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ASYMPTOTICALLY STABLE DYNAMICAL SYSTEMS ARE LINEAR

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If π is a dynamical system on a locally compact metric space X which has a globally asymptotically stable critical point, then π can be embedded into a dynamical system on l_2 which is derived from a linear differential equation. If X is n-dimensional, then l_2 may be replaced by R^{2n}.

Throughout this paper R and R^+ will denote the reals and non-negative reals respectively. A dynamical system on a topological space X is a continuous mapping: $\pi: X \times R \to X$ such that (where $\pi(x, t) = x\pi t$)

(i) $x\pi 0 = x$ for all $x \in X$,

(ii) $(x\pi t)\pi s = x\pi(t + s)$ for all $x \in X$ and $s, t \in R$.

A point $p \in X$ is called a critical point of π if $p\pi t = p$ for every $t \in R$. A subset S of X is called a section with respect to π if $(S\pi t) \cap S = \phi$ for every $t \neq 0$. A subset S of X is said to be a section for $Y \subset X$ if S is a section and $\{x\pi t: x \in S,\ t \in R\} = Y$. A compact subset M of X is said to be stable with respect to π if for any neighborhood U of M there is a neighborhood V of M such that $\{x\pi t: x \in V,\ t \in R^+\} \subset U$. The compact subset M of X is said to be a global attractor if for any neighborhood U of M and $x \in X$, there is a $c \in R$ such that $x\pi t \in U$ whenever $c \leqq t$. If M is a stable global attractor, then M is said to be globally asymptotically stable.

Let X and Y be topological spaces on which are defined dynamical systems π and ρ respectively. We say that π can be embedded into ρ if there is a homeomorphism h of X onto a subset of Y such that $h(x\pi t) = h(x)\rho t$ for every $x \in X$ and $t \in R$. In the special case $h(X) = Y$ we will say that π is isomorphic to ρ.

The set of all sequences $x = \{x_1, x_2, \cdots, x_n, \cdots\}$ of real numbers such that $\sum_{n=1}^{\infty} x_n^2$ converges is denoted by l_2. If addition and scalar multiplication are defined coordinatewise and if a norm is defined by $\|x\| = (\sum_{n=1}^{\infty} x_n^2)^{1/2}$, then l_2 is a real Banach space.

Throughout the remainder of this paper X will denote a locally compact metric space.

Let $p \in X$ be a globally asymptotically stable critical point with respect to the dynamical system π and let U be a compact neighborhood of p. It is known ([1, Theorem 2.7.14]) that there is a continuous (Liapunov) function $v: X \to R^+$ such that

(i) $v(x) = 0$ if and only if $x = p$,

(ii) $v(x\pi t) = e^{-t}v(x)$ for $x \in X - \{p\}$ and $t > 0$.

Let $a > 0$ be so small that $v^{-1}(a) \subset U$ and set $S = v^{-1}(a)$. The following lemma is also well known and is easily verified.

LEMMA 1. *S is a compact section for $X - \{p\}$. Moreover, the mapping $\Upsilon: X - \{p\} \to R$ defined by $x\pi\Upsilon(x) \in S$ is continuous.*

Since S is compact it is separable. Let d denote a metric on X and let $\{x_n\}$ be a countable dense subset of S. We define a countable number of continuous functions $f_n: S \to R^+$ by

$$f_n(x) = d(x, x_n) .$$

LEMMA 2. *If $f_n(x) \leqq f_n(y)$ for every n, then $x = y$.*

Proof. Suppose that $x \neq y$. Let $r = d(x, y)$ and $B = \{z: d(z, y) \leqq r/4\}$. Since $\{x_n\}$ is dense in S there is a k such that $x_k \in B$. Then

$$f_k(y) = d(y, x_k) \leqq \frac{1}{4}r < \frac{3}{4}d(x, x_k) = f_k(x) .$$

A similar argument shows that there is a j such that $f_j(x) < f_j(y)$. The desired result follows directly.

LEMMA 3. *The mapping $h: S \to l_2$ defined by*

$$h(x) = \left(f_1(x), \frac{1}{2}f_2(x), \cdots, \frac{1}{n}f_n(x), \cdots\right)$$

is a homeomorphism of S onto $h(S)$.

Proof. Since S is compact the mapping d restricted to $S \times S$ is uniformly continuous and bounded. Hence, the set of mappings $\{f_n\}$ is equicontinuous and equibounded. For each $x \in S$, $h(x) \in l_2$ since $\{f_n\}$ is equibounded. Since $\{f_n\}$ is equicontinuous, h is continuous. It follows immediately from Lemma 2 that h is one-to-one. A continuous one-to-one mapping of a compact space onto a Hausdorff space is a homeomorphism.

Let $c \in (0, 1)$ and define a dynamical system ρ on l_2 by $x\rho t = c^t x$. This dynamical system can be interpreted as being derived from the linear differential equation $dy/dt = ky$, $y(0) = x$, where $k = \ln c$.

LEMMA 4. *If $x, y \in S$ are such that $h(x) = h(y)\rho t$ for some $t \in R$, then $x = y$ and $t = 0$.*

Proof. Suppose that $h(x) = h(y)\rho t = c^t h(y)$ for some $t \in R$. Without loss of generality we may assume that $t \geqq 0$. Then $f_n(x) = c^t f_n(y) \leqq f_n(y)$ for every n. By Lemma 2, $x = y$. If $x = y$, clearly $t = 0$.

LEMMA 5. *The mapping* $H: X \to l_2$ *defined by*

$$H(x) = \begin{cases} 0 & \text{if } x = p\,, \\ c^{-\Upsilon(x)} h(x\pi\Upsilon(x)) & \text{if } x \in X - \{p\} \end{cases}$$

where Υ *is the mapping defined in Lemma* 1, *is a homeomorphism of* X *onto* $h(X)$.

Proof. If $H(x) = H(y)$, $x \neq 0 \neq y$, then

$$c^{-\Upsilon(y)} h(y\pi\Upsilon(y)) = c^{-\Upsilon(x)} h(x\pi\Upsilon(x))$$

so that

$$h(y\pi\Upsilon(y)) = h(x\pi\Upsilon(x))\rho(\Upsilon(x) - \Upsilon(y))\,.$$

By Lemma 4, $y\pi\Upsilon(y) = x\pi\Upsilon(x)$ and $\Upsilon(x) - \Upsilon(y) = 0$. Hence, $x = y$ and H is one-to-one. Since π, Υ, and h are cotinuous on $X - \{p\}$, H is continuous on $X - \{p\}$. We will now show that H is continuous at p. Let $\{z_i\}$ be a sequence in $X - \{p\}$ which converges to p. We will first show that $\Upsilon(z_i) \to -\infty$. Since $z_i\pi\Upsilon(x_i) \in S$ and $V(z) = a$ for each $z \in S$, we have

$$0 < a = V(z_i\pi\Upsilon(z_i)) = e^{-\Upsilon(z_i)} v(z_i)\,.$$

We must have $\Upsilon(z_i) \to -\infty$ since $v(z_i) \to 0$. Now

$$H(z_i) = c^{-\Upsilon(z_i)} h(z_i\pi\Upsilon(z_i)) \longrightarrow 0$$

because $c \in (0, 1)$, $\Upsilon(z_i) \to -\infty$, and $h(S)$ is compact with $0 \notin h(S)$. This proves that H is continuous at p so that H is continuous. Note that $H(x) = h(x\pi\Upsilon(x))\rho(-\Upsilon(x))$. A short calculation shows that $H^{-1}(H(x)) = h^{-1}[H(x)\rho\Upsilon(x)]\pi(-\Upsilon(x))$ whenever $x \neq p$. Since h^{-1}, H, ρ, Υ, and π are continuous on their respective domains, H^{-1} is continuous on $H(X) - \{0\}$. Let $\{x_i\}$ be any sequence such that $H(x_i) \to 0$. Since $H(x_i) = c^{-\Upsilon(x_i)} h(x_i\pi\Upsilon(x_i))$ and $h(S)$ is compact with $0 \notin h(S)$ we must have $\Upsilon(x_i) \to -\infty$. Then

$$0 < a = v(z_i\pi\Upsilon(z_i)) = e^{-\Upsilon(x_i)} v(x_i)$$

so that we must have $v(x_i) \to 0$. Thus, $x_i \to p$. This proves that H^{-1} is continuous at 0. H is a homeomorphism.

THEOREM 6. *Let* π *be a dynamical system on a locally compact metric space* X *and let* ρ_c, $0 < c < 1$, *be the dynamical system on* l_2 *defined by* $x\rho_c t = c^t x$. *If* π *has a globally asymptotically stable critical point, then* π *can be embedded into* ρ_c.

Proof. In light of Lemma 5 it remains to show that $H(x\pi t) =$

$h(x)\rho t$. It is easy to show that $\Upsilon(x\pi t) = \Upsilon(x) - t$. Hence,

$$\begin{aligned} H(x\pi t) &= c^{-\Upsilon(x)+t}h((x\pi t)\pi(\Upsilon(x) - t)) \\ &= c^t c^{-\Upsilon(x)}h(x\pi\Upsilon(x)) \\ &= c^t h(x) \\ &= h(x)\rho t \ . \end{aligned}$$

If X is of finite dimension n, then l_2 can be replaced by R^{2n} in Theorem 6. This may be proved as follows. Let S be a compact section for π. It is known that if A is compact and B is one dimensional, then $\dim(A \times B) = \dim A + \dim B$. This is cited in [2, page 34] and [5, page 302], and referenced as [3] in [5]. Since $S\pi R$ is homeomorphic with $S \times R$, we have $\dim S + 1 = \dim S + \dim R = \dim(S \times R) = \dim(S\pi R) \leqq n$. Hence $\dim S \leqq n - 1$. It is known that a k-dimensional space can be embedded in R^{2k+1}, [2, page 60]. Hence, S can be embedded into R^{2n-1}. The one point compactification of R^{2n-1} is S^{2n-1}, the unit sphere in R^{2n}. Thus, there is an imbedding $g: S \to S^{2n-1} \subset R^{2n}$. Consider the dynamical system α_c defined by the linear differential equation

$$\frac{dy}{dt} = ky \ , \qquad y(0) = x$$

where $y: R \to R^{2n}$ and $k < 0$. Then $x\alpha_c t = c^t x$ for $t \in R$, $x \in R^{2n}$, and $c = e^k$. Define $G: X \to R^{2n}$ by

$$G(x) = \begin{cases} 0 & \text{if } x = p \ , \\ c^{-\Upsilon(x)}g(x\pi\Upsilon(x)) & \text{if } x \in X - \{p\} \ . \end{cases}$$

The proof that G is a homeomorphism is essentially the same as the proof of Lemma 5. With this result the proof of the following theorem is identical with that of Theorem 6.

THEOREM 7. *Let π be a dynamical system on an n-dimensional locally compact space X and α_c, $0 < c < 1$, be the dynamical system on R^{2n} defined by $x\alpha_c t = c^t x$. If π has a globally asymptotically stable critical point, then π can be embedded into α_c.*

If S can be embedded into S^{k-1}, then obvious modifications of the proof of Theorem 7 show that π can be embedded into the dynamical system on R^k defined by $x\alpha_c t = c^t x$, $0 < c < 1$. If X has dimension n, what is the smallest integer k such that S can be embedded into S^{k-1}? The author does not know, but conjectures that if $X = R^n$ then S can be embedded into S^{n-1}. If this conjecture were true then S would be homeomorphic to S^{n-1}. The proof of this, or the construction of a counterexample, appears to be difficult. However, in the case $n = 2$, the conjecture is true.

THEOREM 8. *Let π be a dynamical system on R^2 which has a globally asymptotically stable point p. If S is any section for $X - \{p\}$, then S is homeomorphic to S^1.*

Proof. Evidently S is compact and connected. Let x and y be any two points of S. Since p is asymptotically stable $L^-(x) = L^-(y) = \phi$. It is easy to show that $D = \{p\} \cup \{x\pi R\} \cup \{y\pi R\}$ is a curve which separates the plane into exactly two components. Moreover, $S \cap D = \{x, y\}$. Hence, $S - \{x, y\}$ has exactly two components. A continuum whose connection is destroyed by the removal of two arbitrary points is a simple closed curve, [5, page 99].

COROLLARY 9. *Let π be a dynamical system on R^2 and let α_c, $0 < c < 1$, be the dynamical system on R^2 defined by $x\alpha_c t = c^t x$. If π has a globally asymptotically stable critical point, then π is isomorphic to α_c.*

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