There is a commutative diagram of functors*

$$\begin{tikzcd}
\mathscr{G} \arrow[r, "D"] \arrow[d, "\pi"'] & \mathscr{N} \arrow[d, "\tilde{\pi}"] \\
\mathscr{B} \arrow[r, "D'"] & \mathscr{N}' .
\end{tikzcd}$$

In the next theorem we assert that the normal system is a complete invariant of a smooth G-manifold.

THEOREM 4.3. *The functor $D: \mathscr{G} \to \mathscr{N}$ defines a bijection between equivariant diffeomorphism classes of smooth G-manifolds* *and isomorphism classes of $\mathscr{G}$-normal systems.*

For the special case of compact regular $O(n)$-manifolds, this theorem is the main result of Jänich's paper [7]. We shall also prove the following two results. Theorem 4.5 is perhaps the most interesting of the three.

THEOREM 4.4. *The functor $D'\colon \mathscr{B} \to \mathscr{N}'$ defines a bijection between isomorphism classes of local G-orbit spaces and isomorphism classes of $\mathscr{B}$-normal system.*

THEOREM 4.5. *Let B be a local G-orbit space. Then the functor D defines a bijection between isomorphism classes in $\pi^{-1}(B)$ and isomorphism classes in $\tilde{\pi}^{-1}(D'(B))$, (where $\pi^{-1}(B)$ is the category of G-manifolds over B defined in* (III.1.5)).

Before proving these three theorems, we will introduce the notion of an "augmented normal system," which takes into account the equivalence class of the appropriate collared neighborhoods. Recall that in order to define the normal system of M, we had to choose equivariant tubular maps $T_\alpha\colon \nu_\alpha(M) \to M(\alpha)$. Let us consider what information such a tubular map gives us in terms of normal systems, that is, let us apply the functor D to T_α. The closed β-normal orbit bundle of ν_α is $P^\alpha_\beta \times_{S_\alpha} P_\alpha$. So, $D(T_\alpha)$ is essentially a collection of bundle maps $D(T_\alpha)_\beta = \tau_{\alpha,\beta}\colon P^\alpha_\beta \times_{S_\alpha} P_\alpha \to P_\beta$, indexed by J^α. Also, consider the fact that T_α and T_β differ by an equivariant diffeomorphism where they overlap. Another way to express this fact is that for each pair (α, β) with $\beta > \alpha$, T_α and T_β define the germ of an equivariant diffeomorphism $\mu_{\alpha,\beta}\colon X^\beta \times_{S_\beta} (P^\alpha_\beta \times_{S_\alpha} P_\alpha) \to \nu_\alpha(\beta)$, which should be thought of as a tubular map from the closed β-stratum of ν_α into $\nu_\alpha(\beta)$. The map $\mu_{\alpha,\beta}$ is defined by the following diagram of (germs of) equivariant embeddings.

$$\begin{tikzcd} X^\beta \times_{S_\beta} (P^\alpha_\beta \times_{S_\alpha} P_\alpha) \arrow[r, "\times\tau_{\alpha,\beta}"] \arrow[d, dashed, "\mu_{\alpha,\beta}"] & \nu_\beta \arrow[d, "T_\beta"] \\ \nu_\alpha(\beta) \arrow[r, "T_\alpha(\beta)"] & M(\beta) . \end{tikzcd}$$

We should also point out that $\mu_{\alpha,\beta}$ is completely determined by $\{\tau_{\alpha,\gamma}, \tau_{\beta,\gamma} | \gamma \geqq \beta\}$. For if $\gamma \geqq \beta$, then the restriction of $\mu_{\alpha,\beta}$ to the closed γ-stratum is determined by the diagram

$$(4.6) \qquad \begin{tikzcd} X^\beta_\gamma \times_{S_\beta} (P^\alpha_\beta \times_{S_\alpha} P_\alpha) \arrow[r, "\times\tau_{\alpha,\beta}"] \arrow[d, "(\mu_{\alpha,\beta})_\gamma"] & X^\beta_\gamma \times_{S_\beta} P_\beta \arrow[d, "\hat{\tau}_{\beta,\gamma}"] \\ X^\alpha_\gamma \times_{S_\alpha} P_\alpha \arrow[r, "\hat{\tau}_{\alpha,\gamma}"'] & M_\gamma , \end{tikzcd}$$

where $\hat{\tau}_{\alpha,\gamma}$ and $\hat{\tau}_{\beta,\gamma}$ are the maps covered by $\tau_{\alpha,\gamma}$ and $\tau_{\beta,\gamma}$, respectively. These considerations motivate the following definition.

DEFINITION 4.7. A *collaring* τ for a $\mathscr{G}$-normal system is a collection of smooth bundle maps $\tau_{\alpha,\beta}\colon P^{\alpha}_{\beta} \times_{S_\alpha} P_\alpha \to P_\beta$ indexed by $\{(\alpha, \beta) \in J \times J \mid \beta > \alpha\}$ and satisfying the following two conditions:

(a) The restriction of $\tau_{\alpha,\beta}$ to $\partial_\alpha P^{\alpha}_{\beta} \times_{S_\alpha} P_\alpha$ is $\theta_{\alpha,\beta}$.

(b) If $\mu_{\alpha,\beta}\colon X^{\beta} \times_{S_\beta} (P^{\alpha}_{\beta} \times_{S_\alpha} P_\alpha) \to X^{\alpha}(\beta) \times_{S_\alpha} P_\alpha$ is the map defined (on each stratum) by diagram (4.6), then $\mu_{\alpha,\beta}$ is the germ of a G-equivariant diffeomorphism on some neighborhood of $\partial_\alpha X^{\alpha}_{\beta} \times_{S_\alpha} P_\alpha$.

If τ' is another collaring for the same normal system, then τ' is *equivalent* to τ if for each $\alpha \in J$, there is an equivariant diffeomorphism $\psi_\alpha\colon X^\alpha \times_{S_\alpha} P_\alpha \to X^\alpha \times_{S_\alpha} P_\alpha$ (defined on some neighborhood of the zero-section) with $(\psi_\alpha)_* = \mathrm{id}$ and such that the following diagram commutes for each $\beta \geqq \alpha$.

$$\begin{tikzcd}
P^{\alpha}_{\beta} \times_{S_\alpha} P_\alpha \arrow[d, "D(\psi_\alpha)_\beta"'] \arrow[dr, "\tau_{\alpha,\beta}"] & \\
P^{\alpha}_{\beta} \times_{S_\alpha} P_\alpha \arrow[r, "\tau'_{\alpha,\beta}"'] & P_\beta
\end{tikzcd}\ .$$

DEFINITION 4.8. An *augmented $\mathscr{G}$-normal system* is a $\mathscr{G}$-normal system together with an equivalence class of collarings.

Suppose that ξ and ξ' are normal systems with collarings τ and τ'. A morphism $\phi\colon \xi \to \xi'$ is said to be a *morphism of augmented normal systems* if for each $\alpha \in J(\xi)$, there is a stratified map $G_\alpha\colon X^\alpha \times_{S_\alpha} P_\alpha(\xi) \to X^\alpha \times_{S_\alpha} P_\alpha(\xi')$ so that the following diagram commutes.

$$\begin{tikzcd}
P^{\alpha}_{\beta} \times_{S_\alpha} P_\alpha(\xi) \arrow[r, "\tau_{\alpha,\beta}"] \arrow[d, "D(G_\alpha)_\beta"] & P_\beta(\xi) \arrow[d, "\phi_\beta"] \\
P^{\alpha}_{\beta} \times_{S_\alpha} P_\alpha(\xi') \arrow[r, "\tau'_{\alpha,\beta}"'] & P_\beta(\xi')
\end{tikzcd}\ .$$

As usual, the map G_α is only required to be defined on some neighborhood of the zero-section. The notion of a morphism of augmented $\mathscr{G}$-normal systems clearly depends only on the equivalence class of the collarings. For, if we alter τ or τ' by an equivalence, the effect is to alter G_α by composition with an equivariant diffeomorphism.

Augmented $\mathscr{B}$-normal systems can be defined in a similar fashion.

Let $\mathscr{N}_a$ and $\mathscr{N}'_a$ be the categories of augmented $\mathscr{G}$-normal systems and of augmented $\mathscr{B}$-normal systems, respectively. As we indicated in the remarks following (4.5), choosing tubular neighborhoods for the strata of a G-manifold amounts to choosing a collaring for the associated normal system. The definition of equivalence of collarings mirrors the fact that any two tubular maps differ by an equivariant diffeomorphism. Hence, we can associate an augmented $\mathscr{G}$-normal system to each smooth G-manifold and this correspondence is independent of the choices of tubular maps. In other words, we have a functor $D_a: \mathscr{G} \to \mathscr{N}_a$. Similarly, there is a functor $D'_a: \mathscr{B} \to \mathscr{N}'_a$. Also, there is a canonical functor $\pi_a: \mathscr{N}_n \to \mathscr{N}'_a$ such that the following diagram commutes

$$\begin{array}{ccc} \mathscr{G} & \xrightarrow{D_\alpha} & \mathscr{N}_a \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi_a} \\ \mathscr{B} & \xrightarrow{D'_\alpha} & \mathscr{N}'_a \end{array} .$$

THEOREM 4.9. *The functors* $D_a: \mathscr{G} \to \mathscr{N}_a$ *and* $D'_a: \mathscr{B} \to \mathscr{N}'_a$ *are equivalences of categories.*

The theorem asserts that there are functors $A: \mathscr{N}_a \to \mathscr{B}$ and $A': \mathscr{N}'_a \to \mathscr{G}$ (called *assembling functors*) and natural isomorphisms $A \circ D_a \cong 1_{\mathscr{G}}$, $D_a \circ A \cong 1_{\mathscr{N}_a}$, $A' \circ D'_a \cong 1_{\mathscr{B}}$ and $D'_a \circ A' \cong 1_{\mathscr{N}'_a}$. The functors A and A' are defined by reversing the process by which we defined D_a and D'_a. In the case of two strata, this process of building a G-manifold was described in § 1. The details in the general case are sketched below.

Proof of 4.9. Suppose that $\xi = \{J, B_\alpha, P_\alpha, \theta_{\alpha,\beta}, [\tau]\}$ is an augmented $\mathscr{G}$-normal system. We wish to reconstruct a smooth G-manifold $M = A(\xi)$. Let $\nu_\alpha = X^\alpha \times_{S_\alpha} P_\alpha$ and let $M_\alpha = G/H^\alpha \times_{S_\alpha} P_\alpha$. Define

$$M(1) = \bigcup_{d(\alpha)=1} M_\alpha .$$

Suppose, by induction, that we have defined $M(i)$. For each $\alpha \in J$ with $d(\alpha) = i + 1$, we wish to define an equivariant collared neighborhood $t_\alpha: C_+\nu_\alpha \to M(i)$. The point is that we can define a map by defining its restriction to each stratum. If τ is a collaring in the given equivalence class, then $\tau_{\alpha,\beta}$ is a bundle map from the β-normal orbit bundle of $C_+\nu_\alpha$ to the β-normal orbit bundle of $M(i)$. Thus, $\tau_{\alpha,\beta}$ induces a map $(t_\alpha)_\beta: (C_+\nu_\alpha)_\beta \to M(i)_\beta$. We define t_α by requiring that its restriction to the β-stratum be $(t_\alpha)_\beta$. It follows easily from our definition of a collaring that t_α is a diffeomorphism. As a set,

$M(i+1)$ will now be defined to be the disjoint union

$$M(i+1) = (M(i) - \partial M(i)) \cup \bigcup_{d(\alpha)=i+1} M_\alpha .$$

Define $T_\alpha\colon \nu_\alpha \to M(i+1)$ by

$$T_\alpha(x) = \begin{cases} t_\alpha([x, 1]); & x \in \nu_\alpha - M_\alpha \\ x \quad ; & x \in M_\alpha . \end{cases}$$

We give $M(i+1)$ the structure of a smooth manifold with $J(i+1)$-faces by requiring that each T_α be an equivariant tubular map. This completes the inductive step. Thus, on the objects of $\mathscr{N}_a$, A is given by

$$A(\xi) = M(m) = M$$

where m is the maximum length of any chain in J. If $\phi\colon \xi \to \xi'$ is any morphism, then we must also show how to define a stratified map $A(\phi)\colon A(\xi) \to A(\xi')$. Essentially, this is done by the same procedure we used to construct t_α from $\{\tau_{\alpha,\beta}\}$, that is, the restriction of $A(\phi)$ to the α-stratum is defined to be the equivariant map

$$\times\, \phi_\alpha\colon G/H^\alpha \times_{S_\alpha} P_\alpha(\xi) \longrightarrow G/H^\alpha \times_{S_\alpha} P_\alpha(\xi') .$$

This defines an equivariant function $A(\phi)\colon A(\xi) \to A(\xi')$, which we leave to the reader to check is a stratified map of G-manifolds. The functor A' is defined similarly.

To finish the proof, we must exhibit natural isomorphisms of functors $A \circ D_a \cong 1_{\mathscr{G}}$, etc. Let M be a smooth G-manifold and let $\xi = D_a(M)$. Then $P_\alpha(\xi)$ is the closed α-normal orbit bundle of M. Therefore, there is a canonical isomorphism $\Phi_\alpha\colon X^\alpha \times_{S_\alpha} P_\alpha \xrightarrow{\cong} \nu_\alpha(M)$ (and this is the main point). The restriction of Φ_α to the zero-section yields a canonical isomorphism $\psi_\alpha\colon A(\xi)_\alpha \to M_\alpha$. Hence, we have defined a function $\psi\colon A(\xi) \to M$, which, again, is easily checked to be an equivariant diffeomorphism. It follows that the natural transformation which assigns to each $M \in ob\mathscr{G}$ the morphism ψ: $A(D_a(M)) \to M$ is a natural isomorphism $A \circ D_a \cong 1_{\mathscr{G}}$. The descriptions of the other natural isomorphisms are similar.

In order to prove Theorems (4.3) and (4.4), we need two lemmas (see (4.10) and (4.12), below). These lemmas essentially are the existence and uniqueness parts of the Equivariant Collared Neighborhood Theorem in somewhat disguised form. We shall also need more delicate versions of these lemmas (see (4.10)′ and (4.12)′, below) in order to prove Theorem 4.5.

Let $f\colon \mathscr{N}_a \to \mathscr{N}$ and $f'\colon \mathscr{N}'_a \to \mathscr{N}'$ be the functors which associate to an augmented normal system the underlying normal system.

LEMMA 4.10 (*Existence of Collared Neighborhoods*). *Let Λ be a $\mathscr{G}$-normal system. Then there is an augmented $\mathscr{G}$-normal system ξ, with $f(\xi) = \Lambda$. The analogous statement is true for $\mathscr{B}$-normal systems.*

Proof. The lemma asserts that we can choose a collaring τ for Λ. As before, let $M_\alpha = G/H^\alpha \times_{S_\alpha} P_\alpha(\Lambda)$ and let $M(1)$ be the union of those M_α with $d(\alpha) = 1$. By the Equivariant Collared Neighborhood Theorem, for each α with $d(\alpha) = 2$, we can choose a collared neighborhood $t_\alpha: C_+\nu_\alpha \to M(1)$ such that $t_\alpha|_{\Sigma\nu_\alpha}$ is the natural identification. Applying D to t_α, we get a bundle map $t_{\alpha,\beta}: P^\alpha_\beta \times_{S_\alpha} P^\alpha \to P_\beta$ where $d(\beta) = 1$ (as in the remarks following 4.5). We can also use the t_α's to construct $M(2)$ (as in the proof of (4.9)). One continues in this fashion, by inductively constructing $M(i)$ and then choosing collared neighborhoods $t_\alpha: C_+\nu_\alpha \to M(i)$ with $d(\alpha) = i + 1$, in order to define $\{\tau_{\alpha,\beta}\}$ and to construct $M(i + 1)$. The maps $\mu_{\alpha,\beta}$ defined by diagram (4.6) are clearly equivariant diffeomorphisms. Hence, $\{\tau_{\alpha,\beta}\}$ defines a collaring. The proof for $\mathscr{B}$-normal systems is similar.

LEMMA 4.10′. *Let Λ be a $\mathscr{G}$-normal systems and let η be an augmented $\mathscr{B}$-normal system with $f'(\eta) = \tilde{\pi}(\Lambda)$. Then there is an augmented $\mathscr{G}$-normal system ξ with $f(\xi) = \Lambda$ and with $\pi_a(\xi) = \eta$.*

Proof. Let $\bar{\tau}$ be a collaring for η in the given equivalence class. The lemma asserts that we can find a collaring τ for λ which "covers" $\bar{\tau}$. The collaring $\bar{\tau}$ defines collared neighborhoods of the form $\bar{t}_\alpha: c_+C_\alpha \to B(i)$ with $\bar{t}_\alpha|_{\sigma C_\alpha}$ being the canonical identification. (Here $C_\alpha = X^\alpha/G \times_{T_\alpha} Q_\alpha(\eta)$ and c_+C_α and σC_α are as defined in 1.3.) The lemma will follow, if we can modify the proof of Lemma 4.10 by choosing collared neighborhoods $t_\alpha: C_+\nu_\alpha \to M(i)$ with $\pi(t_\alpha) = \bar{t}_\alpha$. We know that $C_+\nu_\alpha \cong \Sigma\nu_\alpha \times [0, \infty)$ and that $c_+C_\alpha \cong \sigma C_\alpha \times [0, \infty)$. Also, the restriction of t_α to $\Sigma\nu_\alpha$ must be the canonical identification; hence, it must cover $\bar{t}_\alpha|_{\sigma C_\alpha}$. Thus, it follows from the Covering Homotopy Theorem (III.2.2) that there is a stratified map t_α covering $\bar{t}_\alpha$.

$$\begin{tikzcd}
\Sigma\nu_\alpha \times [0, \infty) \arrow[r, dashed, "t_\alpha"] \arrow[d] & M(i) \arrow[d] \\
\sigma C_\alpha \times [0, \infty) \arrow[r, "\bar{t}_\alpha"] & B(i)
\end{tikzcd}$$

Since $\bar{t}_\alpha$ maps c_+C_α isomorphically, it follows that t_α maps $C_+\nu_\alpha$ isomorphically; hence $t_\alpha: C_+\nu_\alpha \to M(i)$ is a collared neighborhood. The augmented $\mathscr{G}$-normal system ξ so produced will then lie over η.

DEFINITION 4.11. Let $\Lambda = \{J, B_\alpha, P_\alpha, \theta_{\alpha,\beta}\}$ be a normal system. There is a normal system $\Lambda \times I$ with data $\{J, B_\alpha \times I, P_\alpha \times I, \theta_{\alpha,\beta} \times \mathrm{id}\}$. If $\phi: \Lambda \to \Lambda'$ is an isomorphism then as *isotopy of* ϕ is a morphism $\Phi: \Lambda \times I \to \Lambda'$ which is an isomorphism on each level and which satisfies $\Phi|_{\Lambda \times \{0\}} = \phi$.

LEMMA 4.12 (*Uniqueness of Collared Neighborhoods*). *Suppose that ξ and ξ' are augmented $\mathscr{G}$-normal systems and that $\phi: f(\xi) \to f(\xi')$ is an isomorphism of the underlying normal systems. Then there is an isomorphism $\theta: \xi \to \xi'$ and an isotopy of ϕ to $f(\theta)$. The analogous result is true for augmented $\mathscr{B}$-normal systems.*

Proof. The procedure will be to modify ϕ by a sequence of isotopies. At the first stage, we shall only change those ϕ_β with $d(\beta) = 1$. Associated to $P_\beta(\xi)$ and $P_\beta(\xi')$ we have the G-manifolds with J_β-faces M_β and M'_β. Let $\hat{\phi}_\beta: M_\beta \to M'_\beta$ be the equivariant diffeomorphism induced by ϕ_β. Let τ and τ' be collarings for ξ and ξ'. Let (α, β) be a pair with $d(\beta) = 1$, $d(\alpha) = 2$ and with $\alpha < \beta$. The maps $\tau_{\alpha,\beta}$ and $\tau'_{\alpha,\beta}$ induce collared neighborhoods $t_\alpha: C_+\nu_\alpha \to M_\beta$ and $t'_\alpha: C_+\nu'_\alpha \to M'_\beta$. There is an equivariant diffeomorphism $F: C_+\nu'_\alpha \to C_+\nu_\alpha$ which is defined on some neighborhood of $\Sigma\nu'_\alpha$ and which makes the following diagram commute

$$\begin{tikzcd} M_\beta \arrow[r, "\hat{\phi}_\beta"] & M'_\beta \\ C_+\nu_\alpha \arrow[u, "t_\alpha"] & C_+\nu'_\alpha \arrow[u, "t'_\alpha"'] \arrow[l, dashed, "F"'] \end{tikzcd}$$

Also, since $C_+\nu_\alpha$ is a bundle associated to $P_\alpha(\xi)$, there is an induced bundle isomorphism $(\times\phi_\alpha): C_+\nu_\alpha \to C_+\nu'_\alpha$. Consider the map $G = F \circ (\times\phi_\alpha): C_+\nu_\alpha \to C_+\nu_\alpha$. It follows from the definition of morphism, that G is the identity on $\Sigma\nu_\alpha$ and on $C_+(\partial_\gamma\nu_\alpha)$, for any $\gamma < \alpha$. We claim that there is an equivariant isotopy $\psi_t: C_+\nu_\alpha \to C_+\nu_\alpha$ such that $\psi_1 = G$, such that ψ_0 is the identity on some small collared neighborhood of$\Sigma\nu_\alpha$ in $C_+\nu_\alpha$, and such that $\psi_t = G$ on the complement of a slightly larger neighborhood. Moreover, ψ_t will be the identity on $C_+(\partial_\gamma\nu_\alpha)$. In order to construct ψ, first let $r: C_+\nu_\alpha \to [0, 1]$ be a smooth invariant function, which is 1 on a neighborhood of $\Sigma\nu_\alpha$ and which vanishes outside a slightly larger neighborhood, and let $q: C_+\nu_\alpha \times I \to I$ be given by $q(x, t) = 1 + r(x)(t - 1)$. Recall that $C_+\nu_\alpha$ is a bundle over $\Sigma\nu_\alpha$ and that $[0, \infty)$ acts on it by fiberwise scalar multiplication. Consider the isotopy $\psi_t(x) = q(x, t)^{-1}G(q(x, t)\cdot x)$. This is well-defined when $r(x) < 1$ or when $t > 0$. For $r(x) = 1$ it converges smoothly to the bundle map $\psi_0(x) = \lim_{t\to 0} t^{-1}G(tx) = G_*(x)$.

But a bundle map differs from the identity by a scalar multiple on each fiber, so by a possible further isotopy we may assume that ψ_0 is the identity near $\Sigma\nu_\alpha$. Clearly $\psi_1 = G$ and $\psi_t|_{C_+(\partial_\gamma\nu_\alpha)} = \text{id}$. Consider the isotopy $H\colon C_+\nu_\alpha \times I \to C_+\nu'_\alpha$ defined by $H(x, t) = \psi_t \circ (\times \phi_\alpha)^{-1}(x)$. Since $H(x, t) = F(x)$ off of a neighborhood of $\Sigma\nu_\alpha$, it extends to an ambient isotopy on M_β. That is to say, there is an equivariant isotopy $\hat{\Phi}_\beta\colon M_\beta \times I \to M'_\beta$ of $\hat{\phi}_\beta$ to $\hat{\theta}_\beta$ so that the following diagram commutes,

$$\begin{array}{ccc} M_\beta \times I & \xrightarrow{\hat{\Phi}_\beta} & M'_\beta \\ \uparrow{\scriptstyle t_\alpha \times 1} & & \uparrow{\scriptstyle t'_\alpha} \\ C_+\nu_\alpha \times I & \xrightarrow{H} & C_+\nu'_\alpha\,. \end{array}$$

We see that $\hat{\theta}_\beta = \hat{\Phi}_\beta|_{M_\beta \times \{1\}}$ is a bundle map on a small collared neighborhood of $\Sigma\nu_\alpha$. Moreover, $\hat{\Phi}_\beta(x, t) = \hat{\phi}_\beta(x)$ for $x \in \partial M_\beta$ and for $x \in M_\beta - t_\alpha(C_+\nu_\alpha)$. The equivariant map $\hat{\Phi}_\beta$ induces an isotopy of the associated normal orbit bundles, and we denote this map by Φ_β: $P_\beta(\xi) \times I \to P_\beta(\xi')$. Since Φ_β is constant on $\partial P_\beta(\xi)$ we can find an isotopy $\Phi\colon f(\xi) \times I \to f(\xi')$ which is equal to Φ_β on the β-stratum and which is $\phi \times \text{id}_I$ on every other stratum. This completes the description of the first in our sequence of isotopies. It should be clear that we can continue this process inductively. At the next stage, for example, we consider collared neighborhoods of the form $t_\gamma\colon C_+\nu_\alpha \to M(2)$, where $d(\gamma) = 3$, and we alter $\phi(2)\colon M(2) \to M'(2)$ by an isotopy so that it will be a bundle map on $C_+\nu_\alpha$. This will lead to an isotopy of ϕ which will only change those ϕ_α with $d(\alpha) \leqq 2$. This completes the proof. The argument for $\mathscr{B}$-normal systems is essentially identical.

LEMMA 4.12′. *Suppose that ξ and ξ' are augmented $\mathscr{G}$-normal systems lying over the same augmented $\mathscr{B}$-normal system η* (i.e., $\pi_a(\xi) = \pi_a(\xi') = \eta$). *Suppose further that $\phi\colon f(\xi) \to f(\xi')$ is an isomorphism lying over the identity* (i.e., $\tilde{\pi}(\phi) = \text{id}_{f'(\eta)}$). *Then the isotopy Φ constructed in the proof of* (4.12) *also lies over the identity* (i.e., $\tilde{\pi}(\Phi_t) = \text{id}$).

Proof. Let us reconsider the proof of (4.12). There we constructed an equivariant diffeomorphism $G\colon C_+\nu_\alpha \to C_+\nu_\alpha$. With the hypotheses of the above lemma, it will follow that $\pi(G) = \text{id}\colon c_+C_\alpha \to c_+C_\alpha$. We assert that this implies that the isotopy $\psi_t(x) = q(x, t)^{-1} \cdot G(q(x, t) \cdot x)$ also satisfies $\pi(\psi_t) = \text{id}$. For indeed, $\pi(\psi_t)(z) = \bar{q}(z, t)^{-1} \circ \pi(G)(\bar{q}(z, t) \circ z) = \bar{q}(z, t)^{-1} \circ \bar{q}(z, t) \circ z = z$, where $\bar{q}\colon c_+C_\alpha \times I \to I$ is the map induced by q. (Also notice that no further correction of ψ_t is necessary; for, it follows from the fact that $\pi(G_*) = \mathrm{id}$ that the bundle map G_* is automatically the identity.) The above observations imply that the sequence of isotopies constructed in (4.12) will all cover the identity on η.

Proof of 4.3 (*and of* 4.4). The proofs of (4.3) and (4.4) are essentially identical. The functor $D\colon \mathscr{G} \to \mathscr{N}$ clearly induces a function from the set of equivariant diffeomorphism classes of smooth G-manifolds to the set of isomorphism classes of $\mathscr{G}$-normal systems. It follows from Lemma 4.10 that this function is a surjection. For, if $\Lambda \in ob\mathscr{N}$, then, by (4.10), we can find an augmented normal system ξ lying over it. Then $A(\xi)$ is a G-manifold and by (4.9), $D_a(A(\xi))$ is isomorphic to ξ. Hence, $D(A(\xi))$ is isomorphic to $f(\xi) = \Lambda$. Similarly, it follows from Lemma 4.12 that this function is an injection. For, if $D(M) \cong D(M')$, then, by (4.12), $D_a(M) \cong D_a(M')$. But then (4.9) implies that $M \cong M'$. This completes the proof.

Proof of 4.5. Again, D clearly defines a function from isomorphism classes in $\pi^{-1}(B)$ to isomorphism classes in $\tilde{\pi}^{-1}(D'(B))$. First we show that this function is surjective. Suppose that Λ is a $\mathscr{G}$-normal system with $\tilde{\pi}(\Lambda) = D'(B)$. By (4.10)$'$, there is an augmented $\mathscr{G}$-normal system ξ with $f(\xi) = \Lambda$ and with $\pi_a(\xi) = D'_a(B)$. Then $A(\xi) \in ob\pi^{-1}(B)$ and $D(A(\xi)) \in ob\tilde{\pi}^{-1}(B)$; so the function is surjective. Similarly, it follows from (4.12)$'$ that the function is injective.

REMARK 4.13. Since $\mathrm{Hom}_{\mathscr{G}}(M, M')$ is a subset of the space of all C^∞ maps from M to M' (with the coarse C^∞-topology), it has an induced topology, also called the *coarse C^∞-topology*. Similarly it makes sense to speak of the "coarse C^∞-topology" on $\mathrm{Hom}_{\mathscr{B}}(B, B')$ and on $\mathrm{Diff}_1^G(M)$ $(=\mathrm{Hom}_{\pi^{-1}(B)}(M, M))$. The theory of normal systems can be used to gain insight into the homotopy theory of these three spaces of stratified maps. We sketch a few ideas below.

Let ξ and ξ' be $\mathscr{G}$-normal systems with bundles indexed by $J = J(\xi) = J(\xi')$. Let $P_\alpha = P_\alpha(\xi)$ and $P'_\alpha = P_\alpha(\xi')$. Recall that a morphism $\phi\colon \xi \to \xi'$ is a J-tuple of bundle maps $\phi_\alpha\colon P_\alpha \to P'_\alpha$ satisfying the compatibility condition expressed by the following commutative diagram.

$$\begin{array}{ccc} \partial_\alpha P^\alpha_\beta \times_{S_\alpha} P_\alpha & \xrightarrow{\times\phi_\alpha} & \partial_\alpha P^\alpha_\beta \times_{S_\alpha} P'_\alpha \\ \downarrow & & \downarrow \\ \partial_\alpha P_\beta & \xrightarrow[\phi_\beta]{} & \partial_\alpha P'_\beta \end{array}$$

In other words, $\mathrm{Hom}_{\mathscr{N}}(\xi, \xi')$ is a certain subset of $\prod_{\alpha\in J} \mathrm{Hom}(P_\alpha, P'_\alpha)$, where $\operatorname{Hom}(P_\alpha, P'_\alpha)$ denotes the space of smooth S_α-bundle maps from P_α to P'_α with coarse C^∞-topology. The induced topology on $\operatorname{Hom}_{\mathscr{N}}(\xi, \xi')$ is again called the *coarse C^∞-topology*. In a similar fashion, we can define the "coarse C^∞-topology" on $\operatorname{Hom}_{\mathscr{N}'}(\eta, \eta')$ and on $\operatorname{Hom}_{\tilde{\pi}^{-1}(\eta)}(\xi, \xi)$. We assert that with these topologies, the maps

$$D\colon \operatorname{Hom}_{\mathscr{G}}(M, M') \longrightarrow \operatorname{Hom}_{\mathscr{N}}(D(M), D(M'))$$
$$D'\colon \operatorname{Hom}_{\mathscr{B}}(B, B') \longrightarrow \operatorname{Hom}_{\mathscr{N}'}(D'(B), D'(B'))$$
$$D\colon \operatorname{Diff}_1^G(M) \longrightarrow \operatorname{Hom}_{\tilde{\pi}^{-1}(D'(B))}(D(M), D(M))$$

are homotopy equivalences. (This assertion is a generalization of Lemmas 4.12 and 4.12′.) The map $D\colon \operatorname{Hom}_{\mathscr{G}}(M, M') \to \operatorname{Hom}_{\mathscr{N}}(D(M), D(M'))$ is an embedding and may be regarded as the inclusion. One proves that D is a homotopy equivalence by showing that both spaces are homotopy equivalent to a common subspace; namely, the subspace consisting of those maps which are linear bundle maps on some prescribed tubular neighborhoods of the strata. The argument, which is a fairly standard application of some ideas of Cerf, is omitted. Of course, the proof that the other two maps are also homotopy equivalences is entirely analogous.

The homotopy of such space of morphisms of normal systems, e.g., of $\operatorname{Hom}_{\mathscr{N}}(\xi, \xi')$, can be analyzed via a "stratum by stratum" approach. Set $J^i = \{\alpha \in J \mid d(\alpha) \leqq m - i\}$ where m is the maximum length of any chain of normal orbit types in J. Let $h^i(\xi, \xi')$ be the space of J^i-tuples of bundle maps $(\phi_\alpha)_{\alpha \in J^i}$, which satisfy the above compatibility condition. It is easy to see that there is a sequence of fibrations $p_i\colon h^{i+1}(\xi, \xi') \to h^i(\xi, \xi')$ converging to $h^m(\xi, \xi') = \operatorname{Hom}_{\mathscr{N}}(\xi, \xi')$ (p_i is the obvious map). The point is that the first space in this sequence and each of the fibers are fairly well-understood spaces. To be precise, $h^0(\xi, \xi') = \prod_{\alpha \in J^m} \operatorname{Hom}(P_\alpha, P'_\alpha)$ and the fiber of p_i over a given path component is $\prod \operatorname{Hom}(P_\alpha, P'_\alpha)_\partial$, where the product is over all $\alpha \in J^i - J^{i-1}$ and where $\operatorname{Hom}(P_\alpha, P'_\alpha)_\partial$ denotes the subspace of $\operatorname{Hom}(P_\alpha, P'_\alpha)$ consisting of those maps which carry ∂P_α to $\partial P'_\alpha$ and which are equal to some fixed map on ∂P_α.

This approach works best in the category $\tilde{\pi}^{-1}(\eta)$. If P is a principal S-bundle, then it is well-known (and easy to see) that the space of bundle automorphisms of P can be identified with the space of sections of an associated bundle $P^* = S \times_S P$, where the action of S on itself is via conjugation. If ξ is a $\mathscr{G}$-normal system lying over η, then we can form the associated bundles P^*_α. Let Z_α denote the kernel of the natural projection $S_\alpha \to T_\alpha$. Define a subbundle R_α of P^*_α, by $R_\alpha = Z_\alpha \times_{S_\alpha} P_\alpha$. Then the sections of R_α correspond to those automorphisms of P_α which cover the identity on Q_α ($= Q_\alpha(\eta)$). If $\beta < \alpha$, then a section of R_β induces a section of $\partial_\beta R_\alpha$.

Thus, an automorphism $\phi \in \mathrm{Hom}_{\tilde{\pi}^{-1}(\eta)}(\xi, \xi)$ can be identified with a J-tuple of sections satisfying the compatibility condition that the section of $\partial_\beta R_\alpha$ is induced by a section of R_β. We can let Γ^i be the space of J^i-tuples of sections satisfying this same compatibility condition. Then, as in the above paragraph, we get a sequence of fibrations $q_i\colon \Gamma^{i+1} \to \Gamma^i$ (this sequence of fibrations was suggested by Bredon in [3]). The fiber of q_i is $\prod \Gamma(R_\alpha)$, where $\alpha \in J^i - J^{i-1}$ and where $\Gamma(R_\alpha)$ denotes the space of sections of R_α which are the "identity section" on ∂R_α. Thus, both Γ^0 and the fiber of q_i are spaces with homotopy groups which can be computed by standard methods. In this way, the homotopy of $\mathrm{Diff}_1^G(M)$ can be studied by methods which are completely bundle theoretic. When there are relatively few strata and when B is a simple enough space, this procedure can actually be to compute $\pi_0 \mathrm{Diff}_1^G(M)$, e.g., see [3].

REMARK 4.14. The theory of normal systems clarifies the problem of constructing classifying spaces for smooth G-manifolds. Theorem 4.3 shows that we must actually construct a classifying space for $\mathscr{G}$-normal systems. But it is fairly clear how to do this. We hope to give the details in a later paper. In [4], the theory of normal systems is used to construct classifying spaces for the special case of regular $O(n)$, $U(n)$ and $Sp(n)$-manifolds.

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