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A TREE-LIKE TSIRELSON SPACE

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An example is given of a reflexive Banach space X such that $(X \oplus X \oplus \cdots \oplus X)_{l_1^n}$, $n = 1, 2, \cdots$, are uniformly isomorphic to X. Some related examples are also given.

1. Introduction. In [4] Lindenstrauss observed that a Banach space X such that $(X \oplus X \oplus \cdots \oplus X)_{l_1^n}$ is isometric to a subspace of X for every n must contain an isometric copy of l_1. This gives a very simple proof to the fact that there exists no separable reflexive Banach space which is isometrically universal for all the separable reflexive Banach spaces. Lindenstrauss asked whether the isomorphic version of this result is true; i.e., does the fact that X contains uniformly isomorphic images of $(X \oplus X \oplus \cdots \oplus X)_{l_1^n}$, $n = 1, 2, \cdots$, imply that X contains l_1 isomorphically? An affirmative answer would give an alternative proof to the nonexistence of an isomorphically universal space in the family of all separable reflexive spaces as well as in the family of all spaces with a separable dual. (The nonexistence of these spaces was proved by W. Szlenk [8] by a completely different method.) Unfortunately the answer to Lindenstrauss' question is negative in a very strong sense.

THEOREM. *Let $1 \leqq p \leqq \infty$ and $\lambda > 1$. There exists a Banach space X with a 1-unconditional basis $\{e_i\}_{i=1}^{\infty}$ with the following properties:*

(a) *X is reflexive.*

(b) *X does not contain a subspace isomorphic to l_p (c_0 in the case $p = \infty$).*

For every $n = 1, 2, \cdots$ there exist n disjoint subsequences of the natural numbers $N_1, N_2, \cdots, N_n$ such that

(c) *$\{e_i\}_{i \in N_j}$; is isometrically equivalent to $\{e_i\}_{i=1}^{\infty}$, and*

(d) *If $x_j \in [e_i]_{i \in N_j}$; $j = 1, 2, \cdots, n$ then*

$$\lambda^{-1}\Big(\sum_{j=1}^{n} ||x_j||^p\Big)^{1/p} \leqq \Big|\Big|\sum_{j=1}^{n} x_j\Big|\Big| \leqq \lambda\Big(\sum_{j=1}^{n} ||x_j||^p\Big)^{1/p}$$

$$\Big(\lambda^{-1} \max_{1 \leqq j \leqq n} ||x_j|| \leqq \Big|\Big|\sum_{j=1}^{n} x_j\Big|\Big| \leqq \lambda \max_{1 \leqq j \leqq n} ||x_j|| \text{ if } p = \infty\Big).$$

(e) *There exists a $K < \infty$ such that X is K-isomorphic to $(X \oplus X \oplus \cdots \oplus X)_{l_p^n}$ for every n.*

The construction uses ideas from [9] and [1] as well as the basic idea of James to construct Banach spaces on trees. The notations are standard and can be found in [5] or [6].

Proof of the theorem. We first deal with the case $p = \infty$. Let $(T, \leqq)$ be the set

$$T = \{(n, i); n = 0, 1, \cdots, i = 1, \cdots, 2^n\}.$$

With the partial order

$$(n, i) \leqq (m, j) \text{ if and only if } n \leqq m \text{ and } (i-1)2^{m-n} < j \leqq i2^{m-n}.$$

Let L be the linear space of all the functions on T which differ from zero only on a finite number of points of T. For $n = 0, 1, \cdots$ and $i = 1, \cdots, 2^n$ define $e_{n,i} \in L$ by

$$e_{n,i}(m, j) = \begin{cases} 1 & (n, i) = (m, j) \\ 0 & \text{otherwise}. \end{cases}$$

And define the operators $P_{n,i}$, $S_{n,i}$, and P_n from L to L by

$$(P_{n,i}x)(m, j) = \begin{cases} x(m, j) & (n, i) \leqq (m, j), \quad x \in L \\ 0 & \text{otherwise} \end{cases}$$

$$(S_{n,i}x)(m, j) = x(m+n, (i-1)2^m + j), \quad x \in L$$

and

$$P_n = \sum_{i=1}^{2^n} P_{n,i}.$$

Now, we define on L a sequence of norms $\|\cdot\|_n$ by induction

$$\|x\|_0 = \|x\|_{l_1} = \sum_{n,i} |x(n, i)|$$

$$\|x\|_m = \inf\left\{\|x_0\|_{m-1} + \lambda \sum_{k=1}^{K} \max_{1 \leqq i \leqq 2^k} \|P_{k,i}x_k\|_{m-1}\right\}$$

where the inf is taken over all finite sequence $x_0, \cdots, x_K$ in L which satisfy

$$\sum_{k=0}^{K} x_k = x \quad \text{and} \quad P_k x_k = x_k, \quad k = 0, \cdots, K.$$

It is easy to prove by induction that for every $x \in L$ and every m

$$\|x\|_{c_0} \leqq \|x\|_m \leqq \|x\|_{m-1}.$$

So that we can define

$$\|x\| = \lim_{m \to \infty} \|x\|_m.$$

$||\cdot||$ is a norm. Let Y_m be the completion of L with respect to $||\cdot||_m$ and let Y be the completion of L with respect to $||\cdot||$.

LEMMA 1. (a) $\{e_{n,i}\}_{n=0,i=1}^{\infty\ \ 2^n}$ *is a 1-unconditional basis for* Y_m *and for* Y.

(b) *If* R *is a norm one projection on* $l_1(T)$ *such that* $P_{k,i}R = RP_{k,i}$, *for all* $k = 0, 1, \cdots$ *and* $i = 1, \cdots, 2^k$, *then* R *is a norm one projection on* Y_m *and on* Y.

(c) $S_{n,j}$ *is an isometry from* $P_{n,j}Y_m$ (resp. $P_{n,j}Y$) *onto* Y_m(resp. Y) *for all* $n = 0, 1, \cdots$, $j = 1, \cdots, 2^n$.

(d) *For every* $x \in L$ *the infimum in the definition of* $||x||_m$ *is attained.*

(e) *For every* $x \in L$

$$||x|| = \min\left\{||x_0||_{l_1} + \lambda \sum_{k=1}^{K} \max_{1\leq i\leq 2^k} ||P_{k,i}x_k||;\, x = \sum_{k=0}^{K} x_k,\ P_kx_k = x_k\right\}.$$

Proof. (a) and (b) are proven by induction and passing to the limit. (d) is a simple consequence of (b) (for $R = I - P_n$). We prove now (e). For every $\{x_k\}_{k=0}^K$ such that $x = \sum_{k=0}^K x_k$ and $P_kx_k = x_k$, $k = 0, \cdots, K$ and for all m

$$\begin{aligned}||x|| \leqq ||x||_m &\leqq ||x_0||_{m-1} + \lambda \sum_{k=1}^{K} \max_{1\leqq i\leqq 2^k} ||P_{k,i}x_k||_{m-1}\\ &\leqq ||x_0||_{l_1} + \lambda \sum_{k=1}^{K} \max_{1\leqq i\leqq 2^k} ||P_{k,i}x_k||_{m-1}\,.\end{aligned}$$

So, passing to the limit and using (b) to prove that the infimum is attained, we get

$$||x|| \leqq \min\left\{||x||_{l_1} + \lambda \sum_{k=1}^{K} \max_{1\leqq i\leqq 2^k} ||P_{k,i}x_k||;\, x = \sum_{k=0}^{K} x_k,\ P_kx_k = x_k\right\}.$$

In order to prove the other side inequality it is enough to prove that for all m and all $x \in L$

$$||x||_m \geqq \min\left\{||x_0||_{l_1} + \lambda \sum_{k=1}^{K} \max_{1\leqq i\leqq 2^k} ||P_{k,i}x_k||;\, x = \sum_{k=0}^{K} x_k,\ P_kx_k = x_k\right\}.$$

We prove this by induction on m. This is obvious for $m = 0$, assume it is true for $m - 1$ and assume that

$$||x||_m = ||x_0||_{m-1} + \lambda \sum_{k=1}^{K} \max_{1\leqq i\leqq 2^k} ||P_{k,i}x_k||_{m-1}$$

where $x = \sum_{k=0}^K x_k$ and $P_kx_k = x_k$, $k = 0, \cdots, K$.

By the induction hypothesis

$$||x_0||_{m-1} \geqq ||y_0||_{l_1} + \lambda \sum_{h=1}^{H} \max_{1\leqq i\leqq 2^h} ||P_{h,i}y_h||$$

for some $\{y_h\}_{h=0}^H$ such that $x_0 = \sum_{h=0}^H y_h$ and $P_h y_h = y_h$, $h = 0, \cdots, H$. We assume as we may that $H = K$, then $x = y_0 + \sum_{k=1}^K (x_k + y_k)$, $P_k(x_k + y_k) = x_k + y_k$, $k = 1, \cdots, K$ and

$$||x||_m \geqq ||y_0||_{l_1} + \lambda \sum_{k=1}^K \max_{1 \leqq i \leqq 2^k} ||P_{k,i}(x_k + y_k)|| .$$

To prove (c) it is clearly enough to show that for every x such that $P_{n,j}x = x$ and for every m

$$||x||_m = \min \left\{ ||x_n||_{m-1} + \lambda \sum_{k=n+1}^K \max_{1 \leqq i \leqq 2^k} ||P_{k,i}x_k||_{m-1} \right\}$$

where the minimum is over all the sequences $\{x_k\}_{k=n}^K$ such that $x = \sum_{k=n}^K x_k$ and $P_{n,j}P_k x_k = x_k$, $k = n, n+1, \cdots, K$.

Let x satisfy $P_{n,j}x = x$ and let $\{y_h\}_{h=0}^H$ be such that

$$||x||_m = ||y_0||_{m-1} + \lambda \sum_{h=1}^H \max_{1 \leqq i \leqq 2^h} ||P_{h,i}y_h||_{m-1} ,$$

$$x = \sum_{h=0}^H y_h \quad \text{and} \quad P_h y_h = y_h , \quad h = 0, \cdots, H .$$

We can assume that $H > n$ and by (a), we can also assume that $P_{n,j}y_h = y_h$, $h = 0, \cdots, H$.

$$\begin{aligned} ||x||_m &= ||y_0||_{m-1} + \lambda \sum_{h=1}^n \max_{1 \leqq i \leqq 2^h} ||P_{h,i}y_h||_{m-1} + \lambda \sum_{h=n+1}^H \max_{1 \leqq i \leqq 2^h} ||P_{h,i}y_h|| \\ &= ||y_0||_{m-1} + \lambda \sum_{h=1}^n ||y_h||_{m-1} + \lambda \sum_{h=n+1}^H \max_{1 \leqq i \leqq 2^h} ||P_{h,i}y_h|| . \end{aligned}$$

If $\sum_{h=1}^n ||y_h||_{m-1} > 0$ then since $\lambda > 1$

$$||x||_m > ||y_0 + y_1 + \cdots + y_n||_{m-1} + \lambda \sum_{h=n+1}^H \max_{1 \leqq i \leqq 2^h} ||P_{h,i}y_h||$$

in contradiction to the fact that the minimum is attained at $y_0, \cdots, y_H$. This concludes the proof of Lemma 1.

PROPOSITION 2. (a) *For every* $n = 0, 1, \cdots$ *and* $\{y_i\}_{i=1}^{2^n}$ *such that* $P_{n,i}y_i = y_i$, $i = 1, \cdots, 2^n$,

$$\max_{1 \leqq i \leqq 2^n} ||y_i|| \leqq \left\| \sum_{i=1}^{2^n} y_i \right\| \leqq \lambda \max_{1 \leqq i \leqq 2^n} ||y_i|| .$$

(b) *Y does not contain an isomorphic image of* c_0.

Proof. (a) The left hand side follows from the 1-unconditionality of $\{e_{n,i}\}_{n=0, i=1}^{\infty, 2^n}$. For the right hand side put

$$x_n = \sum_{i=1}^{2^n} y_i \quad \text{and} \quad x_k = 0 \quad \text{for} \quad k \neq n$$

then, by Lemma 1.e,

$$\left\|\sum_{i=1}^{2^n} y_i\right\| \leqq \lambda \max_{1\leqq i\leqq 2^n} \|P_{n,i}x_n\| = \lambda \max_{1\leqq i\leqq 2^n} \|y_i\| .$$

(b) Assume that Y contains an isomorph of c_0. Since the unit vector basis of c_0 tends weakly to zero, we can assume that there exist a sequence $\{u_n\}_{n=1}^{\infty}$ of norm one elements in Y, an increasing sequence $\{m_n\}_{n=1}^{\infty}$ of positive integers and a constant K such that

$$(P_{m_n} - P_{m_{n+1}})u_n = u_n , \quad n = 1, 2, \cdots$$

and

$$\max_{1\leqq n<\infty} |a_n| \leqq \left\|\sum_{n=1}^{\infty} a_n u_n\right\| \leqq K \max_{1\leqq n<\infty} |a_n|$$

for every sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n \to 0$ as $n \to \infty$. For every n let $1 \leqq i_n \leqq 2^{m_n}$ be such that

$$\|P_{m_n,i_n}u_n\| = \max_{1\leqq i\leqq 2^{m_n}} \|P_{m_n,i}u_n\|$$

and put

$$v_n = P_{m_n,i_n}u_n .$$

By part (a) and Lemma 1.a.

$$1 = \|u_n\| \leqq \lambda\|v_n\| \leqq \lambda\|u_n\| \leqq \lambda$$

and

$$\lambda^{-1} \max_{1\leqq n<\infty} |a_n| \leqq \left\|\sum_{n=1}^{\infty} a_n v_n\right\| \leqq \left\|\sum_{n=1}^{\infty} a_n u_n\right\| \leqq K \max_{1\leqq n<\infty} |a_n|$$

for every sequence $\{a_n\}_{n=1}^{\infty}$ such that $a_n \to 0$ as $n \to \infty$. We also have $P_{m_n,i_n}v_n = v_n$ $n = 1, 2, \cdots$. By passing to a subsequence we can also assume that

$$P_{m_n,i_n}v_r = v_r \quad \text{for all} \quad r \geqq n .$$

This last property (with other m_n's) remains true for every block basis of the u_n's. Thus, by a theorem of James [3], we may assume that there exist an n, a $1 \leqq j \leqq 2^n$ and two normalized vectors w_1, w_2 in Y such that

$$(I - P_n)w_1 = w_1 , \quad P_{n,j}w_2 = w_2 \quad \text{and} \quad \|w_1 + w_2\| < \lambda - \varepsilon \quad \text{where}$$

$\varepsilon > 0$ satisfies $1 < \lambda - \varepsilon < 1 + \varepsilon/\lambda$. Let $\{x_k\}_{k=0}^{K}$ be such that $w_1 + w_2 = \sum_{k=0}^{K} x_k$, $P_k x_k = x_k$, $k = 0, \cdots, K$ and

$$(*) \qquad \|w_1 + w_2\| = \|x_0\|_{l_1} + \lambda \sum_{k=1}^{K} \max_{1\leqq i\leqq 2^k} \|P_{k,i}x_k\|$$

(such x_k's exist by Lemma 1.e). We can also assume that $K \geqq n$ and that $\operatorname{supp} x_k \subseteqq \operatorname{supp}(w_1 + w_2)$, $k = 0, \cdots, K$. We first prove that

$$(**) \qquad \left\|\sum_{k=1}^{n-1} P_n x_k\right\| \leqq \frac{\lambda - \varepsilon}{\lambda}.$$

If this were not true then, since $P_{n,j} P_n x_k = P_n x_k$ for $k = 0, \cdots, K$,

$$\begin{aligned} \lambda - \varepsilon > \|w_1 + w_2\| &\geqq \lambda \sum_{k=1}^{n-1} \max_{1 \leqq i \leqq 2^k} \|P_{k,i} P_n x_k\| \\ &= \lambda \sum_{k=1}^{n-1} \|P_n x_k\| \geqq \lambda \left\|\sum_{k=1}^{n-1} P_n x_k\right\| > \lambda - \varepsilon . \end{aligned}$$

From (**), we get that

$$(***) \qquad \left\|P_n x_0 + \sum_{k=n}^{K} x_k\right\| \geqq \frac{\varepsilon}{\lambda}.$$

Indeed,

$$\begin{aligned} \left\|P_n x_0 + \sum_{k=n}^{K} x_k\right\| &= \left\|P_n x_0 + \sum_{k=n}^{K} P_n x_k\right\| \\ &\geqq \left\|P_n\Big(\sum_{k=0}^{K} x_k\Big)\right\| - \left\|\sum_{k=1}^{n-1} P_n x_k\right\| \\ &= \|w_2\| - \left\|\sum_{k=1}^{n-1} P_n x_k\right\| \geqq 1 - \frac{\lambda - \varepsilon}{\lambda} = \frac{\varepsilon}{\lambda}. \end{aligned}$$

Now, by Lemma 1.e, the equalities

$$w_1 = \sum_{k=0}^{n-1} (I - P_n) x_k , \quad P_k(I - P_n) x_k = (I - P_n) x_k , \quad k = 0, \cdots, n-1$$

and

$$P_n x_0 + \sum_{k=n}^{K} x_k = P_n x_0 + \sum_{k=n}^{K} x_k , \quad P_k x_k = x_k , \quad k = 0, n, n+1, \cdots, K ,$$

(*) and (***) we get

$$\begin{aligned} \lambda - \varepsilon > \|w_1 + w_2\| &\geqq \|(I - P_n) x_0\|_{l_1} + \lambda \sum_{k=1}^{n-1} \max_{1 \leqq i \leqq 2^k} \|P_{k,i}(I - P_n) x_k\| \\ &\quad + \|P_n x_0\| + \lambda \sum_{k=n}^{K} \max_{1 \leqq i \leqq 2^k} \|P_{k,i} x_k\| \\ &\geqq \|w_1\| + \left\|P_n x_0 + \sum_{k=n}^{K} x_k\right\| \geqq 1 + \frac{\varepsilon}{\lambda} \end{aligned}$$

which contradicts the choice of ε. This concludes the proof of Proposition 2.

The space Y satisfies (b), (c) and (d) of the theorem for $p = \infty$ this follows from 2.b, 1.c and 2.a, respectively it is also not hard to see that Y satisfies (e), however (a) is not satisfied, indeed, if $\{(n_k, i_k)\}_{k=1}^{\infty}$ is a totally ordered sequence in T then it is not difficult to see (using 1.e.) that $[e_{n_k,i_k}]_{k=1}^{\infty}$ is isometric to l_1, so some additional work is needed.

Proof of theorem for $p = \infty$. Define on L a new norm by

$$|||x||| = ||\,|x|^2||^{1/2} \quad x \in L$$

(for $x = \sum_{n,i} a_{n,i}e_{n,i}$ $|x|^{\alpha}$ is defined to be $\sum_{n,i} |a_{n,i}|^{\alpha} e_{n,i}$), and let X be the completion of L with respect to this norm. It is easy to check that $\{e_{n,i}\}_{n=0,i=1}^{\infty,\ 2^n}$ constitutes a 1-unconditional basis for X. Now, if $\{x_m\}_{m=1}^{M}$ is a block basis of $\{e_{n,i}\}_{n=0,i=1}^{\infty,\ 2^n}$ then

$$a \max_{1 \leqq m \leqq M} |a_m| \leqq \left\| \sum_{m=1}^{M} a_m x_m \right\| \leqq b \max_{1 \leqq m \leqq M} |a_m| \quad \text{for all} \quad a_1, \cdots, a_M$$

if and only if

$$a^{1/2} \max_{1 \leqq m \leqq M} |a_m| \leqq \left|\left|\left| \sum_{m=1}^{M} a_m |x_m|^{1/2} \right|\right|\right| \leqq b^{1/2} \max_{1 \leqq m \leqq M} |a_m| \quad \text{for all} \quad a_1, \cdots, a_M \,.$$

This proves that (b), (c) and (d) of the Theorem remain valid for X (with $\lambda^{1/2}$ instead of λ). In order to prove (a) it is enough, by James theorem [2] to prove that X does contain an isomorph of ℓ_1. This in turn is a consequence of the following simple fact: if $\{x_m\}_{m=1}^{M}$ are disjointly supported with respect to $\{e_{n,i}\}_{n=0,i=1}^{\infty,\ 2^n}$ then

$$\left|\left|\left| \sum_{m=1}^{M} x_m \right|\right|\right| \leqq \left(\sum_{m=1}^{M} |||x_m|||^2 \right)^{1/2} .$$

To prove (e) it is enough, in view of (c), (d) and Pelczynski's decomposition method [7], to prove that X is isomorphic to $X \oplus X$. Now, as we mentioned above for any totally ordered sequence $\{(n_k, i_k)\}_{k=1}^{\infty}$ in T $\{e_{n_k,i_k}\}_{k=1}^{\infty}$ in Y is equivalent to the unit vector basis in l_1 thus, $\{e_{n_k,i_k}\}_{k=1}^{\infty}$ in X is equivalent to the unit vector basis in l_2. So, X contains a copy of ℓ_2 and therefore is isomorphic to each of its one co-dimensional subspaces. In particular to $[e_{n,i}]_{n=1,i=1}^{\infty,\ 2^n}$ which, in turn is isomorphic to $X \oplus X$.

Proof of the theorem for $1 \leqq p < \infty$. Let X and $\{e_i\}_{i=1}^{\infty}$ be the space and the basis which satisfy the theorem for $p = \infty$ and let $\{f_i\}_{i=1}^{\infty}$ be the biorthogonal basis of $\{e_i\}_{i=1}^{\infty}$ then clearly X^* and $\{f_i\}_{i=1}^{\infty}$ satisfy the theorem for $p = 1$.

For $p > 1$ define, for every eventually zero sequence $\{a_i\}_{i=1}^{\infty}$,

$$||\{a_i\}_{i=1}^{\infty}||_p = \left\| \sum_{i=1}^{\infty} |a_i|^p f_i \right\|^{1/p} .$$

Considerations similar to those in the proof of the $p = \infty$ case show that the completion of the space of finite sequences under $\|\cdot\|_p$ satisfies the theorem.

REMARK. It may be useful to know what is the dual norm to $\|\cdot\|$. Define on L a sequence of norms as follows

$$|x|_0 = \|x\|_{c_0}$$
$$|x|_m = \max\left\{|x|_{m-1}, \lambda^{-1} \max_{1 \leq k < \infty} \sum_{i=1}^{2^n} |P_{k,i}x|_{m-1}\right\}$$

and define

$$|x| = \lim_{m\to\infty} |x|_m .$$

It can be shown that for every $x \in L$

$$|x| = \max\left\{\|x\|_{c_0}, \lambda^{-1} \max_{1 \leq k < \infty} \sum_{i=1}^{2^k} |P_{k,i}x|\right\}$$

and that $\{[e_{n,i}]_{n=0,i=1}^{N\ \ 2^n}, |\cdot|\}$ is the dual of $\{[e_{n,i}]_{n=0,i=1}^{N\ \ 2^n}, \|\cdot\|\}$.

Once this duality is proved it can be used to simplify the proof of the theorem, in particular the proof of Proposition 2.b. We prefered, however, to give a proof which avoids the routine proof of the duality.

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