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A PRIMENESS PROPERTY FOR CENTRAL POLYNOMIALS

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In this note we prove an anolog of of Amitsur's theorem for central polynomials.

THEOREM. **Let F be an infinite field, $f(x) = f(x_1, \cdots, x_r)$, $g(x) = g(x_{r+1}, \cdots, x_s)$ two noncommutative polynomials in disjoint sets of variables. Assume that $f(x_1, \cdots, x_r) \cdot g(x_{r+1}, \cdots, x_s)$ is central but not an identity for F_k. Then both $f(x)$ and $g(x)$ are central polynomials for F_k.**

Note. Since $[x, y]^2$ is central for F_2 while $[x, y]$ is not, the assumption of disjointness of the variables cannot be removed.

Central polynomials that are not identities of the $k \times k$ matrices F_k were constructed in [2], [3]. In [1] Amitsur proved the following primeness property of the polynomial identities of F_k:

THEOREM (Amitsur). *Let F be an infinite field, $f(x) = f(x_1, \cdots, x_n)$, $g(x) = g(x_1, \cdots, x_n)$ two noncommutative polynomials over F. If $f(x) \cdot g(x)$ is an identity for F_k, then either $f(x)$ or $g(x)$ is an identity for F_k.*

Proof of the theorem. Since F is infinite, by standard arguments we may assume it is algebraically closed. Hence every matrix in F_k is conjugate to its Jordan canonical form. We show (W.L.O.G.) that $f(x)$ is central. By assumption there are $y_1, \cdots, y_s \in F_k$ such that

$$f(y_1, \cdots, y_r) \cdot g(y_{r+1}, \cdots, y_s) = \alpha I \neq 0 .$$

Denote $A = g(y_{r+1}, \cdots, y_s)$, then $\det A \neq 0$ since $\det \alpha I \neq 0$, so that $A^{-1} = B \in F_k$ exist. Thus deduce the identity

$$f(y_1, \cdots, y_r) = \alpha(y_1, \cdots, y_r) \cdot B \tag{1}$$

where $\alpha(y)$ is a scalar function on $(F_k)^r$, not identically zero. Conjugate both sides of (1) by a matrix $D \in F_k$ so that DBD^{-1} is in a Jordan canonical form. Since $f(x)$ is a polynomial,

$$Df(y_1, \cdots, y_r)D^{-1} = f(Dy_1D^{-1}, \cdots, Dy_rD^{-1}) = f(\bar{y}_1, \cdots, \bar{y}_r) .$$

By (1), $Df(y)D^{-1} = \alpha(y)DBD^{-1}$. Since

$$(y_1, \cdots, y_r) = (D^{-1}\bar{y}_1 D, \cdots, D^{-1}\bar{y}_r D) ,$$

we can write

$$\alpha(y_1, \cdots, y_r) = \bar{\alpha}(\bar{y}_1, \cdots, \bar{y}_r) ,$$

so we may finally assume in (1) that B is in its Jordan canonical form:

$$(1') \qquad f(y_1, \cdots, y_r) = \alpha(y_1, \cdots, y_r)\begin{pmatrix} \beta_1 & \varepsilon_1 & & 0 \\ & \ddots & \ddots & \\ 0 & & \beta_2 \ddots & \varepsilon_{k-1} \\ 0 & & & \beta_k \end{pmatrix}$$

each $\varepsilon_i = 0$ or 1 and $\alpha(y)$ is a scalar function on $(F_k)^r$, not identically zero.

We proceed to show that all $\varepsilon_i = 0$, for example, that $\varepsilon_1 = 0$. Choose $(y_1, \cdots, y_r) = (y)$ so that $\alpha(y) \neq 0$. Next, let S_k be the Symmetric group on $1, \cdots, k$. If $\eta \in S_k$ and A_η denotes the matrix $(\delta_{\eta(i),j})$, then it is well known that A_η^{-1} exists and for any matrix $(a_{i,j}) \in F_k$, $A_\eta(a_{i,j})A_\eta^{-1} = (a_{\eta(i),\eta(j)})$.

To show $\varepsilon_1 = 0$, choose the transposition $\sigma = (1, 2) \in S_k$ and conjugate (1′) by A_σ. Denoting $y_i' = A_\sigma y_i A_\sigma^{-1}$ we obtain the equation

$$(2) \qquad \alpha(y_1', \cdots, y_r')\begin{pmatrix} \beta_1 & \varepsilon_1 & \\ 0 & \beta_2 & \\ & & \ddots \end{pmatrix} = \alpha(y_1, \cdots, y_r)\begin{pmatrix} \beta_2 & 0 & \\ \varepsilon_1 & \beta_1 & \\ & & \ddots \end{pmatrix} .$$

Equating the $(2, 1)$ entry on both sides we deduce that $\varepsilon_1 = 0$.

Thus B in (1′) is diagonal. Since $\det B \neq 0$, we have $\beta_1, \cdots, \beta_k \neq 0$. By equating the $(1, 1)$ and the $(2, 2)$ entries in (2) we get

$$\alpha(y')\beta_1 = \alpha(y)\beta_2$$
$$\alpha(y')\beta_2 = \alpha(y)\beta_1$$

and all terms are $\neq 0$. Hence $\beta_2 = \pm\beta_1$. Similarly, $\beta_i = \pm\beta_1$, $2 \leqq i \leqq k$. We want to show that $\beta_1 = \cdots = \beta_k$. If $\operatorname{Char} F = 2$, we are already done. Assume therefore that $\operatorname{Char} F \neq 2$. Assume for example that $\beta_2 = -\beta_1$. Let

$$H = \begin{pmatrix} 1 & 1 & & & 0 \\ 0 & 1 & & & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} . \quad \text{Clearly } H^{-1} = \begin{pmatrix} 1 & -1 & & & 0 \\ 0 & 1 & & & \\ & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{pmatrix} .$$

Write $z_i = Hy_iH^{-1}$ and conjugate (1′) (with B diagonal) by H to obtain

$$\alpha(z_1, \cdots, z_r)B = \alpha(y_1, \cdots, y_r)\begin{pmatrix} \beta_1 & -2\beta_1 & & 0 \\ 0 & -\beta_1 & & \\ & 0 & & \ddots \end{pmatrix}.$$

This is contradiction since $-2\beta_1 \neq 0$, hence the right hand side is not diagonal while the left is.

References

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