PACIFIC JOURNAL OF MATHFMATICS
Vol. 84, No. 2, 1979

OPERATORS SATISFYING A G_1 CONDITION

C. R. PUTNAM

An operator T on a Hilbert space is said to be G_1 if $||(T-z)^{-1}||=1/\mathrm{dist}(z, \sigma(T))$ for $z \notin \sigma(T)$ and completely G_1 if, in addition, T has no normal part. Certain results are obtained concerning the spectra of completely G_1 operators and of their real parts. It is shown in particular that there exist completely G_1 operators having spectra of zero Hausdorff dimension. Some sparseness conditions on the spectrum are given which assure that a G_1 operator has a normal part.

1. **Introduction.** All operators considered in this paper will be bounded (linear) on a Hilbert space $\mathfrak{H}$ of elements x. For any such operator T it is well-known (and due to Wintner [26]) that

$$||(T-z)^{-1}|| \geqq 1/\mathrm{dist}(z, \sigma(T))$$

for $z \notin \sigma(T)$ and $||(T-z)^{-1}|| \leqq 1/\mathrm{dist}(z, W^-(T))$ for $z \notin W^-(T)$, where $\sigma(T)$ denotes the spectrum of T and $W^-(T)$ denotes the (convex) closure of the numerical range $W(T) = \{(Tx, x): ||x|| = 1\}$. An operator T is said to be G_1 (or to satisfy a G_1 condition, or to be of class G_1) if

$$(1.1) \qquad ||(T-z)^{-1}|| = 1/\mathrm{dist}(z, \sigma(T)) \quad \text{for} \quad z \notin \sigma(T) .$$

For instance, (1.1) holds for operators T which are normal ($T^*T - TT^* = 0$), more generally, for those which are subnormal (T has a normal extension on a larger Hilbert space), and still more generally, for hyponormal operators ($T^*T - TT^* \geqq 0$). The inclusions indicated here,

$$(1.2) \qquad \text{normals} \subset \text{subnormals} \subset \text{hyponormals} \subset (G_1) ,$$

are all proper and, needless to say, the simple stratification (1.2) can be interstitially (and endlessly) refined. In this connection, see the brief survey in Putnam [16].

An operator T will be called completely G_1 if T is G_1 and if, in addition, T has no normal part, that is, T has no reducing subspace on which it is normal. Similarly, one has corresponding definitions of completely subnormal or completely hyponormal operators. It is well-known that every compact set of the plane is the spectrum of some normal operator. Moreover, necessary and sufficient conditions are known in order that a compact set be the spectrum of a completely subnormal operator (Clancey and Putnam [4]) or of a completely hyponormal operator (Putnam [15], [17]). On the other hand, no such conditions are known for the class of completely G_1 operators.

It may be noted that if T is G_1 and if $\sigma(T)$ is finite, in particular, if $\mathscr{H}$ is finite-dimensional, then necessarily T is normal. In fact, Stampfli [20], p. 473, shows that if T is G_1 and if z_0 is an isolated point of $\sigma(T)$ then z_0 is a normal eigenvalue of T, that is, $z_0 \in \sigma_p(T)$, the point spectrum of T, and the corresponding eigenvectors form a reducing space of T on which T is normal. (For some related results, see also Hildebrandt [8], p. 234, and Luecke [10], p. 631.) More generally, it was shown by Stampfli ([22], [23]) that if T is G_1 and if $\sigma(T)$ is a subset of a smooth (C^2) curve then T is normal. In fact, he even obtains a local version of this result. Thus, if $z_0 \in \sigma(T)$ and if D is an open disk centered at z_0 for which $\sigma(T) \cap D$ lies on a smooth curve and for which T is only locally G_1, so that (1.1) is assumed only in $D - \sigma(T)$, then T has a representation $T = T_1 \oplus T_2$ where T_1 is normal with spectrum $(\sigma(T) \cap D)^-$ and T_2 has a spectrum contained in $\sigma(T) - D$. On the other hand, as Stampfli has shown ([20], p. 474; [22], p. 9), it is possible that (1.1) holds and that $\sigma(T)$ is even a countable subset of a curve $z = z(t)$, $0 \leqq t \leqq 1$, where $z(t)$ is C^2 for $0 \leqq t < 1$, but T fails to be normal. In [10], Luecke shows that if $\sigma(T)$ is countable and has the property that for any $z \in \sigma(T)$ there exists some $w \notin \sigma(T)$ for which $|z - w| = \mathrm{dist}(w, \sigma(T))$, then, in general, T need not be normal. However, if, in addition, T is assumed to be a scalar operator, then it must indeed be normal.

All of this suggests that a simple necessary and sufficient condition on a compact set in order that it be the spectrum of a completely G_1 operator is not easily obtained. In fact, even such a condition on a countable compact set in order that it be the spectrum of a nonnormal operator of class G_1 is not known. (A sufficient condition for normality is that of Luecke [10] mentioned above; another is given in Theorem 2 below.) Of course, any G_1 operator having a countable spectrum certainly has a normal part. It is thus clear that a necessary condition on a compact set, X, in order that it be the spectrum of a completely G_1 operator is that X be perfect. In order to describe certain types of sets X occurring below, it will be convenient to recall the definition of Hausdorff measure.

A "measure function" $h(t)$ is an increasing continuous function on $0 \leqq t < \infty$ satisfying $h(0) = 0$. For a bounded set, X, of the complex plane and a fixed $\delta > 0$ let $\Gamma = \{D_1, D_2, \cdots\}$ be any countable covering of X by open disks D_j of radius $\delta_j \leqq \delta$. Then $\Lambda_h(X) = \lim_{\delta \to 0} [\inf \sum_{j=1}^{\infty} h(\delta_j)]$ exists and is the Hausdorff h-measure of X. (See Garnett [5], p. 58; also Carleson [2], Rogers [19].) If $h(t) = t^r$, $r > 0$, then $\Lambda_h(X)$ is the r-dimensional Hausdorff measure of X. In particular, a nonempty set X is said to have Hausdorff dimension $= 0$ if $\Lambda_h(X) = 0$ for all $h = t^r$, $r > 0$.

2. THEOREM 1. *For any given measure function h there exists a perfect set X of the complex plane and a completely G_1 operator T for which $X = \sigma(T)$ has Hausdorff h-measure $= 0$.*

It may be noted that, in particular, there exist completely G_1 operators with spectra of Hausdorff dimension $= 0$. That the function h of Theorem 1 be preassigned is an essential requirement however. In fact, the condition that $\Lambda_h(\sigma(T)) = 0$ for all measure functions h is sufficient (as well as necessary) in order that $\sigma(T)$ be countable; see Rogers [19], p. 67.

Proof. As in Stampfli ([20], [22]), consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{2.1}$$

acting on a two-dimensional Hilbert space, so that $(A - z)^{-1} = \begin{pmatrix} -1/z & -1/z^2 \\ 0 & -1/z \end{pmatrix}$, and hence $\|(A - z)^{-1}\| \leqq 1/|z| + 1/|z|^2$ for all $z \notin \sigma(A) = \{0\}$. Note also that $W(A)(= W^-(A)) = \{z: |z| \leqq 1/2\}$ and $\|A\| = 1$. Then $\|(A - z)^{-1}\| \leqq (|z| - 1/2)^{-1}$ for $|z| > 1/2$ and clearly there exists a countable set $\alpha = \{z_1, z_2, \cdots\} \subset \{z: 0 < |z| < 1\}$ satisfying $z_n \to 0$ as $n \to \infty$ and such that

$$\|(A - z)^{-1}\| \leqq 1/\mathrm{dist}(z, \alpha) \quad \text{for} \quad z \neq 0\,. \tag{2.2}$$

Next, choose a sequence of nonoverlapping open disks $\{D_1, D_2, \cdots\}$, where each D_n has center z_n and is contained in $\{z: 0 < |z| < 1\}$. Let $A_n = a_n A + z_n$, where $0 < a_n <$ radius D_n, so that $\|A_n - z_n\| =$ radius D_n and $\sigma(A_n) = \{z_n\}$. Then, for each $n = 1, 2, \cdots$, choose a countable set $\alpha_n = \{z_{n1}, z_{n2}, \cdots\} \subset D_n$ satisfying $z_{nk} \neq z_n$ and $z_{nk} \to z_n$ as $k \to \infty$ and the inequality $\|(A_n - z)^{-1}\| \leqq 1/\mathrm{dist}(z, \alpha_n)$ for $z \neq z_n$. Thus, if $T_0 = A$ and $T_1 = \sum \oplus A_n$, one sees that

$$\begin{aligned} &T = T_0 \oplus T_1 \text{ satisfies } \|(T - z)^{-1}\| \leqq 1/\mathrm{dist}(z, \cup\, \alpha_n) \text{ for} \\ &\qquad z \notin \sigma(T) = \{0\} \cup \alpha\,. \end{aligned} \tag{2.3}$$

In the next step each of the disks D_n plays the role of the containing disk $\{z: |z| < 1\}$ in the previous construction. Thus, for each $n = 1, 2, \cdots$, one chooses a sequence of nonoverlapping open disks $\{D_{n1}, D_{n2}, \cdots\}$, contained in D_n and clustering at z_n, and obtains a new operator T_2 for which $T = T_0 \oplus T_1 \oplus T_2$ satisfies a condition analogous to (2.2) for $T = T_0$ and to (2.3) for $T = T_0 \oplus T_1$. Continuation of this process leads to an operator $T = \sum_{k=0}^{\infty} \oplus T_k$ satisfying

$$\|(T - z)^{-1}\| \leqq 1/\mathrm{dist}(z, X) \quad \text{for} \quad z \notin X\,, \tag{2.4}$$

where X is the closure of the set of all centers of circles occurring in the above construction. Since $X \subset \sigma(T)$ then, by (2.4), $\sigma(T) = X$ and T satisfies (1.1). Moreover, it is clear that T is a completely G_1 operator. Further, the inclusions

$$\{z: z < 1\} \supset [\cup D_n \cup \{0\}] \supset [\cup D_{nk} \cup \{0, z_1, z_2, \cdots\}] \supset \cdots \supset \sigma(T)$$

show that, for any given measure function h, one can always choose the countable collection of disks $\{D_n\}$, $\{D_{nk}\}$, $\cdots$, in such a way that $\sigma(T)$ has Hausdorff h-measure $= 0$. This completes the proof of Theorem 1.

COROLLARY 1. *If X denotes an arbitrary compact set of the plane and if h is any measure function, then there exists a perfect set $P \supset X$ and a completely G_1 operator T such that $P - X$ has Hausdorff h-measure $= 0$ and $\sigma(T) = P$.*

Proof. Let $\{z_1, z_2, \cdots\}$ be any countable subset of X dense in X. For each $n = 1, 2, \cdots$, let D_n be an open disk centered at z_n and suppose that diam $D_n \to 0$ as $n \to \infty$. Then let T_n be a completely G_1 operator having spectrum of Hausdorff h-measure $= 0$ and such that $z_n \in \sigma(T_n) \subset D_n$. One need only choose T_n, for instance, to be an appropriate linear function of the operator T constructed in the proof of Theorem 1. (Note that the G_1 property is invariant under linear transformations; see Luecke [11], p. 36.) If $T = \sum \oplus T_n$ then, since each T_n is G_1, $\sigma(T) = (\cup \sigma(T_n))^-$ and hence, since diam $D_n \to 0$ as $n \to \infty$, $\sigma(T) = \bigcup \sigma(T_n) \cup X = P$ satisfies the conditions stated in the corollary.

A related result is the following

COROLLARY 2. *If B is any operator and h is any measure function there exists a completely G_1 operator T for which $B \oplus T$ is also G_1 and $\sigma(T) \subset \{\partial(\sigma(B)) \cup \beta\}$ where β has Hausdorff h-measure $= 0$.*

Proof. Choose a sequence of points $\alpha = \{z_1, z_2, \cdots\}$ in such a way that no z_n lies in $\sigma(B)$, $\mathrm{dist}(z_n, \sigma(T)) \to 0$ as $n \to \infty$, and such that $\|(B - z)^{-1}\| \leqq 1/\mathrm{dist}(z, \alpha)$ for $z \notin \sigma(B)$. Then choose a sequence of open disks $\{D_1, D_2, \cdots\}$, where z_n is the center of D_n, satisfying $D_n \cap \sigma(B) = \varnothing$ and diam $D_n \to 0$ as $n \to \infty$, so that the D_n's cluster only on the set $\partial(\sigma(B))$. If $T_1, T_2, \cdots$ are G_1 operators such that $z_n \in \sigma(T_n) \subset D_n$ and $\sigma(T_n)$ has Hausdorff h-measure $= 0$, then $T =$

$\sum \oplus T_n$ satisfies the conditions stated in the corollary.

3. Some lemmas. If $\{A_1, A_2, \cdots\}$ is a decreasing sequence of self-adjoint operators then the A_n converge strongly to a (self-adjoint) operator A, a result due to Vigier (see Riesz and Sz.-Nagy [18], p. 263). In particular, if each A_n is an orthogonal projection, so also is A. Further, it is well-known that a projection $P (P = P^2)$ is orthogonal if and only if $||P|| \leqq 1$. We shall need need the following generalization to arbitrary projections P_n of the above results.

LEMMA 1. *Let $\{P_1, P_2, \cdots\}$ be a sequence of projections ($P_n = P_n^2$) satisfying*

$$P_n P_{n+p} = P_{n+p} \quad (n = 1, 2, \cdots; p = 0, 1, 2, \cdots) \tag{3.1}$$

and

$$\limsup_{n\to\infty} ||P_n|| \leqq 1 . \tag{3.2}$$

Then the P_n converge strongly as $n \to \infty$ to an orthogonal projection.

Proof. First, let P denote any projection and let $t \geqq 0$ satisfy

$$||P|| \leqq 1 + t . \tag{3.3}$$

Since $P^2 = P$, the range of P^* is orthogonal to the range of $I - P$ and hence, if x is arbitrary in $\mathfrak{H}$ and $y = P^*x$, then $y = P^*y \perp (I - P)y$. Since $Py = y - (I - P)y$, then

$$||y||^2 + ||(I - P)y||^2 = ||Py||^2 \leqq (1 + t)^2 ||y||^2 ,$$

and so $||(I - P)P^*x||^2 \leqq (2t + t^2)||P^*x||^2$. Consequently,

$$||P - PP^*|| = ||P^* - PP^*|| \leqq t^{1/2}(2 + t)^{1/2}(1 + t) , \tag{3.4}$$

and hence

$$||P - P^*|| \leqq 2t^{1/2}(2 + t)^{1/2}(1 + t) . \tag{3.5}$$

Relations (3.2) and (3.5) (with P replaced by P_n) imply that $||P_n - P_n^*|| \to 0$ as $n \to \infty$. Further, if $Q_n = P_n P_n^*$, also $||Q_n - P_n|| \to 0$ as $n \to \infty$, and hence, by (3.1), $||Q_n Q_{n+p} - Q_{n+p}|| \to 0$ as $n \to \infty$ (uniformly in $p \geqq 0$). Similarly, $||Q_n Q_{n+p} - Q_{n+p} Q_n|| \to 0$ as $n \to \infty$ (uniformly in $p \geqq 0$) and hence also $||Q_n(I - Q_{n+p}) - Q_n^{1/2}(I - Q_{n+p})Q_n^{1/2}|| \to 0$ (uniformly in $p \geqq 0$). It follows that there exists a sequence of positive numbers $\{t_1, t_2, \cdots,\}$ with limit 0 for which

$$A_{np} \equiv Q_n - Q_{n+p} + t_n \geqq 0 \text{ for all } n \geqq 1 \text{ and } p \geqq 0 . \tag{3.6}$$

If x is arbitrary in $\mathfrak{H}$, then clearly one can choose integers $n = n_k \to \infty$ and $p = p_k \to \infty$ so that $(Q_{n_k}x, x) \to \liminf_{n\to\infty}(Q_n x, x)$ and also $(Q_{n_k+p_k}x, x) \to \limsup_{n\to\infty}(Q_n x, x)$. Hence, by (3.6),

$$\lim_{n\to\infty}(Q_n x, x) \text{ exists, for each } x \text{ in } \mathfrak{H} \,. \tag{3.7}$$

An argument like that in Riesz and Sz.-Nagy [18], p. 263, shows that $\|A_{np}x\|^4 = (A_{np}x, A_{np}x)^2 \leqq (A_{np}x, x)(A_{np}^2x, A_{np}x)$ and hence, by (3.7) and the definition of A_{np} in (3.6), $(Q_n - Q_{n+p})x \to 0$ (strongly) as $n \to \infty$ (uniformly in $p \geqq 0$), so that $Q = s\text{-}\lim_{n\to\infty} Q_n$ exists and is self-adjoint. Since $\|Q_n - P_n\| \to 0$, then $s\text{-}\lim_{n\to\infty} P_n = Q$ is an orthogonal projection and the proof of Lemma 1 is complete.

LEMMA 2. *Let T be a G_1 operator and suppose that $z_0 \in \sigma(T)$. In addition, suppose that there exists a sequence of circles $C_n = \{z: |z - z_0| = r_n\}$, $n = 1, 2, \cdots$, lying in the resolvent set of T, and for which $r_1 > r_1 > \cdots \to 0$ and*

$$r_n/\text{dist}(C_n, \sigma(T)) \longrightarrow 1 \quad \textit{as} \quad n \longrightarrow \infty \,. \tag{3.8}$$

If each C_n is positively oriented and if P_n denotes the projection

$$P_n = -(2\pi i)^{-1}\int_{C_n}(T - z)^{-1}dz \quad (n = 1, 2, \cdots)\,, \tag{3.9}$$

then $P_n \to P$ (strongly), where P is an orthogonal projection commuting with T, and

$$(T - z_0)P = 0 \,. \tag{3.10}$$

Proof. That the P_n satisfy (3.1) follows from a computation similar to that in Riesz and Sz.-Nagy [18], p. 419. In addition, it is clear that

$$\|P_n\| \leqq (2\pi)^{-1}\Big(\max_{z \text{ on } C_n}\|(T - z)^{-1}\|\Big)2\pi r_n \leqq r_n/\text{dist}(C_n, \sigma(T))\,, \tag{3.11}$$

so that (3.8) implies (3.2). Thus, by Lemma 1, $P_n \to P$ (strongly), where P is an orthogonal projection. Since $P_nT = TP_n$, then also $PT = TP$. Relation (3.10) follows from the limit relation $r_n \to 0$ and an estimate of $(T - z_0)P = -(2\pi i)^{-1}\int_{C_n}(z - z_0)(T - z)^{-1}dz$ similar to that of (3.11).

LEMMA 3. *Let T be an arbitrary operator and suppose that $z_0 \in \sigma_p(T)$. In addition, suppose that there exist $z_n \notin \sigma(T)$ such that $z_n \to z_0$ and $|z_n - z_0|\,\|(T - z_n)^{-1}\| \to 1$ as $n \to \infty$. Then z_0 is a normal eigenvalue of T.*

Proof. The result was given in Putnam [14] and, before this, implicitly in Stampfli [21] (cf. Stampfli's remark in [24], p. 135). A A variation appears earlier in Sz.-Nagy and Foiaş [25], p. 93. See also Hildebrandt [8], p. 234.

REMARK. Let T be G_1. It is clear from Lemma 3 that if $z_0 \in \sigma_p(T)$ and if

$$z_n \notin \sigma(T),\ z_n \longrightarrow z_0 \text{ and } \operatorname{dist}(z_n, \sigma(T))/|z_n - z_0| \longrightarrow 1 \quad \text{as } n \longrightarrow \infty\,, \tag{3.12}$$

then z_0 is a normal eigenvalue of T. In Lemma 2, it is assumed only that z_0 is in $\sigma(T)$ but not necessarily in $\sigma_p(T)$. On the other hand, the condition (3.8) for such a z_0 is clearly much stronger than (3.12). Since T commutes with P, relation (3.10) implies that if $P \neq 0$ then necessarily z_0 is a normal eigenvalue of T.

If only $z_0 \in \sigma(T)$ is assumed, it may be noted that (3.12) may hold for a completely G_1 operator, so that, in particular, $z_0 \notin \sigma_p(T)$. For example, let T be a completely G_1 operator as constructed in the proof of Theorem 1, so that T has the form $T = \sum \oplus (b_n A + w_n)$, where $b_n > 0$ and A is given by (2.1). If $s = \sup \operatorname{Re} \sigma(T)$, then there exists some $z_0 \in \sigma(T)$ with $s = \operatorname{Re} z_0$, and hence (3.12) holds with, say, $z_n = z_0 + c_n$, where $0 < c_n \to 0$.

Further, note that it is possible that T is G_1 with $z_0 \in \sigma_p(T)$ and that there exist circles $C_n = \{z: |z - z_0| = r_n\}$, $n = 1, 2, \cdots$, lying in the resolvent set of T and satisfying $r_1 > r_2 > \cdots \to 0$ and for which the projections P_n of (3.9) are orthogonal and converge strongly to an orthogonal projection $P \neq 0$, but for which z_0 is not a normal eigenvalue of T. Thus, (3.10) need not hold if (3.8) is not assumed, even though the other hypotheses of Lemma 2 are retained.

A simple example is obtained by considering the construction of Stampfli ([20], [22]), with

$$T = A \oplus N\,, \tag{3.13}$$

where A is given by (2.1) and N is normal with spectrum α^-. Here α is defined as in the beginning of the proof of Theorem 1 and, in particular, (2.2) holds. Clearly, for $z_0 = 0$, there exist circles $C_n = \{z: |z| = r_n\}$ lying in the resolvent set of T with $r_1 > r_2 > \cdots \to 0$. It is seen that each P_n is an orthogonal projection. Further, if A acts on the two-dimensional space $\mathfrak{H}_0$ then $P_n \to P$ (strongly), where P is the projection of $\mathfrak{H}$ onto $\mathfrak{H}_0$. Although $z_0 \in \sigma_p(T)$, it is clear that z_0 is not a normal eigenvalue of T.

The above procedure can be modified so as to yield a completely G_1 operator T. One need only consider the operator T constructed in the proof of Theorem 1 above where the numbers $z_1, z_2, \cdots$, and the first sequence of disks $\{D_1, D_2, \cdots\}$, with $D_n = \{z: |z - z_n| < r_n\}$, are chosen so that $(0, t) \cap \bigcup_{n=1}^{\infty} (|z_n| - r_n, |z_n| + r_n) \neq (0, t)$ for all $t > 0$. This enables one to choose circles C_n as in the preceding paragraph and to proceed in a manner similar to that described there.

4. THEOREM 2. *Let T be G_1 and suppose that $\sigma(T)$ is not a perfect set and that for each $z_0 \in \sigma(T)$ there exists a sequence of circles $C_n = \{z: |z - z_0| = r_n\}$, $n = 1, 2, \cdots$, lying in the resolvent of T for which $r_1 > r_2 > \cdots \to 0$ and (3.8) holds. Then*

$$T \text{ is normal if } \sigma(T) \text{ is countable}, \tag{4.1}$$

and

$$T = T_1 \oplus T_2 \text{ if } \sigma(T) \text{ is not countable}, \tag{4.2}$$

where T_1 is normal with $\sigma(T_1) = \alpha^-$ and α a countable set, and where $\sigma(T_2)$ is perfect and $\sigma(T_2) \cap \alpha = \varnothing$.

Proof. Since $\sigma(T)$ is not perfect, $\sigma(T)$ contains a nonempty (countable) set, S_0, of isolated points. Hence, as noted earlier, T has a normal part N_0 corresponding to these points with $\sigma(N_0) = S_0^-$. In case $S_0 = \sigma(T)$, the proof is complete. Otherwise, as will be assumed, $T = N_0 \oplus A_0$, where $\sigma(A_0) \cap S_0 = \varnothing$, and we let S_1 denote the (countable) set of isolated points of the first derivative, $\sigma'(T)$, of $\sigma(T)$. If S_1 is empty the proof is over and so we can suppose that $S_1 \neq \varnothing$. It follows from (3.10) of Lemma 2 that each point z_0 of S_1 either corresponds to a normal eigenvalue (if $P \neq 0$), or, if $P = 0$, can simply be ignored. Thus, at the end of the second stage we have $T = N_1 \oplus A_1$ where $\sigma(N_1) = S_0^- \cup S_1^-$ and, if A_1 is present, $\sigma(A_1) \cap (S_0 \cup S_1) = \varnothing$. One then repeats this process. It should be noted that for $n = 0, 1, 2, \cdots$, $S_n = \sigma^{(n)}(T) - \sigma^{(n+1)}(T)$, where $\sigma^{(n)}(T)$ denotes the nth derived set of $\sigma(T) \equiv \sigma^{(0)}(T)$. If for any positive integer n, S_n is empty, the process terminates. In addition, if $\sigma(T) = \bigcup_{n=0}^{\infty} S_n$, the process also terminates, and, of course, implies that T is normal and that $\sigma(T)$ is countable. Otherwise, the process continues via transfinite induction as noted below.

The νth derived set of $\sigma(T)$ can be defined, in the manner of Cantor using transfinite induction, for any ordinal ν; see Kamke [9], p. 127. It follows from a transfinite induction argument ([9], pp. 132–133) that there is a least ordinal γ, where $0 \leqq$ cardinality of $\gamma \leqq \aleph_0$, with the property that $\sigma^{(\gamma)}(T) = \sigma^{(\alpha)}(T)$ for all ordinals $\alpha \geqq \gamma$. In particular, if $\sigma^{(\gamma)}(T)$ is not empty then it is perfect. It follows (cf. [9], p. 133) that if $\sigma(T)$ is countable, then $\sigma^{(\gamma)}(T)$ is empty and, by the process described in the preceding paragraph, (4.1) is established. If $\sigma(T)$ is not countable then $\sigma^{(\gamma)}(T)$ is perfect and so (4.2) holds with the properties described in Theorem 2.

5. THEOREM 3. *Let T be G_1. Suppose that for every $\varepsilon > 0$ there exists a countable covering of $\sigma(T)$ by open disks $D_n = \{z: |z - z_n| < r_n\}$, $n = 1, 2, \cdots$, with the properties that, for each n, $D_n \cap \sigma(T) \neq \emptyset$ and $C_n = \{z: |z - z_n| = r_n\}$ lies in the resolvent set of T, and that*

$$(5.1)\qquad \sum_n (r_n/d_n - 1)^{1/2} < \varepsilon\ , \quad \textit{where} \quad d_n = \operatorname{dist}(C_n, \sigma(T)) \quad (\leqq r_n)\ ,$$

and

$$(5.2)\qquad \sum_n r_n < \varepsilon\ .$$

Then T is normal.

Proof. Let $\varepsilon > 0$ be fixed. In view of the Heine-Borel theorem it may be suppose that the covering of Theorem 3 is finite, say $\{D_1, \cdots, D_N\}$, and that $D_n \not\subset D_m$ for $n \neq m$. For $n = 1, \cdots, N$, define $P_n = -(2\pi i)^{-1}\int_{C_n}(T - z)^{-1}dz$, where the C_n are regarded as positively oriented, so that, by an estimate similar to that of (3.11), $||P_n|| \leqq r_n/d_n$. (Note that in the present case, $D_n \cap \sigma(T) \neq \emptyset$ but it is not assumed as in Lemma 2 that the center of C_n is in $\sigma(T)$.) Next, if $t_n = r_n/d_n - 1$ then $||P_n|| \leqq 1 + t_n$ (cf. (3.3)). It follows from (3.5) with P and t replaced by P_n and t_n that

$$(5.3)\qquad ||P_n - P_n^*|| \leqq \operatorname{const}(r_n/d_n - 1)^{1/2} \quad (n = 1, \cdots, N)\ ,$$

provided, say, $0 < \varepsilon \leqq 1/2$, as will be assumed. Thus, in view of (5.1).

$$(5.4)\qquad \sum_{n=1}^{N} ||P_n - P_n^*|| \leqq \operatorname{const} \varepsilon\ .$$

Next, consider any pair or circles, say C_1 and C_2. It will be shown that if $D_1 \cap D_2 \neq \emptyset$ then either one circle, say C_2, can be discarded or it can be deformed into a rectifiable simple closed curve C_2' lying in the resolvent set of T and with the properties that

$$(5.5)\qquad P_2 = P_{C_2'} = -(2\pi i)^{-1}\int_{C_2'}(T - z)^{-1}dz$$

and

$$(5.6)\qquad \operatorname{int} C_2' \subset D_2 \quad \text{and} \quad D_1 \cap \operatorname{int} C_2' = \emptyset\ .$$

To see this, note first that $\sigma(T) \cap \{z: r_1 - d_1 < |z - z_1| < r_1 + d_1\} = \emptyset$. If $D_2 \subset \{z: |z - z_1| < r_1 + d_1\}$, then $D_2 \cap \sigma(T) \subset D_1 \cap \sigma(T)$ and so C_2 can be discarded. Also, in case $D_2 \cap \{z: |z - z_1| \leqq r_1 - d_1\} = \emptyset$, then, since $D_2 \not\subset D_1$, C_2 can be deformed into C_2' so as to satisfy both (5.5) and (5.6). The remaining possibility is that

$$D_2 \cap \{z: |z - z_1| \leqq r_1 - d_1\} \neq \emptyset \quad \text{and} \quad D_2 \not\subset \{z: |z - z_1| < r_1 + d_1\} .$$

It may be supposed, however, that $\{z: |z - z_1| \leqq r_1 - d_1\} \not\subset D_2$ since, otherwise, $D_1 \cap \sigma(T) \subset D_2 \cap \sigma(T)$ and C_1 can be discarded. Consequently, $r_2 > d_1$ and $d_2 < 2(r_1 - d_1)$, so that $r_2/d_2 > d_1/2(r_1 - d_1) = 1/2(r_1/d_1 - 1)^{-1}$. Hence, $r_2/d_2 > 1/2\varepsilon^2$, in view of, and in contradiction to (5.1) (with $\varepsilon \leqq 1/2$).

Repeated applications of the above argument show that the circles $C_1, \cdots, C_N$ may be replaced by rectifiable simple closed curves, say, $\gamma_1, \cdots, \gamma_M (M \leqq N)$, where each γ_i is some C_j or some C_j', and where $\operatorname{int} \gamma_n \cap \operatorname{int} \gamma_m = \emptyset$ for $m \neq n$ and $\sigma(T) \subset \bigcup_{n=1}^M \operatorname{int} \gamma_n$. It is seen from relations corresponding to (5.5) and (5.6) that $\sum_{n=1}^M P_n = I$, where $P_n = -(2\pi i)^{-1} \int_{\gamma_n} (T - z)^{-1} dz$, and hence that $\sum' P_n = I$ where the prime denotes that the summation is over a subset of $\{1, \cdots, N\}$. As a result, we revert to the original notation and suppose without loss of generality, that

$$I = \sum P_n \qquad \left(\sum = \sum_1^N\right) . \tag{5.7}$$

It is now easy to complete the proof of Theorem 3. For,

$$T = TI = \sum TP_n = \sum z_n P_n + \sum (T - z_n) P_n . \tag{5.8}$$

But $||(T - z_n)P_n|| \leqq r_n ||P_n|| \leqq r_n(r_n/d_n) < r_n(1 + \varepsilon^2)$, the last inequality by (5.1). Since $\varepsilon \leqq \frac{1}{2}$, (5.2) shows that $\sum ||(T - z_n)P_n|| \geqq 2\varepsilon$. Also, $\sum z_n P_n = \sum z_n P_n^* + \sum z_n(P_n - P_n^*)$ and, by (5.4), $\sum ||z_n(P_n - P_n^*)|| \leqq (\max |z_n|) \operatorname{const} \varepsilon$. Since each D_n contains part of $\sigma(T)$ it is clear from (5.2) that $\max |z_n| \leqq ||T|| + 2\varepsilon \leqq ||T|| + 1$, and so, by (5.8),

$$T = \sum {}_n P_n^* + A , \quad \text{where } ||A|| \leqq \operatorname{const} \varepsilon . \tag{5.9}$$

Hence, $T^*T = \sum z_n T^* P_n^* + T^*A = \sum z_n[\bar{z}_n P_n^* + (T_n^* - \bar{z}_n)P_n^*] + T^*A$. But $||T^*A|| \leqq \operatorname{const} \varepsilon$ and, as above, $\sum ||z_n(T_n^* - \bar{z}_n)P_n^*|| \leqq (\max |z_n|)2\varepsilon$, and so another application of (5.4) yields $||T^*T|| - \sum |z_n|^2 P_n|| < \operatorname{const} \varepsilon$. A similar argument yields the same inequality with T and T^* interchanged, hence T is normal, and the proof is complete.

REMARKS. It is readily seen that Theorem 3 implies the assertion of Theorem 2 when $\sigma(T)$ is countable. We do not know whether the hypothesis of Theorem 2 implies that T is normal even when $\sigma(T)$

is not countable, in which case Theorem 2 would imply Theorem 3. The hypothesis (3.8) of Theorem 2 is of course a "sparseness" condition on $\sigma(T)$ and, conceivably, is restrictive enough to imply normality of T. In the same vein, we do not know whether the condition (5.2) in the hypothesis of Theorem 3 is essential, although, of course, at least a boundedness restriction must be placed on the r_n's of (5.1). (Note that if C_r is the circle with center at $z = 0$ and radius r then $r/\text{dist}(C_r, \sigma(T)) \to 1$ as $r \to \infty$.) It is clear, of course, that (5.2) alone is not enough, since this condition amounts only to requiring that $\sigma(T)$ be of one-dimensional Hausdorff measure 0.

It may be noted that there exist uncountable sets, corresponding to $\sigma(T)$, for which (3.8) holds. To see this, one need only modify the construction of the standard Cantor set so that the length of each removed complementary open interval is a fraction sufficiently close to 1 of the length of the (closed) interval from which it was removed.

6. Real parts of G_1 operators. If T is G_1 then, as was shown in Putnam [13], p. 509,

$$\text{Re}\,\sigma(T) \subset \sigma(\text{Re}\, T)\,. \tag{6.1}$$

For another proof, see Berberian [1], where it is also shown that, if $\sigma(T)$ is connected,

$$\text{Re}\,\sigma(T) = \sigma(\text{Re}\, T)\,. \tag{6.2}$$

That (6.2) need not hold in general, however, can be deduced from the example of Stampfli of (3.13) above, simply by choosing the sequence $\{z_1, z_2, \cdots\}$ so that, for instance, $\text{Re}\, z_n \neq \pm 1/2$ for all n. Then $\text{Re}\,\sigma(T)$ consists of 0 and the real parts of the z_n's while $\sigma(\text{Re}\, T) = \text{Re}\,\sigma(T) \cup \{\pm 1/2\}$. A consideration of the operator T constructed in Theorem 1, where now the disks D_n are chosen so that $\text{Re}\, z \neq \pm 1/2$ for $z \in D_n (n = 1, 2, \cdots)$, shows that (6.1) may hold properly also if T is completely G_1.

It is known that (6.2) always holds for hyponormal operators; see Putnam [12], p. 46. In view of certain known results concerning the spectra of completely subnormal and completely hyponormal operators one has the following

THEOREM 4. *Let T have the rectangular form $T = H + iJ$ and let X be a compact subset of the real line. Then:*

(i) *X is the spectrum of $H = \text{Re}\, T$ for some completely subnormal T if and only if X is the closure of an open subset of the real line;*

(ii) *X is the spectrum of $H = \operatorname{Re} T$ for some completely hyponormal T if and only if, for every open interval I,* $\operatorname{meas}_1(X \cap I) > 0$ *whenever $X \cap I \neq \varnothing$, where* meas_1 *denotes linear Lebesgue measure.*

Proof of (i). First, let X be the closure of an open set of real numbers, so that $X = (\cup I_n)^-$, where $I_1, I_2, \cdots$ is a countable set of pairwise disjoint open intervals. Since the unilateral shift V is subnormal and $\sigma(V)$ is the closed unit disk (see, e.g., Halmos [7]), one need only put $T = \sum \oplus (a_n V + b_n)$ where a_n, b_n are real, $a_n > 0$, and $I_n = (-a_n + b_n, a_n + b_n)$. Clearly, $X \subset \sigma(T)$, while the reverse inclusion follows from the fact that each term $a_n V + b_n$ is G_1.

Conversely, suppose that $H = \operatorname{Re} T$ where T is completely subnormal and let $X = (\operatorname{int} \sigma(H))^-$. It will be shown that $X = \sigma(H)$. If $X \neq \sigma(H)$, then there exists some $c \in \sigma(H) - X$ and an open interval I_c containing c such that $\sigma(H) \cap I_c$ has no interior. In view of (6.2), there exists an open disk D intersecting $\sigma(T)$ for which $Y = \sigma(T) \cap D^-$ is nowhere dense and has a connected complement. Hence $C(Y) = P(Y)$, by Lavrentiev's theorem (cf. Gamelin [5], y. 48), and hence T has a normal part with spectrum Y; see Clancey and Putnam [4]. Thus, T is not completely subnormal, a contradiction.

Proof of (ii). First, suppose that $X \cap I$ has positive linear measure whenever I is an open interval and $X \cap I$ is not empty. Let $T = H + iJ$ on $\mathfrak{H} = L^2(X)$, where $(Hx)(t) = tx(t)$ and $(Jx)(t) = -(i\pi)^{-1} \int_X (s - t)^{-1} x(s) ds$, the integral regarded as a Cauchy principal value. Then T is completely hyponormal, $\sigma(T) = X \times [-1, 1]$, and $\operatorname{Re} \sigma(T) = X$; cf. Clancey and Putnam [3], p. 452.

Next, suppose that $H = \operatorname{Re} T$ where T is completely hyponormal. Then $\sigma(T) \cap D$ has positive planar measure whenever D is an open disk for which $\sigma(T) \cap D$ is not empty; see Putnam [15], p. 324. Since T satisfies (6.2), it is clear that $\sigma(H) \cap I$ has positive linear measure whenever I is an open interval for which $\sigma(H) \cap I$ is not empty. This completes the proof of Theorem 4.

As was noted in § 1, a necessary and sufficient condition on a compact set of the plane in order that it be the spectrum of a completely G_1 operator is not known. Also, we do not have an analogue of Theorem 4. However, it is possible to prove the following

THEOREM 5. *In order that a compact set X of the real line be the spectrum of the real part of a completely G_1 operator T it is necessary that X be uncountable (equivalently, that X contain a perfect set).*

Proof. In view of (6.1) it is clear that if T is any G_1 operator and if $X=\sigma(\operatorname{Re} T)$ then $\sigma(T)$ is contained in the set consisting of all lines $\{z: \operatorname{Re} z = c\}$ where $c \in X$. Further, since T of the theorem is completely G_1, then $\{z: \operatorname{Re} z = c\} \cap \sigma(T)$ is empty whenever c is an isolated point of X, as can be seen from (6.1) and Stampfli's result ([22], [23]) mentioned in §1. Consequently, $\sigma(T)$ is contained in the union of lines $\{z: \operatorname{Re} z = c\}$ where $c \in X'$, the first derived set of X. As above, no point of $\sigma(T)$ can lie on $\{z: \operatorname{Re} z = c\}$ if z is an isolated point of X', that is if $c \notin X''$. It follows as in the proof of Theorem 2 that if γ is the least ordinal (necessarily of finite or denumerable cardinality) with the property that $X^{(\gamma)} = X^{(\gamma+1)}$ then necessarily $\sigma(T)$ is contained in the union of lines $\{z: \operatorname{Re} z = c\}$ with $c \in X^{(\gamma)}$. Consequently, $X^{(\gamma)} \neq \emptyset$, hence is perfect, and the proof of Theorem 5 is complete.

REMARKS. In Theorem 5 it is possible that X contains some isolated points. One need only consider the example mentioned at the beginning of this section illustrating that (6.1) may be a proper inclusion with T completely G_1. We do not know whether the condition of Theorem 5 on X is also sufficient, that is, whether any uncountable compact set of the real line must be the spectrum of the real part of some completely G_1 operator.

REFERENCES

1. S. K. Berberian, *Conditions on an operator implying Re $\sigma(T)=\sigma(\text{Re } T)$*, Trans. Amer. Math. Soc., **154** (1971), 267–272.

2. L. Carleson, *Selected Problems on Exceptional Sets*, D. van Nostrand Co., Inc., Princeton, 1967.

3. K. F. Clancey and C. R. Putnam, *The spectra of hyponormal integral operators*, Comm. Math. Helv., **46** (1971), 451–456.

4. ———, *The local spectral behavior of completely subnormal operators*, Trans. Amer. Math. Soc., **163** (1972), 239–244.

5. T. W. Gamelin, *Uniform Algebras*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1969.

6. J. Garnett, *Analytic Capacity and Measure*, Lecture notes in mathematics, **297**, Springer, Berlin, 1972.

7. P. R. Halmos, *A Hilbert Spac3 Problem Book*, D. van Nostrand Co., Inc., Princeton, 1967.

8. S. Hildebrandt, *Über den numerischen Wertebereich eines Operators*, Math. Ann., **163** (1966), 230–247.

9. E. Kamke, *Theory of Sets*, Dover Publications, New York, 1950.

10. G. R. Luecke, *Operators satisfying condition (G_1) locally*, Pacific J. Math., **40** (1972), 629–637.

11. ———, *Topological properties of paranormal operators on Hilber space*, Trans. Amer. Math. Soc., **172** (1972), 35–43.

12. C. R. Putnam, *Commutation Properties of Hilbert Space Operators and Related Topics*, Ergebnisse der Math., **36**, Springer, Berlin, 1967.

13. ———, *The spectra of operators having resolvents of first-order growth*, Trans.

Amer. Math. Soc., **133** (1968), 505–510.
14. ———, *Eigenvalues and boundary spectra*, Illinois J. Math., **12** (1968), 278–282.
15. ———, *An inequality for the area of hyponormal spectra*, Math. Zeits., **116** (1970), 323–330.
16. ———, *Almost normal operators, their spectra and invariant subspaces*, Bull. Amer. Math. Soc., **79** (1973), 615–624.
17. ———, *The role of zero sets in the spectra of hyponormal operators*, Proc. Amer. Math. Soc., **43** (1974), 137–140.
18. F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Frederick Ungar Pub. Co., New York, 1955.
19. C. A. Rogers, *Hausdorff Measures*, Cambridge Univ. Press, 1970.
20. J. G. Stampfli, *Hyponormal operators and spectral density*, Trans. Amer. Math. Soc., **117** (1965), 469–476.
21. ———, *Analytic extensions and spectral localization*, J. Math. Mech., **16** (1966), 287–296.
22. ———, *A local spectral theory for operators*, J. Functional Anal., **4** (1969), 1–10.
23. ———, *A local spectral theory for operators, II*, Bull. Amer. Math. Soc., **75** (1969), 803–806.
24. ———, *A local spectral theory for operators, III: resolvents, spectral sets and similarity*, Trans. Amer. Math. Soc., **168** (1972), 133–151.
25. B. Sz.-Nagy and C. Foiaș, *Une relation parmi les vecteurs propersd'un opérateur de l'espace de Hilbert et de l'opérateur adjoint*, Acta Sci. Math. (Szeged), **20** (1959), 91–96.
26. A. Wintner, *Zur Theorie der beschrankten Bilinearformen*, Math. Zeits., **30** (1929), 228–282.

Received September 23, 1977. This work was supported by a National Science Foundation research grant.

PURDUE UNIVERSITY
WEST LAFAYETTE, IN 47907