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SUPPORT POINT FUNCTIONS AND THE LOEWNER VARIATION

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1. Introduction. Let $U = \{z: |z| < 1\}$ and $\mathscr{S}$ the set of functions $f, f(z) = z + a_2z^2 + \cdots$, that are analytic and 1:1 in U. Denote by σ the collection of support point functions of $\mathscr{S}$, i.e., functions $f \in \mathscr{S}$ that satisfy

$$\operatorname{Re} L(f) = \max_{g \in \mathscr{S}} \operatorname{Re} L(g)$$

for some nonconstant continuous (in the topology of local uniform convergence) linear functional on $\mathscr{S}$. Finally, denote by $E(\mathscr{S})$ the set of extreme point functions of $\mathscr{S}$.

It is well known that if $f \in \sigma \cup E(\mathscr{S})$, then $f(U)$ is the complement of a single Jordan arc extending from some finite point to ∞ and along which $|w|$ is strictly increasing. Indeed, this has been demonstrated for the class $E(\mathscr{S})$ by L. Brickman [1] and for the class σ by A. Pfluger [5] (see also L. Brickman and D. Wilken [2]). Consequently, if $f \in \sigma \cup E(\mathscr{S})$, there is a Loewner chain

$$f(z, t) = e^t\left[z + \sum_{n=2}^{\infty} a_n(t)z^n\right] \quad (0 \leqq t < \infty)$$

with $f(z, 0) = f(z)$ and $f(z, t_1)$ subordinate to $f(z, t_2)$ if $0 \leqq t_1 < t_2 < \infty$ (see [6, p. 157]). Note that $e^{-t}f(z, t) \in \mathscr{S}$. Let $w(z, t) = e^{-t}(z + \hat{b}_2(t)z^2 + \hat{b}_3(t)z^3 + \cdots)$ be analytic for $t \in \{t: 0 \leqq t < \infty\}$ and $z \in U$, 1:1 in U with $|w(z, t)| < 1$, and such that $f(z) = f(w(z, t), t)$ for each $t \in \{t: 0 \leqq t < \infty\}$ and all $z \in U$. Observe that we define $\hat{w}(z, t) \equiv e^t w(z, t) = z + \hat{b}_2(t)z^2 + \cdots \in \mathscr{S}$ and that $|\hat{w}(z, t)| < e^t$ for $z \in U$.

In § 2 it is shown that if $f \in E(\mathscr{S})$, then $e^{-t}f(z, t) \in E(\mathscr{S})$ and also that if $f \in \sigma$, then $e^{-t}f(z, t) \in \sigma$. This latter result is a generalization of a theorem due to S. Friedland and M. Schiffer [3, p. 143]. Also, in the process of generalizing this theorem a fairly easy method is established for finding for each t, $0 \leqq t < \infty$, a continuous linear functional which $e^{-t}f(z, t)$ maximizes.

2. Preservation of the sets σ and $E(\mathscr{S})$ under the Loewner variation. It is easy to show that if $f \in E(\mathscr{S})$, then $e^{-t}f(z, t) \in E(\mathscr{S})$ also. Indeed, if this were not the case, then there would exist distinct functions $f_1, f_2 \in \mathscr{S}$ and $\lambda_1, \lambda_2 > 0$ with $\lambda_1 + \lambda_2 = 1$ for which $\lambda_1 f_1(z) + \lambda_2 f_2(z) = e^{-t}f(z, t)$. This would imply that $e^t\lambda_1 f_1(w(z, t)) + e^t\lambda_2 f_2(w(z, t)) = f(w(z, t), t) = f(z)$. Since $e^t f_1(w(z, t))$ and $e^t f_2(w(z,t))$ are in $\mathscr{S}$, the fact that $f(z) \in E(\mathscr{S})$ is contradicted and therefore

$e^{-t}f(z, t) \in E(\mathscr{S})(0 \leqq t < \infty)$.

The following theorem contains the analogous result for the class σ.

THEOREM. *Let $f \in \sigma \subset \mathscr{S}$. Then $e^{-t}f(z, t) \in \sigma$ for all t such that $0 \leqq t < \infty$.*

Proof. Since $f \in \sigma$, there exists a nonconstant continuous linear functional, L, for which

$$\operatorname{Re} L(f) = \max_{g \in \mathscr{S}} \operatorname{Re} L(g) .$$

At this point we need a representation theorem due to O. Toeplitz [7].

THEOREM (*Toeplitz*). *Let $f(z) = z + a_2z^2 + \cdots \in \mathscr{S}$. Then $L(f)$ is a continuous linear functional on $\mathscr{S}$ if and only if there exists a sequence $\{b_n\}$ with $\limsup_{n\to\infty} |b_n|^{1/n} < 1$ such that $L(f) = \sum_{n=1}^{\infty} a_nb_n$.*

Now, $f(z) = f(w(z, t), t)$ where $e^tw(z, t) = \hat{w}(z, t) = z + \hat{b}_2(t)z^2 + \cdots \in \mathscr{S}$ and $|\hat{w}(z, t)| < e^t$ for $z \in U$. Since

$$\begin{aligned} f(w(z, t), t) &= e^t[w(z, t) + a_2(t)w^2(z, t) + \cdots + a_n(t)w^n(z, t) + \cdots] \\ &= \hat{w}(z, t) + a_2(t)e^{-t}\hat{w}^2(z, t) + \cdots \\ &\qquad + a_n(t)e^{-(n-1)t}\hat{w}^n(z, t) + \cdots, \end{aligned}$$

and if $L(f) = \sum_{n=1}^{\infty} a_nb_n$, then it follows that

$$\begin{aligned} \sum_{n=1}^{\infty} a_nb_n &= \sum_{n=1}^{\infty} [\hat{b}_n^{(1)} + a_2(t)e^{-t}\hat{b}_n^{(2)} + a_3(t)e^{-2t}\hat{b}_n^{(3)} + \cdots \\ &\qquad + a_n(t)e^{-(n-1)t}\hat{b}_n^{(n)}]b_n \\ &= \sum_{n=1}^{\infty} \left[\sum_{k=1}^{n} a_k(t)e^{-(k-1)t}\hat{b}_n^{(k)}b_n \right] \end{aligned}$$

where $\hat{b}_n^{(k)}$ is the nth coefficient of $\hat{w}^k(z, t) = [z + \hat{b}_2(t)z^2 + \cdots]^k$. However, since $\hat{w}^k(z, t)$ is analytic in U and bounded by e^{kt}, it follows from Cauchy's formula that

$$\begin{aligned} |\hat{b}_n^{(k)}| &= \left| \frac{1}{2\pi i} \int_{|\varepsilon|=1} \frac{\hat{w}^k(\varepsilon, t)d\varepsilon}{\varepsilon^{n+1}} \right| = \left| \frac{1}{2\pi} \int_0^{2\pi} \frac{\hat{w}^k(e^{i\theta}, t)}{e^{in\theta}} d\theta \right| \\ &\leqq \frac{1}{2\pi} \int_0^{2\pi} |\hat{w}^k(e^{i\theta}, t)|\, d\theta \leqq e^{kt} \end{aligned}$$

for all $n = 1, 2, \cdots$. Also, since $e^{-t}f(z, t) = z + a_2(t)z^2 + \cdots \in \mathscr{S}$, it follows from Littlewood's theorem [4] that $|a_k(t)| \leqq ke$. Therefore,

$$\begin{aligned}\sum_{k=1}^{n} |a_k(t)e^{-(k-1)t}\hat{b}_n^{(k)}b_n| &\leqq \sum_{k=1}^{n} |ke\cdot e^{-(k-1)t}\cdot e^{kt}\cdot b_n| \\ &= e^{(t+1)}\,|b_n|\left(\frac{n(n+1)}{2}\right).\end{aligned}$$

Notice also that $\limsup_{n\to\infty} |e^{(t+1)}\cdot b_n\cdot n(n+1)/2|^{1/n} = \limsup_{n\to\infty} |b_n|^{1/n} < 1$. Consequently, the double summation, $\sum_{n=1}^{\infty} [\sum_{k=1}^{n} a_k(t)e^{-(k-1)t}\hat{b}_n^{(k)}b_n]$, converges absolutely and therefore the order of summation can be reversed and one obtains

$$\begin{aligned}\sum_{n=1}^{\infty} a_n b_n &= \sum_{n=1}^{\infty}\left[\sum_{k=1}^{n} a_k(t)e^{-(k-1)t}\hat{b}_n^{(k)}b_n\right] \\ &= \sum_{k=1}^{\infty}\left[\sum_{n=k}^{\infty} a_k(t)e^{-(k-1)t}\hat{b}_n^{(k)}b_n\right] \\ &= \sum_{k=1}^{\infty}\left[\sum_{n=k}^{\infty} \hat{b}_n^{(k)}b_n e^{-(k-1)t}\right]a_k(t)\ .\end{aligned}$$

Now, for $f\in\mathscr{S}$ define $L_t(f) \equiv \sum_{k=1}^{\infty} (\sum_{n=k}^{\infty} \hat{b}_n^{(k)}b_n e^{-(k-1)t})a_k$. From the theorem of Toeplitz it follows that L_t will be a continuous linear functional on $\mathscr{S}$ provided that

$$\limsup_{k\to\infty}\left|\sum_{n=k}^{\infty} \hat{b}_n^{(k)}b_n e^{-(k-1)t}\right|^{1/k} < 1\ .$$

Since $\limsup_{k\to\infty} |b_k|^{1/k} = \rho < 1$, there exists an N and an r such that $\rho<r<1$ and $|b_k|\leqq r^k$ for all $k\geqq N$. Therefore, $|\sum_{n=k}^{\infty} \hat{b}_n^{(k)}b_n e^{-(k-1)t}|^{1/k}\leqq (e^{kt}\cdot e^{-(k-1)t}\sum_{n=k}^{\infty} r^n)^{1/k} = e^{t/k}r/(1-r)^{1/k}$ for all $k\geqq N$. Since

$$\limsup_{k\to\infty}\left[e^{t/k}\frac{r}{(1-r)^{1/k}}\right] = r < 1\ ,$$

it follows that $\limsup_{k\to\infty} |\sum_{n=k}^{\infty} \hat{b}_n^{(k)}b_n e^{-(k-1)t}|^{1/k} \leqq r < 1$.

Since $\operatorname{Re} L(f) = \operatorname{Re}(\sum_{n=1}^{\infty} a_n b_n)$ is a maximum for the class $\mathscr{S}$, it follows easily that $\operatorname{Re} L_t(e^{-t}f(z,t))$ is also a maximum for the class $\mathscr{S}$. In order to see this one needs only to observe that if f and $\hat{f}$ are any two functions in $\mathscr{S}$ related by a relation of the form $f(z) = e^t\hat{f}(w(z,t))$, then $L(f) = L_t(\hat{f})$. This completes the proof of the theorem.

REMARKS. Since $f(z) \equiv f(w(z,t),t)$ for some $w(z,t)$, one can express $L_t(e^{-t}f(z,t)) = \sum_{k=1}^{\infty} (\sum_{n=k}^{\infty} \hat{b}_n^{(k)}b_n e^{-(k-1)t})a_k(t)$ in terms of the coefficients of the functions $f(z)$ and $e^{-t}f(z,t)$. This can easily be done provided that $L(f)$ $(L(f) = \sum_{n=1}^{\infty} a_n b_n)$ does not contain too many terms. Then for each t, $0<t<\infty$, the corresponding Schiffer differential equation which $e^{-t}f(z,t)$ must satisfy can then be computed with little difficulty. Unfortunately, extracting useful information from these new equations is not an easy task.

Suppose, however, that it is known that $\operatorname{Re} L(f)$ is a maximum for the class $\mathscr{S}$ when f is one of the Koebe functions, $f(z) = z/(1 - e^{i\theta}z)^2 (0 \leqq \theta < 2\pi)$. Then since $e^{-t}f(z, t) = f(z)$ in this case, it follows that $\operatorname{Re} L_t(f)$ is a maximum for the class $\mathscr{S}$ for all t $(0 \leqq t < \infty)$. From this one can establish a one parameter family of new coefficient inequalities for the class $\mathscr{S}$. S. Friedland and M. Schiffer [3, p. 149] have done this for the case where $L(f) = a_4$.

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