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COMPLEX BASES OF CERTAIN SEMI-PROPER HOLOMORPHIC MAPS

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The existence theorem of complex bases of quasi-proper holomorphic maps was studied by N. Kuhlmann. In this paper the existence of the complex bases in a more general case will be shown.

0. Introduction. In the function theory of several complex variables, the complex bases of holomorphic maps of analytic spaces have been introduced as a generalized concept of Riemann surfaces defined by inverse functions of given holomorphic functions of one complex variable.

Let $f: X \rightarrow Y$ be a holomorphic map of analytic spaces. How does f have a complex base? Authors have discussed the sufficient conditions which allow for the existence of a complex base of f (cf. for example, [3], [5], [6], [7]). If f is proper, then f has a complex base ([7]). N. Kuhlmann [3] showed existence theorems in the case of quasi-proper (N-quasi-proper). f is called *quasi-proper* (resp. *N-quasi-proper*) if, for every compact subset K of Y, there exists a compact subset $\tilde{K}$ of X such that each of the irreducible branches (resp. each of the connected components) of fibres on K intersects $\tilde{K}$.

On this subject, an attempt will be made to abate the condition, so that each of the given unions of connected components of fibres intersects $\tilde{K}$. For such holomorphic maps, we shall have an existence theorem of complex bases (of type of N. Kuhlmann's).

THEOREM. *Let X be an irreducible normal analytic space, f: $X \rightarrow Y$ be a holomorphic map of X into an analytic space Y and E_f be the set of degeneracy of f. Suppose that f satisfies (C) and that $f(E_f)$ is analytic in Y. Then f has a complex base $(\tilde{Z}, \tilde{\varphi})$ and $\tilde{Z}$ is also normal. Moreover, the natural holomorphic map $\tilde{\psi}$ with $f = \tilde{\psi} \circ \tilde{\varphi}$ is proper and light, and $\tilde{\varphi}$ satisfies (C_1).*

1. Preliminaries. We assume in this paper that all analytic spaces are reduced and have countable bases of open sets.

Let $f: X \rightarrow Y$ and $f_1: X \rightarrow Y_1$ be holomorphic maps of analytic spaces. f_1 is said to *strictly depend* on f, if f_1 is constant on each connected component of fibres of f. f_1 is said to be *analytically related* to f, if f and f_1 strictly depend on each other. A pair (Z, φ) is called a *complex base* of f, if Z is an analytic space, and if $\varphi: X \to Z$ is a surjective holomorphic map which is analytically related to f, and if, for each holomorphic map $h: X \to T$ which strictly depends on f, there exists a unique holomorphic map $\psi: Z \to T$ with $h = \psi \circ \varphi$.

A holomorphic map $f: X \to Y$ is said to be *semi-proper*, if, for each compact subset K of Y, there exists a compact subset $\tilde{K}$ of X such that $f^{-1}(y) \cap \tilde{K} \neq \emptyset$, for $y \in K \cap f(X)$; f is said to be *quasi-proper* if $B \cap \tilde{K} \neq \emptyset$, for each irreducible branch B of $f^{-1}(y)$. N. Kuhlmann modified this definition ([3]); f is said to be *N-quasi-proper*, if $N \cap \tilde{K} \neq \emptyset$, for each connected component N of $f^{-1}(y)$. He showed the existence of complex bases of N-quasi-proper holomorphic maps in [3].

Now, we consider a more general case in which each of the given unions of connected components of $f^{-1}(y)$ $(y \in K \cap f(X))$ intersects $\tilde{K}$.

DEFINITION. A holomorphic map $f: X \to Y$ is said to *satisfy (C)* if f has the following property;

(C) *Given an analytic set A in Y and a commutative diagram of holomorphic maps*

$$\begin{tikzcd} X - f^{-1}(A) \arrow[r, "\varphi"] \arrow[dr, "(f,h)"'] & Z \arrow[d, "\psi"] \\ & (Y - A) \times T \end{tikzcd}$$

where ψ is light (that is, each fibre is discrete) and $h: X \to T$ strictly depends on f and (f, h) is a holomorphic map given by $x \mapsto (f(x), h(x))$, and if K is a compact subset of $(Y - A) \times T$, then there exists a compact subset $\tilde{K}$ of $X - f^{-1}(A)$ such that $\varphi^{-1}(p) \cap \tilde{K} \neq \emptyset$, for $p \in \psi^{-1}(K) \cap \varphi(X - f^{-1}(A))$.

If f satisfies (C), then f satisfies the following (C_1) (we take $A = \emptyset$ and $h = f$);

(C_1) *Given a compact subset K of Y and a commutative diagram of holomorphic maps*

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "f"'] & Z \arrow[d, "\psi"] \\ & Y \end{tikzcd}$$

where ψ is light, then there exists a compact subset $\tilde{K}$ of X such that $\varphi^{-1}(p) \cap \tilde{K} \neq \emptyset$ for $p \in \psi^{-1}(K) \cap \varphi(X)$.

Note that in such cases, f, φ, and (f, h) (in the two diagrams above) are naturally semi-proper, and that if φ is surjective, ψ is proper; in this case ψ is finite! Every N-quasi-proper holomorphic map satisfies (C); for, if $f: X \to Y$ is N-quasi-proper, so is (f, h). Thus we have the following inclusion:

$$\begin{aligned} \text{proper} \Longrightarrow \text{quasi-proper} \Longrightarrow N\text{-quasi-proper} \\ \Longrightarrow (C) \Longrightarrow (C_1) \Longrightarrow \text{semi-proper}\ . \end{aligned}$$

LEMMA 1.1. ([4]) *Let Y be a normal analytic space and $f: X \to Y$ be a proper modification map. If f is nowhere degenerate, then f is a biholomorphic map.*

LEMMA 1.2. *Let X be an irreducible normal analytic space and $f: X \to Y$ be a nowhere degenerate holomorphic map. If f satisfies (C_1), then f has a complex base (Z, φ), and the natural holomorphic map $\psi: Z \to Y$ with $f = \psi \circ \varphi$ is proper and light.*

Proof. f is nowhere degenerate, and so f has a complex base (Z, φ) (cf. [6]). f is semi-proper, so $f(X)$ is analytic in Y (cf. [1]). Thus we may assume that f is surjective. Since f and φ are analytically related and X has a countable basis, $\psi^{-1}(y) = \varphi(f^{-1}(y))$ is discrete for $y \in Y$. Thus ψ is light and therefore, is proper, as desired.

2. Proof of theorem. We shall prove our theorem by introducing modification of the proof of the theorem of N. Kuhlmann.

We may assume that f is surjective as in Lemma 1.2, and moreover, that Y is a connected complex manifold since the set of singular points of Y is a thin analytic set in Y. By [1], Proposition 1.24, $f(E_f)$ is thin of dimension $\leqq \dim Y - 2$.

Let $Y' = Y - f(E_f)$, $X' = X - f^{-1}(f(E_f))$ and $f' = f \mid X' \to Y'$. Since f' satisfies (C_1) and is nowhere degenerate, f' has a complex base (Z', φ') and the natural holomorphic map ψ' with $f' = \psi' \circ \varphi'$ is proper and light by Lemma 1.2. Z' is a normal analytic space. By [5], Satz 1 (or [2], Satz A), we have a (unique up to biholomorphic equivalence) normal analytic space $\tilde{Z}$ with a holomorphic map $\tilde{\psi}: \tilde{Z} \to Y$ which is proper, light and surjective tuch that $Z' = \tilde{\psi}^{-1}(Y')$ and Z' is dense in $\tilde{Z}$ and $\psi' = \tilde{\psi} \mid Z'$.

We have to show that there exists a surjective holomorphic map $\tilde{\varphi}: X \to Z$ such that $(\tilde{Z}, \tilde{\varphi})$ is a complex base of f.

(a) A holomorphic map $\varphi': X' \to Z'$ can uniquely be extended to the surjective holomorphic map $\tilde{\varphi}: X \to Z$ such that $\tilde{\varphi}$ is analytically related to f and $f = \tilde{\psi} \circ \tilde{\varphi}$: Let $G \subset X \times Y$ be a graph of $f: X \to Y$ and $G' \subset X' \times Z'$ be a graph of $\varphi': X' \to Z'$. Let $\iota \times \tilde{\psi}$: $X \times \tilde{Z} \to X \times Y$ be a holomorphic map given by $(x, p) \mapsto (x, \tilde{\psi}(p))$ and

$G_1: =(\iota \times \tilde{\psi})^{-1}(G)$. There exists an irreducible branch $\tilde{G}$ of G_1 with $\tilde{G} \cap (X' \times Z') = G'$.

The projection $\pi_1: \tilde{G} \to X$ onto 1st component X is a proper light modification map and therefore, π_1 is biholomorphic by Lemma 1.1. Let $\pi_2: \tilde{G} \to \tilde{Z}$ be the projection onto 2nd component $\tilde{Z}$ and $\tilde{\varphi}: = \pi_2 \circ \pi_1^{-1}: X \to \tilde{Z}$. Then $f = \tilde{\psi} \circ \tilde{\varphi}$. And since $\tilde{\psi}$ is light, $\tilde{\varphi}$ is analytically related to f. Uniqueness of $\tilde{\varphi}$ is obvious.

(b) Let $h: X \to T$ be a holomorphic map strictly depending on f. Then there exists a (unique) holomorphic map $\psi: \tilde{Z} \to T$ such that $h = \psi \circ \tilde{\varphi}$: Since f satisfies (C), $(\tilde{\varphi}, h): X \to \tilde{Z} \times T$ is semi-proper and therefore, $G_0: = (\tilde{\varphi}, h)(X)$ is analytic in $\tilde{Z} \times T$. Let $\tilde{\pi}_1$ and $\tilde{\pi}_2$ be projections of G_0 onto 1st and 2nd components, respectively. $\pi_1 | G_0 \cap (Z' \times T) \to Z'$ is a biholomorphic map. In fact, $h': = h | X'$ strictly depends on f' and (Z', φ') is a complex base of f', so there exists a holomorphic map $\psi'': Z' \to T$ with $h' = \psi'' \circ \varphi'$. And then, $G_0 \cap (Z' \times T)$ is a graph of ψ''. h strictly depends on $\tilde{\varphi}$, so $\tilde{\pi}_1$ is light. f satisfies (C_1) and $\tilde{\varphi} = \tilde{\pi}_1 \circ (\tilde{\varphi}, h)$, so $\tilde{\pi}_1$ is proper. Thus $\tilde{\pi}_1: G_0 \to \tilde{Z}$ is a proper light modification map and therefore, it is biholomorphic. Let $\psi: = \tilde{\pi}_2 \circ \tilde{\pi}_1^{-1}: \tilde{Z} \to T$, then $h = \psi \circ \tilde{\varphi}$.

With (a) and (b) we conclude the proof.

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