PACIFIC JOURNAL OF MATHEMATICS
Vol. 80, No. 1, 1979

SPECTRAL SYNTHESIS IN SEGAL ALGEBRAS ON HYPERGROUPS

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Warner (1966), Hewitt and Ross (1970), Yap (1970), and Yap (1971) extended the so-called Ditkin's condition for the group algebra $L^1(G)$ of a locally compact abelian group G to the algebras $L^1(G) \cap L^2(G)$, dense subalgebras of $L^1(G)$ which are essential Banach $L^1(G)$-modules, $L^1(G) \cap L^p(G)(1 \leqq p < \infty)$ and Segal algebras respectively. Chilana and Ross (1978) proved that the algebra $L^1(K)$ satisfies a stronger form of Ditkin's condition at points of the center $Z(\hat{K})$ of $\hat{K}$, where K is a commutative locally compact hypergroup such that its dual $\hat{K}$ is also a hypergroup under pointwise operations. Topological hypergroups have been defined and studied by Dunkl (1973), Spector (1973), and Jewett (1975) to begin with. In this paper we define Segal algebras on K and prove that they satisfy a stronger form of Ditkin's condition at the points of $Z(\hat{K})$. Examples include the analogues of some Segal algebras on groups and their intersections.

1. Introduction. In this paper we define and study Segal algebras on hypergroups with emphasis on spectral synthesis. A good deal of Harmonic Analysis has recently been developed on locally compact hypergroups by Dunkl [5], Spector [21], Jewett [9], and Ross ([17], [18]). Our basic reference will be Jewett [9]. Throughout this paper K will denote a commutative locally compact hypergroup ('Convos' in [9]) such that its dual $\hat{K}$ is a hypergroup under pointwise operations and notation and terminology for Harmonic Analysis on K will be as in [4]. As proved in ([23], Appendix) K is first countable if and only if it is metrizable. Being commutative, K admits a Haar measure m, as shown by Spector [22]. The convolution algebra $L^1(m) = L^1(K)$ can be identified with the pointwise algebra $A(\hat{K})$ of Fourier transforms on $\hat{K}$. Chilana and Ross [4] proved that $A(\hat{K})$ is a regular algebra of functions on $\hat{K}$ with a bounded approximate unit and it satisfies a stronger form of Ditkin's condition at points in the center $Z(\hat{K})$ of $\hat{K}$. They also gave examples to show that not all points in $\hat{K}$ need be spectral sets. This is partially in contrast with the situation in locally compact abelian groups where Ditkin's condition is satisfied for $L^1(G)$ at each point of $\hat{G}$. Warner [24] proved it for the algebra $L^1(G) \cap L^2(G)$, Hewitt and Ross ([7], 39.32) showed that it is true for dense Banach modules $\mathscr{U}$ in $L^1(G)$ such that $\mathscr{U} * L^1(G)$ is dense in $\mathscr{U}$. Also Yap [26] proved the same for the algebra $L^1(G) \cap L^p(G)(1 \leqq p < \infty)$ and then extended it to Segal

algebras in $L^1(G)$ ([27], [28]), which turn out to be Banach $L^1(G)$-modules of ([7], 39.32). In §2 we define a Segal algebra $S(K)$ on K and prove that the algebra $AS(\hat{K})$ of Fourier transforms of functions in $S(K)$ is a regular algebra of functions on $\hat{K}$ with an approximate unit which is bounded in $A(\hat{K})$. We then show that a Banach algebra $(\mathscr{U}, \|\cdot\|_{\mathscr{U}})$ satisfying

$$\{f \in L^1(K),\ \hat{f} \in C_{00}(\hat{K})\} \subset \mathscr{U} \subset L^1(K)$$

is a Segal algebra if and only if it is a Banach $L^1(K)$-module with $L^1(K)*\mathscr{U}$ dense in $(\mathscr{U}, \|\cdot\|_{\mathscr{U}})$ and most of the results in ([7], 39.32) have their analogues for K. We then define locally convex Segal algebras on K and extend the above results to them.

Various stronger forms of Ditkin's condition have been given by Wik [25], Rosenthal [16] and Saeki [19] for the algebra $L^1(G)$ of a locally compact abelian group G and they all coincide when G is σ-compact and metrizable. In §3 we give analogues of their definitions for $S(K)$ which coincide when K is σ-compact and first countable and $S(K)$ is a Banach algebra. We prove that the analogous conditions for $S(K)$ on special hypergroups K are satisfied at the points of the center $Z(\hat{K})$. We note that this is new even in the case of locally compact abelian groups (compare ([7], 39.32) and [28]). We further apply our results to study spectral synthesis in $S(K)$. In the end we indicate that some of the results can be proved for abstract Segal algebras (also compare Burnham [2], [3]).

In §4 we give examples of Segal algebras on hypergroups; they include analogues of some Segal algebras on groups such as $B^p(G)$, $A^p(G)$, and $A_{(p,q)}(G)$ given by Yap [27], Larsen, Liu and Wang [11], (see also Larsen [10]) and Yap [29] respectively.

2. Segal algebras on hypergroups. In this section we will introduce the concept of Segal algebras on K. As stated in §1 we assume throughout that $\hat{K}$ is a commutative hypergroup under pointwise operations. The Plancherel measure on $\hat{K}$ will be denoted by π and the Haar measure on K by m.

DEFINITION 2.1. Let $S(K)$ be a subalgebra of $L^1(K)$ which is a Banach algebra under a norm $\|\cdot\|_S$ such that

S(i) $f \in L^1(K)$ and $\hat{f} \in C_{00}(\hat{K})$ imply that $f \in S(K)$,

S(ii) $S(K)$ is translation invariant and for some $\eta > 0 \|f_x\|_S \leqq \eta \|f\|_S$ for each $f \in S(K)$ and $x \in K$, and

S(iii) for each $f \in S(K)$, the mapping $x \to f_x$ of K into $S(K)$ is continuous.

Then $S(K)$ will be called a *Segal algebra*.

REMARKS 2.2. (i) The Fourier transforms of the functions in $S(K)$ form a subalgebra $AS(\hat{K})$ of $A(\hat{K})$ with the norm carried over from $S(K)$.

Because of ([4], 2.6) S(i) gives that $S(K)$ is $\|\cdot\|_1$-dense in $L^1(K)$.

For the group case this is the condition that is imposed on $S(K)$ rather than our S(i) and then S(i) is proved to be true (for instance, cf. [14] VI, 2.2(iii)) (see Remark 3.1 also).

(ii) In view of ([4], 2.5) for each compact set E of $\hat{K}$ and symmetric set V with compact closure such that $\pi(V) > 0$ there is a function φ in $A_{00}(\hat{K})$ and thus in $AS(\hat{K})$ such that $0 \leqq \varphi \leqq 1$, $\varphi = 1$ on E and $\varphi = 0$ outside $E * V * V$. In particular, for each compact set E in $\hat{K}$ there exists φ in $AS(\hat{K})$ such that $\varphi = 1$ on E.

(iii) In view of (ii) above $AS(\hat{K})$ satisfies ([14], II, 1.1 (iii)) i.e., for any $\gamma \in \hat{K}$ and any neighborhood U of γ there is a function τ_γ in $A_{00}(\hat{K})$ and thus in $AS(\hat{K})$ such that τ_γ is 1 in a neighborhood of γ and zero outside U. So by ([14], II, 1.3) localization lemma is true for $AS(\hat{K})$. This is the property which is used in proving some results on closed ideals in $S(K)$ which we shall discuss later.

The proofs of other results viz. (i), (ii), and (iv) in VI, 2.2 and VI, 2.3 in [14] can be modified to give:

(iv) there exists a constant C such that

$$\|f\|_1 \leqq C\|f\|_S \quad \text{for each} \quad f \in S(K) ;$$

(v) $S(K)$ is an ideal in $L^1(K)$ and $\|h*f\|_S \leqq \eta\|h\|_1\|f\|_S$ for each $h \in L^1(K)$ and $f \in S(K)$;

(vi) for any compact subset F of $\hat{K}$ there is a constant C_F such that for each $f \in S(K)$ with $\hat{f}$ vanishing outside F we have

$$\|f\|_S \leqq C_F\|f\|_1 ;$$

(vii) given $\varepsilon > 0$, $f \in S(K)$ there exists a neighborhood U of the identity e in K such that $\|f*u - f\|_S < \varepsilon$ for $u \in L^1(K)$ with $\operatorname{supp} u \subset U$, $u \geqq 0$ and

$$\int_K u(\check{x})dm(x) = 1$$

(where '$\check{\ }$' signifies involution in K);

(viii) it follows from (ii) and (v) that

$$AS(\hat{K}) \supset AS(\hat{K})\cdot A(\hat{K}) \supset A_{00}(\hat{K})\cdot A(\hat{K}) \supset A_{00}(\hat{K}) .$$

THEOREM 2.3. *$AS(\hat{K})$ is a regular, semi-simple Banach algebra in $C_0(\hat{K})$ which has an approximate unit $\{\varphi_\alpha\}_{\alpha\in D}$ such that φ_α belongs* *to* $C_{00}(\hat{K})$ *and* $\|\varphi_\alpha\|_A = 1$ *for all* $\alpha \in D$. *If* K *is first countable then* $\{\varphi_\alpha\}$ *can be chosen as a sequence.*

Proof. Regularity follows from Remark 2.2 (iii) and we shall give the proof for existence of approximate unit for a more general class of algebras in Theorem 2.8.

REMARKS 2.4. (i) It follows from the above theorem and Remark 2.2(ii), (v), and (viii) that $S(K)$ is a dense subalgebra of $L^1(K)$ such that it is a Banach $L^1(K)$-module and $L^1(K)*S(K)$ is dense in $(S(K), \|\cdot\|_S)$. As proved in ([7], 39.32) such modules are Segal algebras when K is a group. Proofs can be modified to give that a subalgebra $\mathscr{U}$ of $L^1(K)$ that is a Banach algebra with respect to its own norm $\|\cdot\|_{\mathscr{U}}$ such that $\{f \in L^1(K), \hat{f} \in C_{00}(\hat{K})\} \subset \mathscr{U}$ is a Banach $L^1(K)$-module satisfying: $L^1(K)*\mathscr{U}$ is dense in $(\mathscr{U}, \|\cdot\|_{\mathscr{U}})$ if and only if it is a Segal algebra.

(ii) The structure space of $S(K)$ can be identified with $\mathscr{X}_b^c(K)$. The proof follows on the lines of ([7], 39.32) or alternatively of [28].

Burnham [3] has defined locally convex Segal algebras; we impose somewhat different conditions in order to have some interesting results which are satisfied by (Banach) Segal algebras defined above.

REMARK 2.5. We first note a result; let B be a subalgebra of $L^1(K)$ which is a normed algebra under a norm $\|\cdot\|$. The completion A of B lies in $L^1(K)$ if and only if there exists a constant C such that $\|f\|_1 \leqq C\|f\|$ for all f in B and in that case $\|f\|_1 \leqq C\|f\|$ for all f in A. The proof is standard and for example can be obtained by using ([14], II, 3.6).

DEFINITION 2.6. Let $\{(S_\sigma(K), \|\cdot\|_\sigma); \sigma \in \Sigma\}$ be a collection of Banach algebras with $S_\sigma(K) \subset L^1(K)$ for each σ. Let $S_0(K) = \bigcap \{S_\sigma(K): \sigma \in \Sigma\}$ and $S(K)$ be a subalgebra of $S_0(K)$ equipped with the topology given by norms $\{\|\cdot\|_\sigma: \sigma \in \Sigma\}$ restricted to $S(K)$ which satisfies:

L(i) $f \in L^1(K), \hat{f} \in C_{00}(\hat{K})$ imply that $f \in S(K)$,

L(ii) $S(K)$ is a translation invariant ideal in $L^1(K)$ and for each $\sigma \in \Sigma$ there exists an $\eta_\sigma > 0$ such that $\|f_x\|_\sigma \leqq \eta_\sigma \|f\|_\sigma$ for $f \in S(K)$ and $x \in K$, and

L(iii) for each $f \in S(K)$ the mapping $x \to f_x$ of K into $S(K)$ is continuous.

Then $S(K)$ will be called a *locally convex Segal algebra.*

REMARK 2.7. (i) The set $AS(\hat{K})$ of Fourier transforms of functions in $S(K)$ is a subalgebra of $A(\hat{K})$ with topology carried over from that of $S(K)$.

(ii) Because of Remark 2.5, for every $\sigma \in \Sigma$ there exists C_σ such that $\|f\|_1 \leqq C_\sigma \|f\|_\sigma$ for all f in $S_\sigma(K)$.

(iii) For $f \in S(K)$, $h \in L^1(K)$, $\sigma \in \Sigma$, the Bochner integral $\|\cdot\|_\sigma\text{-}\int_K h(y)f_y m(dy)$ exists in $S_\sigma(K)$. As in ([14], VI, 2.2(ii)) $\|\cdot\|_1\text{-}\int_K h(y)f_y m(dy)$ also exists in $L^1(K)$ and it can be proved in a similar manner that the integral is equal to h^*f.

Because of (ii) the two integrals are equal and thus

$$\|\cdot\|_\sigma\text{-}\int_K h(y)f_y m(dy) = h*f\ .$$

So

$$\begin{aligned}\|h*f\|_\sigma &\leqq \int_K \|h(y)f_y\|_\sigma m(dy)\\ &\leqq \int_K |h(y)|\, \|f_y\|_\sigma m(dy)\\ &\leqq \eta_\sigma \int_K |h(y)|\, \|f\|_\sigma m(dy) = \eta_\sigma \|h\|_1 \|f\|_\sigma\ .\end{aligned}$$

Since $S(K)$ is an ideal $h*f \in S(K)$.

(iv) To any compact subset F of $\hat{K}$ and $\sigma \in \Sigma$, there exists a constant $C_{F,\sigma}$ such that for each $f \in S(K)$ with $\hat{f}$ vanishing outside F we have $\|f\|_\sigma \leqq C_{F,\sigma}\|f\|_1$. In fact, $C_{F,\sigma}$ can be chosen to be $\|\tau\|_\sigma$ where τ is a function as in Remark 2.2 (ii) which equals 1 on F.

(v) For a finite subset Σ' of Σ the set $\cap\{S_\sigma(K): \sigma \in \Sigma'\}$ with $\|\cdot\|_{\Sigma'} = \max\{\|\cdot\|_\sigma, \sigma \in \Sigma'\}$ is a Banch algebra and $\|f_x\|_{\Sigma'} \leqq \eta_{\Sigma'}\|f\|_{\Sigma'}$ for $f \in S(K)$ where $\eta_{\Sigma'} = \max\{\eta_\sigma: \sigma \in \Sigma'\}$. So we can assume that Σ is saturated with respect to suprema of finite subsets of Σ.

(vi) Given $\varepsilon > 0$, $f \in S(K)$, $\sigma \in \Sigma$ there exists a neighborhood U of the identity e in K such that $\|f*u - f\|_\sigma < \varepsilon/2$ for $u \in L^1(K)$ with $\operatorname{supp} u \subset U$, $u \geqq 0$ and

$$\int_K u(\check{x})dm(x) = 1\ .$$

THEOREM 2.8. *$AS(\hat{K})$ is a regular, semi-simple locally multiplicatively convex algebra in $C_0(\hat{K})$ which has an approximate unit $\{\varphi_\alpha: \alpha \in D\}$ such that φ_α belongs to $C_{00}(\hat{K})$ and $\|\varphi_\alpha\|_1 = 1$ for all α. If K is first countable then $\{\varphi_\sigma\}$ can be chosen as a sequence.*

Proof. Regularity follows from Remark 2.2 (iii). Let $\mathscr{U}$ be a basis of compact symmetric neighborhoods of identity e in K. We direct the net by

$$D = \{(U, n): U \in \mathscr{U}, n \in N\}$$

where

$$\alpha' = (U', n') \geqq (U, n) = \alpha$$

if and only if $U' \subset U$, $n' \geqq n$.

For $U \in \mathscr{U}$ let $f_U = (m(U))^{-1}\xi_U$, for $n \in N$ use ([4], 2.6) to select v_α in $S(K)$ with $\hat{v}_\alpha$ in $C_{00}(\hat{K})$ such that

$$\|f_U - v_\alpha\|_1 < \frac{1}{n} .$$

Define $u_\alpha = \|v_\alpha\|_1^{-1} v_\alpha$ and $\varphi_\alpha = \hat{u}_\alpha$; then routine estimates using Remark 2.7 (iii) give

$$\|u_\alpha * f - f\|_\sigma < \frac{2}{n}\eta_\sigma \|f\|_\sigma + \|f_E * f - f\|_\sigma$$

$$\text{for each} \quad \sigma \in \Sigma,\ f \in S(K) .$$

Let $\varepsilon > 0$, $f \in S(K)$, $\sigma \in \Sigma$ be arbitrary.

Let $n \in N$ be such that $\eta_\sigma \|f\|_\sigma < n\varepsilon/4$ and $U \in \mathscr{U}$. Using Remark 2.7 (vi), we have for $\alpha' \geqq \alpha = (U, n)$,

$$\begin{aligned} \|u_{\alpha'} * f - f\|_\sigma &\leqq \frac{2}{n'}\eta_\sigma \|f\|_\sigma + \|f_{U'} * f - f\|_\sigma \\ &\leqq \frac{2}{n}\eta_\sigma \|f\|_\sigma + \|f_{T'} * f - f\|_\sigma \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon . \end{aligned}$$

Thus $\{u_\alpha : \alpha \in D\}$ is an approximate unit for $S(K)$ so that $\{\varphi_\alpha : \alpha \in D\}$ is an approximate unit for $AS(\hat{K})$.

REMARK 2.9. If $\{S_\sigma(K) : \sigma \in \Sigma\}$ is a collection of Segal algebras then $S_0(K) = \bigcap \{S_\sigma(K) : \sigma \in \Sigma\}$ with the topology given by norms $\{\|\cdot\|_\sigma : \sigma \in \Sigma\}$ is a sequentially complete locally convex Segal algebra. Also if $\{f \in L^1(K), \hat{f} \in C_{00}(\hat{K})\} \subset S(K) \subset S_0(K)$ and $S(K)$ is a translation invariant ideal then $S(K)$ is a Segal algebra. In fact, all locally convex Segal algebras are essentially of this type as we show below.

THEOREM 2.10. *Let $S(K)$ be as in Definition 2.6. Then there exists a collection $\{T_\sigma(K) : \sigma \in \Sigma\}$ of Segal algebras such that $S(K)$ is a dense subset of $T_0(K) = \bigcap \{T_\sigma(K) : \sigma \in \Sigma\}$.*

Proof. Let $T_\sigma(K)$ be the completion of $S(K)$ in $S_\sigma(K)$. Then $T_\sigma(K)$ is a Banach $L^1(K)$-module. Also in view of Theorem 2.3 and Remark 2.2 (ii), (v), and (viii) $T_\sigma(K) * L^1(K)$ is dense in $(T_\sigma(K), \|\cdot\|_\sigma)$. So by Remark 2.4 (i) $T_\sigma(K)$ is a Segal algebra. Since $S(K)$ is dense in each $(T_\sigma(K), \|\cdot\|_\sigma)$, we have that $S(K)$ is dense in $T_0(K)$.

REMARK 2.11. It follows from the above theorem and Remark 2.4 (ii) that the structure space of $S(K)$ is $\mathscr{X}_b(K)$.

The following result shows that the ideal theory of $S(K)$ is same as that of $L^1(K)$.

THEOREM 2.12. *There exists a bijective correspondence between the family of all closed ideals of $S(K)$ and the family of all closed ideals of $L^1(K)$ in the sense that every closed ideal I_S of $S(K)$ is of the form $I \cap S(K)$ where I is a (unique) closed ideal in $L^1(K)$. In fact I is the closure of I_S in $L^1(K)$. In particular I_S and I have the same zero sets.*

The proof for the Banach algebra case follows from Burnham [2] and also from obvious modification of ([7], 39.32 (u)) which can further be adapted to locally convex case in view of Theorem 2.10 above. For the locally convex Fréchet algebra case the result has also been noted in ([3], p. 49).

COROLLARY 2.13. *A subset I of $S(K)$ is a closed ideal if and only if it is a closed translation-invariant subspace.*

Proof. It is enough to show that a closed translation-invariant subspace I of $S(K)$ is an ideal in $S(K)$. Let $f \in I$ and $h \in L^1(K)$. Then as in Remark 2.7 (iii) for each $\sigma \in \Sigma$ $h*f$ is in the closure of I in $S_\sigma(K)$. Also $h*f$ is in $S(K)$. So $h*f$ is in the closure $\bar{I}$ of I in $S(K)$ and hence in I.

3. Spectral synthesis in Segal algebras. We assume in this section that $\hat{K} = \mathscr{X}_b(K)$ so that $\hat{K}$ is the structure space of $S(K)$ where $S(K)$ is as in Definition 2.6. This assumption is not needed in some of the results proved below.[1]

For a closed subset E of $\hat{K}$ let

$$I(E) = \{f \in S(K) : \hat{f} = 0 \quad \text{on} \quad E\} ,$$
$$J_{00}(E) = \{f \in S(K) : \hat{f} \text{ is zero on an open neighborhoood of } E \text{ in } \hat{K} \text{ and has compact support}\} ,$$

and

$$J(E) = \overline{J_{00}(E)} .$$

[1] For instance, the Wiener Tauberian Theorem is true if the extra condition that $S(K)$ is regular on $\mathscr{X}_b(K)$ is imposed. We take this opportunity to note that this condition for $L'(K)$ in ([4], Theorem 2.12) has been omitted by mistake.

E will be called *spectral* for $S(K)$ if $J(E) = I(E)$ and Ditkin or Calderon for $S(K)$ if each f in $I(E)$ is in the closure of $f * J_{00}(E)$ in $S(K)$.

REMARK 3.1. (i) The proof for the group case can be modified to give that if $\varphi \in A(\hat{K})$ and $\varphi(\gamma) \neq 0$ at a point γ of $\hat{K}$ then there is a ψ in $A_{00}(\hat{K})$ such that $\psi(\chi) = 1/\varphi(\chi)$ for each χ in some neighborhood of γ. In view of Theorem 2.8, $AS(\hat{K})$ is a standard function algebra ([14], II, 1.1) and in particular $A(\hat{K})$ is a standard function algebra. Further if $\mathscr{U}$ is a dense ideal in $A(\hat{K})$ then by ([14], II, 1.4 (iii)) $A_{00}(\hat{K}) \subset \mathscr{U}$ and thus $L(i)$ can be replaced by denseness of $S(K)$ in $L^1(K)$ just as in the group case.

(ii) Theorem 2.8 gives that E is Calderon if and only if each f in $I(E)$ is in the closure of $f * J(E)$. It also gives that the empty set is Calderon; this fact is usually expressed by saying that Ditkin's condition is satisfied at ∞.

(iii) In view of (ii) above and Corollary 2.13 we have *Wiener-Tauberian theorem*: If f belongs to $S(K)$ and if $\hat{f}$ vanishes nowhere on $\hat{K}$; then the closed translation invariant subspace of $S(K)$ generated by f is $S(K)$ itself.

(iv) E is spectral for $S(K)$ if and only if it is so for $L^1(K)$.

(v) If E is a Calderon set for $L^1(K)$ then it is so for $S(K)$. In particular, points in $Z(\hat{K})$ are Calderon for $S(K)$ by ([4], 3.9).

REMARK 3.2. Because of Remark 2.2 (ii) a careful reading of (39.24), (39.39), and (39.42) in [7] gives the following results.

(i) Let Δ denote the set of γ in $\hat{K}$ such that $S(K)$ satisfies Ditkin's condition.

(a) If E is a closed subset of $\hat{K}$ such that the boundary $\partial E \subset \Delta$ and E contains no nonvoid perfect sets then E is a Calderon set for $S(K)$.

(b) If E is a closed nonspectral subset of Δ then there exists a continuum of closed ideals in $S(K)$ with zero set E.

(ii) If E is a closed subset of $\hat{K}$ such that $E \subset Z(\hat{K})$ and ∂E contains no nonvoid perfect sets then E is a Calderon set for $AS(\hat{K})$.

(iii) Suppose that $\hat{K}$ is discrete at points of $\hat{K} \backslash Z(\hat{K})$. If E is a closed set in $\hat{K}$ and $E \cap Z(\hat{K})$ contains no nonvoid perfect sets then E is a Calderon set for $AS(\hat{K})$. In particular if $\hat{K} \backslash Z(\hat{K})$ is discrete and $Z(\hat{K})$ is countable then every closed subset of $\hat{K}$ is Calderon for $AS(\hat{K})$.

(iv) If E is a closed nonspectral set in $Z(\hat{K})$ then there exists a continuum of closed ideals in $AS(\hat{K})$ with zero set E.

DEFINITION 3.3. Let E be a closed subset of $\hat{K}$.

(a) E will be called *strong Ditkin* for $S(K)$ if there exists a net $\{f_\alpha: \alpha \in D\}$ in $S(K)$ such that

(i) for each α, $\hat{f}_\alpha = 0$ in a neighborhood of E and has compact support,

(ii) for each $\sigma \in \Sigma$, $\sup\{\|f_\alpha\|_{E,\sigma}: \alpha \in D\} < \infty$ where $\|f_\alpha\|_{E,\sigma} = \sup\{\|f_\alpha * f\|_\sigma: f \in J(E), \|f\|_\sigma \leq 1\}$ and

(iii) for $f \in I(E)$, $f * f_\alpha \to f$ in $S(K)$.

(b) E will be called *ultra-strong Ditkin* for $S(K)$ if it satisfies (i) and (iii) of (a) above and

(ii)′ $\sup\{\|f_\alpha\|_1: \alpha \in D\} < \infty$.

(c) E will be called *sequentially strong Ditkin* for $S(K)$ if there exists a sequence $\{f_n\}$ in $S(K)$ such that

(i) for each n, $\hat{f}_n = 0$ in a neighborhood of E and has compact support and

(ii) for each $f \in I(E)$, $f * f_n \to f$ in $S(K)$.

REMARKS 3.4. (i) If E is strong Ditkin or sequentially strong Ditkin then it is clearly Calderon. Also by Remark 2.7 (iii) E is ultra-strong Ditkin implies that it is strong Ditkin.

(ii) Wik [25] defined a closed subset E of $\hat{G}$ with $G = Z$ to be strong Ditkin for $L^1(G)$ if there exists a sequence $\{v_n\}$ in $I(E)$ such that $v_n * f \to f$ in $L^1(G)$ for each f in $J(E)$.

(iii) ([16], 2.2 (a)) can be rewritten as;

E is strong Ditkin for $L^1(G)$ if and only if it is sequentially strong Ditkin for $L^1(G)$ if G is separable and metrizable. In fact as argued in ([16], Theorem 1.3) we can apply the Banach Steinhaus theorem and obtain that for a Banach Segal algebra $S(K)$, a sequentially strong Ditkin set for $S(K)$ is strong Ditkin for $S(K)$.

(iv) Because of Remark 2.2 (ii) Rosenthal's proof of Theorem 2.4(b) [16] can be modified to give that if E is a closed subset of $\hat{K}$ such that the boundary of E is sequentially strong Ditkin for $S(K)$ then E is sequentially strong Ditkin for $S(K)$.[2]

(v) Theorem 2.8 gives that ϕ is ultra-strong Ditkin and it is sequentially strong Ditkin in case K is first countable.

(vi) A finite union of strong Ditkin (ultra-strong Ditkin, sequentially strong Ditkin) sets is strong Ditkin (respectively ultra-strong Ditkin, sequentially strong Ditkin).

REMARK 3.5. As already noted in Remark 3.1 (iv) the points in the center $Z(\hat{K})$ of $\hat{K}$ are Calderon for $S(K)$. Since $\gamma \in Z(\hat{K})$ and $f \in S(K)$ need not imply γf is in $S(K)$, we cannot have an analogue of Corollary 3.7 [4] straightaway. However points in $Z(\hat{K})$ are

[2] For related results on ultrastrong Ditkin sets see our forthcoming paper in Proc. Amer. Math. Soc.

ultra-strong Ditkin for $S(K)$ and they are sequentially strong Ditkin in case K is first countable and σ-compact as we show in Theorem 3.6 below. In fact we show in Remark 3.9 that if E is ultra-strong Ditkin (respectively sequentially strong Ditkin) for $L^1(K)$ then it is so for $S(K)$.

THEOREM 3.6. *Let $\gamma \in Z(\hat{K})$. Then there is a net $\{f_\alpha : \alpha \in \Lambda\}$ in $S(K)$ such that*

(i) $\|f_\alpha\|_1 < 3$ *for all* α.

(ii) *If $f \in S(K)$ and $\hat{f}(\gamma) = 0$ then for each $\sigma \in \Sigma$*

$$\lim_\alpha \|f - f*f_\alpha\|_\sigma = 0 .$$

(iii) *Each $\hat{f}_\alpha$ vanishes in a neighborhood of γ in $\hat{K}$ and has compact support.*

If K is first countable and σ-compact then $\{f_\alpha\}$ can be chosen as a sequence.

Proof. By Theorem 2.8 there exists an approximate unit $\{u_\beta\}_{\beta \in D}$ for $S(K)$ such that $\|u_\beta\|_1 = 1$ and $\hat{u}_\beta \in A_{00}(\hat{K})$ for all β. The net $\{f_\alpha\}$ will be directed by the set $\Lambda = \{(F, n, \beta): F \subset K$ compact symmetric, $n \in \boldsymbol{N}, \beta \in D\}$ where $\alpha' = (F', n', \beta') \geqq (F, n, \beta) = \alpha$ signifies $F' \supset F$, $n' \geqq n$, $\beta' \geqq \beta$. Given $\alpha = (F, n, \beta)$ select $g_{F,\delta}$ as in Lemma 3.2 [4] so that $g_{F,\delta} = 1$ in a neighborhood Γ of 1 where $\delta = (1/n)$ and then define

$$h_\alpha = \gamma(u_\beta - u_\beta * g_{F,\delta}) \quad \text{and} \quad f_\alpha = u_\beta * h_\alpha .$$

Then $h_\alpha \in L^1(K)$ and $\|h_\alpha\|_1 = \|u_\beta - u_\beta * g_{F,\delta}\|_1 < 3$ and therefore, $f_\alpha \in L^1(K)$ and $\|f_\alpha\|_1 < 3$. Also $\hat{f}_\alpha = \hat{u}_\beta$. $\hat{h}_\alpha$ has compact support and hence $f_\alpha \in S(K)$. Now by ([4], 2.2)

$$\hat{h}_\alpha(\chi) = (u_\beta - u_\beta * g_{F,\delta})\hat{}\,(\chi * \bar{\gamma}) = \hat{u}_\beta(1 - \hat{g}_{F,\delta})(\chi * \bar{\gamma}) .$$

Thus $\hat{h}_\alpha = 0$ in the neighborhood $\gamma * \Gamma$ of γ. So (i) and (iii) are satisfied.

Now to check (ii) let $f \in S(K)$ be such that $\hat{f}(\gamma) = 0$. Let $\sigma \in \Sigma$ and $\varepsilon > 0$ be arbitrary; then there exists β_1 (depending upon f, ε, and σ) such that

$$\eta_\sigma \|f - f * u_\beta\|_\sigma < \frac{\varepsilon}{8} \quad \text{for all} \quad \beta \geqq \beta_1 .$$

Then

$$\eta_\sigma \|f * u_\beta - f * u_{\beta_1}\|_\sigma < \frac{\varepsilon}{4} \quad \text{for all} \quad \beta \geqq \beta_1 .$$

Now $(f\bar{\gamma})\hat{}(1) = 0$, so from the proof of Theorem 3.3 [4] there exists $\alpha' = (F', n', \beta^1)$ (depending upon β_1, f, ε, and σ and, therefore, on f, ε, and σ) such that

$$\|\bar{\gamma}f - (\bar{\gamma}f)*(\bar{\gamma}h_\alpha)\|_1 < \frac{\varepsilon}{8\|u_{\beta_1}\|_\sigma} \quad \text{for} \quad \alpha \geqq \alpha' .$$

By Theorem 3.6 [4], $(\bar{\gamma}f)*(\bar{\gamma}h_\alpha) = \bar{\gamma}(f*h_\alpha)$ and

$$\begin{aligned}\|\bar{\gamma}f - \bar{\gamma}f*\bar{\gamma}h_\alpha\|_1 &= \|\bar{\gamma}(f - f*h_\alpha)\|_1 \\ &= \|f - f*h_\alpha\|_1 .\end{aligned}$$

Let $\beta_0 \in D$ be such that $\beta_0 \geqq \beta_1$, $\beta_0 \geqq \beta'$ and put $\alpha_0 = (F', n', \beta_0)$. Then for $\alpha = (F, n, \beta) \geqq \alpha_0$,

$$\begin{aligned}&\|f - f^*f_\alpha\|_\sigma \\ &\quad = \|f - f*u_{\beta_1} + f*u_{\beta_1} - f*u_{\beta_1}*h_\alpha + f*u_{\beta_1}*h_\alpha - f^*u_\beta*h_\alpha\|_\sigma \\ &\quad \leqq \|f - f*u_{\beta_1}\|_\sigma + \|u_{\beta_1}*(f - f*h_\alpha)\|_\sigma + \|(f*u_{\beta_1} - f*u_\beta)*h_\alpha\|_\sigma \\ &\quad \leqq \|f - f*u_{\beta_1}\|_\sigma + \eta_\sigma\|u_{\beta_1}\|_\sigma\|f - f*h_\alpha\|_1 + \eta_\sigma\|f*u_{\beta_1} - f*u_\beta\|_\sigma\|h_\alpha\|_1 \\ &\quad < \varepsilon/8 + \varepsilon/8 + \varepsilon/4\cdot 3 = \varepsilon .\end{aligned}$$

REMARK 3.7. A direct proof of the above theorem can also be given along the following lines by first generalizing Lemma 3.2 [4].

(a) If $\gamma \in Z(\hat{K})$, F is a compact symmetric subset of K, Γ_0 is a compact neighborhood of 1 in $\hat{K}$ and $\delta > 0$ then there exists $g \in L^1(K)$ such that

(i) $\hat{g} = 1$ on a neighborhood of γ,

(ii) $\hat{g}$ has compact support contained in $\Gamma = \Gamma_0*\Gamma_0*\gamma$,

(iii) for each $\sigma \in \Sigma$, $\|g\|_\sigma \leqq 2C_{\Gamma,\sigma}$ whereas $\|g\|_1 < 2$ (where $C_{\Gamma,\sigma}$ is as in Remark 2.7 (iv))

and

(iv) $\|g_x^\vee - \overline{\gamma(x)}g\|_1 < \delta$ for all $x \in F$.

The function g is constructed as follows:

Let $H_1 = \{\chi \in \hat{K}: |\chi(y) - \gamma(y)| < (\delta/12) \quad \text{for} \quad y \in F\}$

and $H_2 = \{\chi \in \hat{K}: |\chi(y) - 1| < (\delta/12) \quad \text{for} \quad y \in F\}$.

Then H_1 is a neighborhood of γ and H_2 is a neighborhood of 1 in $\hat{K}$. So there exists an open symmetric neighborhood H of 1 with $H \subset \Gamma_0$ such that $H*\gamma \subset H_1$ and $H \subset H_2$ ([9], 3.2D).

Since π is a regular measure on $\hat{K}$ there exists a compact symmetric neighborhood Φ of 1 such that $\Phi \subset H$ and $\pi(\Phi) > (1/4)\pi(H)$.

By ([9], 3.2D) there exists a neighborhood Ψ of γ such that

$$\Phi*\Psi \subset H*\gamma .$$

Since $\gamma \in Z(\hat{K})$, $\pi(\Phi*H*\gamma) = \pi(\Phi*H)$ and $\pi(H*\gamma) = \pi(H)$. There exists $g_1, g_2 \in L^2(K)$ such that

$$\hat{g}_1 = \xi_\Phi \quad \text{and} \quad \hat{g}_2 = \xi_{H*\gamma} \, .$$

Let

$$g = \frac{1}{\pi(\Phi)} g_1 g_2 \, ,$$

then the rest of the proof involves computations similar to those in Lemma 3.2 [4].

(b) Let $\{u_\beta\}_{\beta \in D}$ and Λ be as in Theorem 3.6 above.

Let $f_\alpha = u_\beta - u_\beta * g_{F,\delta}$ where $g_{F,\delta}$ is selected as in (a) above with $\delta = 1/_n$. Then $f_\alpha \in S(K)$ and $\|f_\alpha\|_1 < 3$, $\hat{f}_\alpha$ has compact support and vanishes in a neighborhood of γ.

Let $f \in S(K)$ with $\hat{f}(\gamma) = 0$, $\sigma \in \Sigma$ and $\varepsilon > 0$ be arbitrary; select β_0 such that $\beta \geqq \beta_0$ implies

$$\|f - f * u_\beta\|_\sigma < \varepsilon/2 \, .$$

There exists a compact symmetric set F_0 in K such that

$$\int_{K \backslash F_0} |f| \, dm < \frac{\varepsilon}{16 C_{\Gamma,\sigma} \eta_\sigma} \, .$$

Let n_0 be such that $4C_{\Gamma,\sigma}\eta_\sigma \|f\|_1 < n_0 \varepsilon$.

Let $\alpha_0 = (F_0, n_0, \beta_0)$. Then for $\alpha = (F, n, \beta) \geqq \alpha_0$,

$$\begin{aligned} \|f - f^* f_\alpha\|_\sigma &= \|f - f^*(u_\beta - u_\beta^* g_{F,\delta})\|_\sigma \\ &\leqq \|f - f^* u_\beta\|_\sigma + \|f^* u_\beta^* g_{F,\delta}\| \\ &\leqq \|f - f^* u_\beta\|_\sigma + \eta_\sigma \|u_\beta\|_1 \|f^* g_{F,\delta}\|_\sigma \\ &< \varepsilon/2 + \eta_\sigma C_{\Gamma,\sigma} \|f^* g_{F,\delta}\|_1 \, . \end{aligned}$$

Since $\hat{f}(\gamma) = 0$, for $\alpha \geqq \alpha_0$

$$\begin{aligned} \|f * g_{F,\delta}\|_1 &\leqq \int_K \int_K |f(x)| \, |g_{F,\delta}(y * \check{x}) - \overline{\gamma(x)} g_{F,\delta}(y)| \, dm(x) dm(y) \\ &= \int_K |f(x)| \int_K |g_{F,\delta}(y * \check{x}) - \overline{\gamma(x)} g_{F,\delta}(y)| \, dm(y) dm(x) \\ &\leqq \int_F |f(x)| \frac{1}{n} dm(x) + \int_{K \backslash F} |f(x)| (\|(g_{F,\delta})_{\check{x}}\|_1 + \|g_{F,\delta}\|)_1 dm(x) \\ &\leqq \frac{1}{n_0} \|f\|_1 + 2\|g_{F,\delta}\|_1 \int_{K \backslash F_0} |f(x)| \, dm(x) \\ &< \frac{1}{4\eta_\sigma C_{\Gamma,\sigma}} \left(\frac{\varepsilon}{4} + \frac{\varepsilon}{4} \right) = \frac{\varepsilon}{2\eta_\sigma C_{\Gamma,\sigma}} \, . \end{aligned}$$

So

$$\|f - f * f_\alpha\|_\sigma < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for} \quad \alpha \geqq \alpha_0 \, .$$

REMARKS 3.8. (i) The hypergroup K defined (cf. [4], 4.6) and studied by Dunkl and Ramirez [6] is first countable and σ-compact. Also $\hat{K}$ is discrete at points of $\hat{K}\backslash Z(\hat{K})$ and $Z(\hat{K}) = \{1\}$. Thus by the above theorem and Remarks 3.5 (iv) and (v) every closed subset of $\hat{K}$ is sequentially strong Ditkin and hence is also strong Ditkin in case $S(K)$ is a Banach algebra, we note that this result is new even when $S(K) = L^1(K)$ and is partially in contrast with the corresponding result for the group case where every nondiscrete locally compact abelian group $\hat{G}$ contains nonspectral closed sets.

(ii) On the other hand, points in $\hat{K}$ need not even be spectral sets for $A(\hat{K})$ by ([4], §4) and therefore, by Remark 3.1 (iv) for $AS(\hat{K})$.

REMARK 3.9. Theorem 3.6 can also be deduced from the following discussion:

Let X be a locally compact Hausdorff space and A a regular Banach algebra in $C_0(X)$ with structure space X. Let B be a subalgebra of A which is either a dense ideal or contains $C_{00}(X) \cap A$ equipped with a locally convex topology given by seminorms $\{\|\cdot\|_\sigma : \sigma \in \Sigma\}$ satisfying:
for some $\eta_\sigma > 0$,

$$\|\varphi\psi\|_\sigma \leqq \eta_\sigma \|\varphi\|_A \|\psi\|_\sigma \quad \text{for} \quad \varphi \in A, \psi \in B$$

such that $\varphi\psi \in B$.

Then B has separately continuous multiplication so that B is a locally convex algebra. We further suppose that B has an approximate unit $\{\varphi_\beta : \beta \in D\}$ such that

$$\rho = \sup\{\|\varphi_\beta\|_A : \beta \in D\} < \infty .$$

Let E be a closed subset of X. E will be said to be *ultra-strong Ditkin* for A if there exists a net $\{\psi_\alpha : \alpha \in \Lambda\}$ in $C_{00}(X) \cap A$ such that

(i) each ψ_α vanishes in a neighborhood of E,

(ii) $\lambda = \sup\{\|\psi_\alpha\|_A : \alpha \in \Lambda\} < \infty$

and (iii) for each φ in A that vanishes on E,

$$\|\varphi\psi_\alpha - \varphi\|_A \longrightarrow 0 .$$

We shall say that E is *ultra-strong Ditkin* for B if there exists a net $\{w_t : t \in T\}$ in $C_{00}(X) \cap B$ satisfying (i) and (ii) with φ_α replaced by w_t and Λ by T and (iii) with ψ_α replaced by w_t, A by B and $\|\cdot\|_A$ by $\|\cdot\|_\sigma$ for each σ.

We shall now show that if E is ultra-strong Ditkin for A then it is so for B.

Let $T = \{(\alpha, \beta) : \alpha \in \Lambda, \beta \in D\}$ and

$$t_1 = (\alpha_1, \beta_1)/(\alpha_2, \beta_2) = t_2$$

if and only if

$$\alpha_1 \geqq \alpha_2\,, \quad \beta_1 \geqq \beta_2\,.$$

For $t = (\alpha, \beta)$, let $w_t = \psi_\alpha \varphi_\beta$; then $w_t \in C_{00}(X) \cap B$ and vanishes in a neighborhood of E.

Also

$$\|w_t\|_A \leqq \|\psi_\alpha\|_A \|\varphi_\beta\|_A \leqq \lambda\rho < \infty\,.$$

Let $\varepsilon > 0, \sigma \in \Sigma$ and $\varphi \in B$ vanishing on E be arbitrary; then there exists β_0 in D (depending upon α, ε, and φ) such that

$$\eta_\sigma \|\varphi - \varphi\varphi_\beta\|_\sigma < \frac{\varepsilon}{4(\lambda + 1)} \quad \text{for} \quad \beta \geqq \beta_0$$

and

$$\eta_\sigma \|\varphi\varphi_\beta - \varphi\varphi_{\beta_0}\|_\sigma < \frac{\varepsilon}{2(\lambda + 1)} \quad \text{for} \quad \beta \geqq \beta_0\,.$$

Since $\|\varphi - \varphi\psi_\alpha\|_A \to 0$ there exists α_0 (depending upon φ, ε, σ, and β_0 and therefore on φ, ε, σ) such that

$$\eta_\sigma \|\varphi - \varphi\psi_\alpha\|_A \|\varphi_{\beta_0}\|_\sigma < \varepsilon/4 \qquad \text{for} \quad \alpha \geqq \alpha_0\,.$$

Now for $t = (\alpha, \beta) \geqq (\alpha_0, \beta_0) = t_0$

$$\begin{aligned} \|\varphi - \varphi w_t\|_\sigma &= \|\varphi - \varphi\varphi_{\beta_0} + \varphi\varphi_{\beta_0} - \varphi\varphi_{\beta_0}\psi_\alpha + \varphi\varphi_{\beta_0}\psi_\alpha - \varphi\varphi_\beta\psi_\alpha\|_\sigma \\ &\leqq \|\varphi - \varphi\varphi_{\beta_0}\|_\sigma + \|\varphi_{\beta_0}(\varphi - \varphi\psi_\alpha)\|_\sigma + \|(\varphi\varphi_{\beta_0} - \varphi\varphi_\beta)\psi_\alpha\|_\sigma \\ &\leqq \|\varphi - \varphi\varphi_{\beta_0}\|_\sigma + \eta_\sigma\|\varphi_{\beta_0}\|_\sigma\|\varphi - \varphi\psi_\alpha\|_A + \eta_\sigma\|\varphi\varphi_{\beta_0} \\ &\quad - \varphi\varphi_\beta\|_\sigma\|\varphi_\alpha\|_A \\ &< \varepsilon/4 + \varepsilon/4 + \varepsilon/2 = \varepsilon\,. \end{aligned}$$

The analogue for sequentially strong Ditkin sets in A and B is clear.

4. Examples.

4.1. In this section we shall give examples of Segal algebras. We shall first recall Lorentz spaces introduced by Lorentz [12] and further studied by Hunt [8], Blozinski [1], O'Neil [13] and Yap [26].

Let $1 < p < \infty, 1 \leqq q < \infty$. Let p' be the conjugate of p i.e., $p' = p/(p-1)$.

Let f be a (complex valued) measurable function defined on a measure space (X, μ). For $y \geqq 0$, we define

$$m(f, y) = \mu\{x \in X \colon |f(x)| > y\}\,.$$

For $x \geqq 0$, let

$$\begin{aligned} f^*(x) &= \inf\{y: y > 0 \quad \text{and} \quad m(f, y) \leqq x\} \\ &= \sup\{y: y > 0 \quad \text{and} \quad m(f, y) > x\} . \end{aligned}$$

For $x > 0$, let

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(t)dt$$

and let

$$\|f\|_{(p,q)} = \left(\int_0^\infty (x^{1/p}f^{**}(x))^q \frac{dx}{x}\right)^{1/q},$$

then $L_{(p,q)}(X) = \{f: \|f\|_{(p,q)} < \infty\}$ equipped with $\|\cdot\|_{(p,q)}$ is called a Lorentz space.

By [13], $\|f\|_p \leqq \|f\|_{(p,p)} \leqq p'\|f\|_p$ so that $L_{(p,p)} = L^p$. The following fact is a special case of ([13], 2.6).

If p, r, s are real numbers such that $1 < r, s < \infty$, $1/r + 1/s > 1$ and $1/p = 1/r + 1/s - 1$ then

$$L^r(K) * L^s(K) \subset L_{(p,1)}(K) .$$

EXAMPLE 4.2. (a) Let $1 \leqq p < \infty$,

$$\begin{aligned} S(K) &= B^p(K) = L^1(K) \cap L^p(K) \\ \|f\|_S &= \|f\|_1 + \|f\|_p, \ f \in S(K) . \end{aligned}$$

Then $S(K)$ is a Segal algebra;

S(i) Let $f \in L^1(K)$ with $\hat{f} \in C_{00}(\hat{K})$.

Then $f \in C_0(K)$ by ([9], 12.2, 7.3) so $f \in L^\infty(K) \cap L^1(K)$ and therefore, $f \in L^p(K)$.

S(ii) follows from ([9], 3.3B).

S(iii) follows from ([9], 5.4H, 2.2B).

We note that we can modify ([27], 2.4) to prove that $B^p(K)$ has factorization property if and only if $p = 1$ (K is arbitrary) or K is discrete ($1 \leqq p < \infty$).

(b) Let $\Lambda \subset [1, \infty)$; by Remark 2.9 $S(K) = \bigcap_{p \in \Lambda} B^p(K)$ is a locally convex Segal algebra with a generating family of norms given by $\{s_p: p \in \Lambda\}$ where

$$s_p(f) = \|f\|_1 + \|f\|_p \quad \text{for} \quad f \in S(K) .$$

EXAMPLE 4.3. (a) Let $1 \leqq p < \infty$ and K be nondiscrete.

Let $S(K) = A^p(K) = \{f: f \in L^1(K), \hat{f} \in L^p(\hat{K})\}$,

$$\|f\|_S = \|f\|_1 + \|\hat{f}\|_p \quad \text{for} \quad f \in S(K) .$$

Then $S(K)$ is a Segal algebra;

S(i) Let $f \in L^1(K)$ with $\hat{f} \in C_{00}(\hat{K})$.

Then $\hat{f} \in L^p(\hat{K})$, so $f \in S(K)$.

S(ii) Let $f \in S(K)$, $x \in K$; then $\|f_x\|_S \leqq \|f\|_S$ using ([4], 2.2).

S(iii) Let $f(\neq 0) \in S(K)$, $x \in K$ and $\varepsilon > 0$. Then ([4], 2.3) implies that there is a neighborhood V of x such that

$$\|f_y - f_x\|_1 < \varepsilon/2 \quad \text{for} \quad y \in V .$$

Choose $\varphi \in C_{00}(\hat{K})$ such that

$$\|\varphi - \hat{f}\|_p < \varepsilon/8 .$$

Let $F = \operatorname{supp} \varphi$; then

$$\int_{\hat{K}\setminus F} |\hat{f}(\gamma)|^p d\pi(\gamma) < (\varepsilon/8)^p .$$

Now $N(F, x, \varepsilon) = \{y \in K: |\gamma(y) - \gamma(x)| < \varepsilon/4\|\hat{f}\|_p$ for all $\gamma \in F\}$ is a neighborhood of x using ([9], 7.3, §12).

So $W = V \cap N(F, x, \varepsilon)$ is a neighborhood of x and $y \in W$, we have by ([4], 2.3)

$$\begin{aligned}\|\hat{f}_y - \hat{f}_x\|_p^p &= \int_{\hat{K}} |\hat{f}_y(\gamma) - \hat{f}_x(\gamma)|^p d\pi(\gamma) \\ &= \int_{\hat{K}} |\gamma(y) - \gamma(x)|^p |\hat{f}(\gamma)|^p d\pi(\gamma) \\ &= \int_F |\gamma(y) - \gamma(x)|^p |\hat{f}(\gamma)|^p \\ &\quad + \int_{\hat{K}\setminus F} |\gamma(y) - \gamma(x)|^p |\hat{f}(\gamma)|^p d\pi(\gamma) \\ &< (\varepsilon/4)^p + 2^p(\varepsilon/8)^p \leqq (\varepsilon/2)^p\end{aligned}$$

so

$$\|\hat{f}_y - \hat{f}_x\|_p < \varepsilon/2 .$$

Thus $\|f_y - f_x\|_S < \varepsilon$.

Hence the mapping $y \rightarrow f_y$ is continuous from K into $S(K)$.

(b) Let $\Lambda \subset [1, \infty)$; by Remark 2.9 $S(K) = \bigcap_{p \in \Lambda} A^p(K)$ is a locally convex Segal algebra with a generating family of norms given by $\{s_p: p \in \Lambda\}$ where,

$$s_p(f) = \|f\|_1 + \|\hat{f}\|_p \quad \text{for} \quad f \in S(K) .$$

As in [11], $A^p(K) \subset A^q(K)$ if $p \leqq q$ so if the infimum p_0 of Λ is in Λ then $\bigcap_{p \in \Lambda} A^p(K) = A^{p_0}(K)$ but still the topologies are different unless Λ is finite.

EXAMPLE 4.5. (a) Let $1<p<\infty$ and $1\leqq q<\infty$; $S(K)=A_{(p,q)}(K)=\{f: f\in L^1(K)$ and $\hat{f}\in L_{(p,q)}(\hat{K})\}$ and for $f\in S(K)$, $\|f\|_S=\|f\|_1+\|\hat{f}\|_{(p,q)}$. Then $S(K)$ is a Segal algebra;

S(i) Let $f\in L^1(K)$ with $\hat{f}\in C_{00}(\hat{K})$ then by [1] $\hat{f}\in L_{(p,q)}(\hat{K})$ so $f\in S(K)$.

S(ii) It is easy to verify that $\|(f_x)^{\wedge}\|_{(p,q)}\leqq\|\hat{f}\|_{(p,q)}$ using $m((f_x)^{\wedge}, y)\leqq m(\hat{f}, y)$ for $y\geqq 0$.

S(iii) The proof of Yap ([29], 2.3) can be modified to prove that the mapping $x\rightarrow f_x$ is continuous from K into $S(K)$.

(b) Let $\Lambda\subset(1,\infty)\times[1,\infty)$.

Then by Remark 2.9 $S(K)=\bigcap_{(p,q)\in\Lambda}A_{(p,q)}(K)$ is a locally convex Segal algebra with a generating family of norms given by

$$\{s_{(p,q)}: (p, q)\in\Lambda\}$$

where $s_{(p,q)}(f)=\|f\|_1+\|\hat{f}\|_{(p,q)}$ for $f\in S(K)$.

The authors would like to thank Mr. Om Prakash for useful discussions and Professor K. A. Ross for encouragement, useful comments and suggestions.

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Received February 16, 1978 and in revised form April 10, 1978.

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