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COMPACT ENDOMORPHISMS OF BANACH ALGEBRAS

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Let T be a compact endomorphism of a commutative semi-simple Banach algebra B. This paper discusses the behavior of the adjoint T^* of T on the set X' of multiplicative linear functionals on B. In particular it is shown that $\cap T^{*n}(X')$ is finite, thus generalizing the example of compact endomorphisms of the disc algebra.

0. **Introduction and preliminaries.** In this paper we discuss maps which are simultaneously endomorphisms of Banach algebras and compact operators. That is, these operators T are linear, satisfy $T(fg) = (Tf)(Tg)$ for all f and g in the algebra and map bounded sets into sequentially compact sets.

As a motivating example, consider the disc algebra A, the sup-norm algebra of functions analytic on the open unit disc D and continuous on $\bar{D}$. Every nonzero endomorphism T of A has the form $Tf = f \circ \varphi$ for $f \in A$, where $\varphi \in A$ and φ maps $\bar{D}$ into $\bar{D}$. It was shown in [3] that if φ is not a constant function, then T is compact if, and only if, $|\varphi(z)| < 1$ for all $z \in \bar{D}$. Moreover, for such φ, if φ_n denotes its nth iterate, then $\cap \varphi_n(\bar{D}) = \{z_0\}$ for some $z_0 \in D$, and further the spectrum $\sigma(T)$ of T satisfies $\sigma(T) = \{(\varphi'(z_0))^n | n$ is a positive integer$\} \cup \{0, 1\}$. When φ is a constant function, the range of T is one-dimensional and T is compact with $\sigma(T) = \{0, 1\}$.

We will now consider compact endomorphisms of other Banach algebras and study to what extent the properties of compact endomorphisms of the disc algebra can be generalized. Our principal results will describe the behavior of the adjoint T^* of T on the maximal ideal space of the algebra.

We first introduce some notation and terminology. Let B be a commutative semi-simple Banach algebra with unit 1 and maximal ideal space X and, in addition, let θ denote the zero functional on B. If $0 \neq T$ is a (necessarily) bounded endomorphism of B, then the adjoint T^* induces a continuous function φ from $X' \equiv X \cup \{\theta\}$ into itself in the following way. For $x \in X$, let $e_x \in B^*$ satisfy $e_x(f) = \hat{f}(x)$, where $f \to \hat{f}$ is the Gelfand transformation of B. It is easy to verify that T^*e_x is multiplicative. There are two possibilities. If $T^*e_x \neq \theta$, then $T^*e_x = e_y$ for some $y \in X$ and we let $\varphi(x) = y$. For the second case, if $T^*e_x = \theta$, we let $\varphi(x) = \theta$. We also define $\varphi(\theta) = \theta$. Since φ is essentially equal to T^* restricted to the set of multiplicative linear functionals on B, φ is a continuous function from X' to X'; φ will be called the map on X or X'

induced by T.

It is useful to note that if $T1 = 1$, then $T^*e_x \neq 0$ for all $x \in X$ since $(T^*e_x)(1) = e_x(T1) = e_x(1) = 1$. Consequently, when $T1 = 1$, φ maps X into X. On the other hand, if $T1 \neq 1$, then $\varphi(x) = \theta$ for some $x \in X$.

If n is a positive integer, we let φ_n denote the nth iterate of φ, i.e., $\varphi_0(x) = x$ and $\varphi_n(x) = \varphi(\varphi_{n-1}(x))$ for $x \in X'$. A routine topological argument shows that $\bigcap \varphi_n(X')$ is a nonempty compact subset of X' and $\bigcap \varphi_n(X')$ is mapped onto itself by φ. Further, when X is connected and $T \neq 0$, then $T1 = 1$, whence φ maps X into X, $\bigcap \varphi_n(X)$ is connected and φ maps $\bigcap \varphi_n(X)$ onto itself.

In the first section we will prove some structure theorems leading to the following theorem.

THEOREM 1.7. *Suppose B is a commutative semi-simple Banach algebra with unit* 1 *and maximal ideal space X and T is a nonzero compact endomorphism of B. If φ is the map on X' induced by T, then $\bigcap \varphi_n(X')$ is finite. If X is connected, then $\bigcap \varphi_n(X)$ is a singleton.*

We recall that we have already characterized the compact endomorphisms of the disc algebra. Moreover, it is easy to verify that for any commutative semi-simple Banach algebra with unit 1 and maximal ideal space X, and any $a \in X$, the endomorphism $T: f \rightarrow \hat{f}(a)1$ is compact. Using Theorem 1.7, we will prove that if X is a compact connected Hausdorff space, then every nonzero compact endomorphism T on $C(X)$ has the form $Tf = f(a)1$ for some $a \in X$. Finally we will discuss some relations between the range $\varphi(X)$ of the induced map φ of a compact endomorphism and the strong and Silov boundaries of other function algebras on X.

1. We begin with the following lemma dealing with the spectral radius $\|T\|_{sp}$ of a compact endomorphism.

LEMMA 1.1. *Suppose B is a commutative semi-simple Banach algebra with unit* 1. *If T is a compact endomorphism of B and T is not nilpotent, then $\|T\|_{sp} = 1$.*

Proof. If B is semi-simple and λ is an eigenvalue of any endomorphism T of B, then for each positive integer n, λ^n is also an eigenvalue. For, if $0 \neq f \in B$ and $Tf = \lambda f$, then $T(f^n) = (Tf)^n = \lambda^n f^n \neq 0$. On the other hand, when T is a compact operator, every nonzero element in the spectrum $\sigma(T)$ is an eigenvalue [4]. Since $\sigma(T)$ is a compact subset of the plane, it follows that if T is a compact endomorphism of B, then $\sigma(T) \subset \{\lambda \mid |\lambda| \leqq 1\}$.

It is easy to see that an endomorphism S of B is zero if, and only if, $S1 = 0$. Thus an endomorphism T is nilpotent if, and only if, $T^m 1 = 0$ for some positive integer m. Assume T is an endomorphism of B which is not nilpotent and set $F_m = T^m 1$. Then for each m, F_m is a nonzero idempotent in B and so

$$1 = \|\hat{F}_m\|_\infty = \|(T^m 1)\hat{}\|_\infty \leqq \|T^m 1\|_B \leqq \|T^m\| \, \|1\| .$$

Since this holds for all positive integers m, it follows that $1 \leqq \lim_{m\to\infty} \|T^m\|^{1/m} = \|T\|_{sp}$. Combining this with the first paragraph gives that if T is a compact endomorphism of a commutative semi-simple Banach algebra with unit, then $\|T\|_{sp} = 1$ if, and only if, T is not nilpotent.

REMARKS. (1) Every quasinilpotent compact endomorphism of a commutative semi-simple Banach algebra with unit is nilpotent.

(2) The hypothesis in Lemma 1.1 that B be semi-simple was needed to indure that $0 \neq f \in B$ implied $0 \neq f^n \in B$ for every positive integer n.

(3) If B is not assumed to be semi-simple, then any denumerable plane set σ with zero as its only limit point can be the spectrum of a compact endomorphism of B. For, it is well known that for each such σ there exists a compact linear operator T on Hilbert space H with $\sigma(T) = \sigma$. If multiplication is defined on H by $fg = 0$ for all $f, g \in H$, then H is a commutative Banach algebra, T is a compact endomorphism on H and $\sigma(T) = \sigma$.

The proof of the next lemma is straightforward.

LEMMA 1.2. *Let B be a commutative semi-simple Banach algebra with unit 1 and maximal ideal space X. If E is a nonzero idempotent in B, then BE and $B(1-E)$ are closed subalgebras of B with units E and $1 - E$, respectively, and $B = BE \oplus B(1 - E)$. If $Z = \{x \in X \mid \hat{E}(x) = 1\}$, then the maximal ideal spaces of BE and $B(1 - E)$ are Z and $X \backslash Z$, respectively. Further, if T is an endomorphism of B with $TE = E$, then BE and $B(1 - E)$ are invariant under T in the sense that $T: BE \to BE$ and $T: B(1 - E) \to B(1 - E)$.*

LEMMA 1.3. *Assume T is a nonzero compact endomorphism of a commutative semi-simple Banach algebra B with unit 1. Then there exists a smallest nonnegative integer M such that $T^M 1 = T^{M+1} 1$. If T is not nilpotent, then $E = T^M 1$ is a nonzero idempotent in B, $TE = E$ and $B = BE \oplus B(1 - E)$ where BE and $B(1 - E)$ are invariant under T and T is nilpotent on $B(1 - E)$.*

Proof. The lemma is trivial if $T1 = 1$. Also if T is nilpotent, then $T^M 1 = 0$ for some positive integer M and there is nothing further to prove.

Assume T is not nilpotent and $T1 \neq 1$. Let X denote the maximal ideal space of B and φ the continuous function on $X' = X \cup \{\theta\}$ induced by T. For each positive integer n, let $Z_n = \{x \in X \mid \varphi_n(x) = \theta\}$. (Since $T1 \neq 1$, $Z_1 \neq \phi$.) For each n, Z_n is both open and closed in X, $\varphi^{-1}(Z_n) = Z_{n+1}$ and $Z_n \subset Z_{n+1}$. Also, $\varphi^{-1}(Z_2 \backslash Z_1) = \varphi^{-1}(Z_2) \backslash \varphi^{-1}(Z_1) = Z_3 \backslash Z_2$ and, in general, $\varphi^{-n}(Z_2 \backslash Z_1) = Z_{n+2} \backslash Z_{n+1}$ for each n.

We assert that $Z_M = Z_{M+1}$ for some positive integer M. To show this, assume $Z_1 \neq Z_2$ and let G be the element in B such that $\hat{G}$ is the characteristic function of $Z_2 \backslash Z_1$. Such an element exists by Silov's Idempotent Theorem [1, p. 88] since $Z_2 \backslash Z_1$ is a subset of X which is both open and closed. By the definition of G, $\hat{G}(x) = 1$ if $x \in Z_2 \backslash Z_1$ and $\hat{G}(x) = 0$ for all other $x \in X$; therefore for each positive integer k, $T^k G\hat{\ }(x) = G(\varphi_k(x)) = 1$ if $x \in \varphi^{-k}(Z_2 \backslash Z_1) = Z_{k+2} \backslash Z_{k+1}$ and $T^k G\hat{\ }(x) = 0$ otherwise. We will now show that if $Z_{k+2} \backslash Z_{k+1} \neq \phi$ for all positive integers k, then $\sigma(T) \supset \{\lambda \mid |\lambda| = 1\}$ which will be a contradiction since T is a compact operator. Thus assume $Z_{k+2} \backslash Z_{k+1} \neq \phi$ for all positive integers k and choose λ with $|\lambda| = 1$. Let n be a positive integer and consider $|[(\lambda + T)^{2n} G]\hat{\ }(x)|$ for some $x \in Z_{n+2} \backslash Z_{n+1} \neq \phi$. Then

$$|[(\lambda + T)^{2n} G]\hat{\ }(x)| = \left| \left[\sum_{k=0}^{2n} \lambda^{2n+k} \binom{2n}{k} T^k G \right]\hat{\ } (x) \right| .$$

But if $x \in Z_{n+2} \backslash Z_{n+1}$, then $T^k G\hat{\ }(x) = 0$ unless $k = n$, and $\hat{G}(\varphi_n(x)) = (T^n G)\hat{\ }(x) = 1$. Therefore

$$\binom{2n}{n} = \left| \binom{2n}{n} \hat{G}(\varphi_n(x)) \right| = |[(\lambda + T)^{2n} G]\hat{\ }(x)| \leqq \|(\lambda + T)^{2n}\| \, \|G\|$$

and so

$$(*) \qquad \binom{2n}{n}^{1/2n} \leqq \|(\lambda + T)^{2n}\|^{1/2n} \|G\|^{1/2n} .$$

If $Z_{n+2} \neq Z_{n+1}$ for all n, we can find such an x for each positive integer n and so (*) holds for all n. Also $\lim_{n \to \infty} \binom{2n}{n}^{1/2n} = 2$. [2, LEMMA 1.2]. Then letting $n \to \infty$ in (*) gives

$$2 = \lim_{n \to \infty} \binom{2n}{n}^{1/2n} \leqq \lim_{n \to \infty} \|(\lambda + T)^{2n}\|^{1/2n} = \|\lambda + T\|_{sp}$$

for all λ, $|\lambda| = 1$. However, from Lemma 1.1, $\|T\|_{sp} = 1$. Therefore if $|\lambda| = 1$, then $\lambda \in \sigma(T)$ and, as a result, every point in $\{\lambda \mid |\lambda| = 1\}$ is in $\sigma(T)$, a contradiction. Therefore the assumption that $Z_{k+2} \backslash Z_{k+1} \neq \phi$ for all positive integers k is false and so there is a least positive integer M for which $Z_M = Z_{M+1}$.

Now let $E = T^M 1$. Since T is not nilpotent, E is a nonzero idempotent in B. Also $\{x \in X \mid \hat{E}(x) = 1\} = \{x \in X \mid (T^M 1)\hat{}(x) = 1\} = \{x \in X \mid \varphi_M(x) \neq \theta\} = X \backslash Z_M = X \backslash Z_{M+1} = \{x \in X \mid \varphi_{M+1}(x) \neq \theta\} = \{x \in X \mid (T^{M+1}1)\hat{}(x) = 1\} = \{x \in X \mid (TE)\hat{}(x) = 1\}$. Therefore $TE = E$.

From Lemma 1.2, we have that BE and $B(1 - E)$ are commutative semi-simple Banach algebras which are invariant under T. The final assertion in the lemma that T is nilpotent on $B(1 - E)$ follows from the fact that $(1 - E)$ is the multiplicative identity in $B(1-E)$ and $T^M(1 - E) = T^M 1 - T^M E = E - E = 0$.

REMARK. Lemma 1.3 shows that $E = T^M 1$ is an eigenvector of T in B and so $1 \in \sigma(T)$ unless T is nilpotent.

Next suppose S is a nonempty closed subset of the maximal ideal space X of a commutative semi-simple Banach algebra B with unit 1. Then the kernel of S, $\ker(S) = \{f \in B \mid \hat{f}(t) = 0 \text{ for all } t \in S\}$ is a closed ideal in B and $B_1 = B/\ker(S)$ is a commutative semi-simple Banach algebra with unit. If X_1 denotes the maximal ideal space of B_1, then X_1 is the hull of $\ker(S)$, i.e., $X_1 = \{x \in X \mid f \in \ker(S) \text{ implies } \hat{f}(x) = 0\}$. X_1 is a closed subset of X and $S \subset X_1 \subset X$. Further, if $x \in X_1$ and $\bar{f} = f + \ker(S) \in B/\ker(S)$, then $\bar{f}\hat{}(x) = \hat{f}(x)$ [1, p. 12].

Now let T be an endomorphism of B with $T1 = 1$ and φ the map of $X \to X$ induced by T. Clearly, if $\varphi(S) \subset S$, then $\ker(S)$ is invariant under T. Also if $\varphi(S) \subset S$, then $\varphi(X_1) \subset X_1$. For, if $\varphi(S) \subset S$, $f \in \ker(S)$ and $x \in X_1$, then $Tf \in \ker(S)$, which implies $(Tf)\hat{}(x) = 0$ and this, in turn, implies $\hat{f}(\varphi(x)) = 0$, i.e., if $x \in X_1$, then $\varphi(x) \in X_1$. Thus $\varphi(X_1) \subset X_1$ if $\varphi(S) \subset S$.

Furthermore, if $\ker(S)$ is invariant under T, then T induces an endomorphism $\bar{T}$ of B_1 into B_1 defined by $\bar{T}\bar{f} = \overline{Tf}$ for $\bar{f} \in B_1$. Let $\bar{\varphi}$ be the map on X_1 induced by $\bar{T}$. Then by definition, $(\bar{T}\bar{f})\hat{}(x) = \bar{f}\hat{}(\bar{\varphi}(x))$ for all $x \in X_1$. We claim that $\bar{\varphi} = \varphi|_{X_1}$. To this end, let $x \in X_1$. Then $\varphi(x) \in X_1$, and so $\bar{f}\hat{}(\bar{\varphi}(x)) = (\bar{T}\bar{f})\hat{}(x) = (\overline{Tf})\hat{}(x) = (Tf)\hat{}(x) = \hat{f}(\varphi(x)) = \bar{f}\hat{}(\varphi(x))$. Since this holds for all $\bar{f} \in B_1$, it follows that $\bar{\varphi}(x) = \varphi(x)$ for each $x \in X_1$, as claimed. We remark, too, that if T is a compact endomorphism, so is $\bar{T}$ [4].

With these observations we now prove the following.

LEMMA 1.4. *Assume B is a commutative semi-simple Banach* *algebra with unit* 1 *and maximal ideal space* X *and suppose* T *is a compact endomorphism of* B *with* $T1 = 1$. *If* φ *is the map on* X *induced by* T *and* $\mathscr{S} = \cap \varphi_n(X)$, *then* $\mathscr{S}$ *is a nonempty compact subset of* X *and* φ *maps* $\mathscr{S}$ *onto itself. Let* $\ker(\mathscr{S}) = \{f \mid \hat{f}(x) = 0$ *for all* $x \in \mathscr{S}\}$ *and set* $B_1 = B/\ker(\mathscr{S})$. *Then* B_1 *is a commutative semi-simple Banach algebra with unit, and* X_1, *the maximal ideal space of* B_1, *satisfies* $\mathscr{S} \subset X_1 \subset X$. *If* $\bar{T}$ *is defined on* B_1 *by* $\bar{T}\bar{f} = \overline{Tf}$ *for* $\bar{f} \in B_1$, *then* $\bar{T}$ *is a compact endomorphism of* B_1 *and* $\sigma(\bar{T}) \subset \{\lambda \mid |\lambda| = 1\} \cup \{0\}$. *Also, if* $\bar{\varphi}$ *is the map on* X_1 *induced by* $\bar{T}$, *then* $\cap \bar{\varphi}_n(X_1) = \mathscr{S}$.

Proof. The properties of B_1 were discussed before the statement of the lemma. Also φ maps X into X since $T1 = 1$, and we have already noted in the introduction that φ maps $\mathscr{S}$ onto $\mathscr{S}$.

To prove that $\sigma(\bar{T}) \subset \{\lambda \mid |\lambda| = 1\} \cup \{0\}$, suppose the contrary that there exists λ, $0 < |\lambda| < 1$, $\lambda \in \sigma(\bar{T})$ and $\bar{T}\bar{f} = \lambda\bar{f}$. For each $x \in \mathscr{S}$ there are two possibilities.

(i) There exists a positive integer N such that $\bar{\varphi}_N(x) = x$. In this case $(\bar{T}^N\bar{f})\hat{}(x) = \bar{f}\hat{}(\bar{\varphi}_N(x)) = \bar{f}\hat{}(x)$ and also $(\bar{T}^N\bar{f})\hat{}(x) = \lambda^N\bar{f}\hat{}(x)$. Therefore $\bar{f}\hat{}(x) = \lambda^N\bar{f}\hat{}(x)$, and since $|\lambda| < 1$, $\bar{f}\hat{}(x) = 0$.

(ii) For all n, $\bar{\varphi}_n(x) \neq x$. Since φ, and therefore $\bar{\varphi}$, maps $\mathscr{S}$ onto itself, we can choose distinct $t_n \in \mathscr{S}$ satisfying $\bar{\varphi}_n(t_n) = x$. Thus if $\bar{T}\bar{f} = \lambda\bar{f}$, then $(\bar{T}^n\bar{f})\hat{}(t_n) = \bar{f}\hat{}(\varphi_n(t_n)) = \bar{f}\hat{}(x)$, while $(\bar{T}^n\bar{f})\hat{}(t_n) = \lambda^n\bar{f}\hat{}(t_n)$, also. Since $\|\bar{f}\hat{}\|_\infty < \infty$ and $\lambda^n \to 0$ it follows that $\bar{f}\hat{}(x) = \lambda^n\bar{f}\hat{}(t_n) \to 0$. Hence if $\bar{\varphi}_n(x) \neq x$ for all n, then $\bar{f}\hat{}(x) = 0$.

Thus we have just shown that if $0 < |\lambda| < 1$ and $\bar{T}\bar{f} = \lambda\bar{f}$, then $\bar{f}\hat{}(x) = 0$ for all $x \in \mathscr{S}$. But this implies $\bar{f} = 0$. Therefore all the nonzero eigenvalues of $\bar{T}$ lie on the unit circle.

All that remains to be shown is that $\mathscr{S} = \cap \bar{\varphi}_n(X_1)$. Now, $\cap \varphi_n(X_1) = \cap \bar{\varphi}_n(X_1)$ since $\bar{\varphi} = \varphi|_{X_1}$. Therefore $\mathscr{S} = \cap \varphi_n(X) \supset \cap \varphi_n(X_1) = \cap \bar{\varphi}_n(X_1) \supset \mathscr{S}$ which proves that $\mathscr{S} = \cap \bar{\varphi}_n(X_1)$.

LEMMA 1.5. *Suppose* B *is a commutative semi-simple Banach algebra with unit* 1 *and maximal ideal space* X. *Let* T *be a compact endomorphism of* B *with* $\sigma(T) = \{0, 1\}$. *Then there exists a finite set of idempotents,* $\{E_1, \cdots, E_m\}$, *in* B *with the following properties.*

(i) $\{E_1, \cdots, E_m\}$ *forms a basis for* $\mathscr{N} = \{f \mid Tf = f\}$ and $E_iE_j = \delta_{ij}E_j$.

(ii) If $E = \sum_{k=1}^m E_k$, then $B = BE_1 \oplus \cdots \oplus BE_m \oplus B(1 - E)$.

(iii) For each k, $k = 1, \cdots, m$, BE_k is a closed subalgebra of B with multiplicative identity E_k. Also BE_k is invariant under T and all the eigenvectors of T in BE_k corresponding to 1 have the form cE_k, c complex.

(iv) If $E = \sum_{k=1}^{m} E_k$, then $B(1-E)$ is a closed subalgebra of B with multiplicative identity $1-E$. $B(1-E)$ is invariant under T and T is nilpotent on $B(1-E)$. Also $T^M 1 = E$ for some positive integer M.

(v) If $T1 = 1$, then $\sum_{k=1}^{m} E_k = 1$ and $B = BE_1 \oplus \cdots \oplus BE_m$.

Proof. (i) Since $\sigma(T) = \{0, 1\}$, $\mathscr{N} = \{f \mid Tf = f\} \neq (0)$. Also $\mathscr{N}$ is closed under multiplication since $T(fg) = (Tf)(Tg) = fg$ whenever $f, g \in \mathscr{N}$. Further, since T is a compact operator, $\mathscr{N}$ is finite dimensional. Therefore $\mathscr{N}$ is a finite dimensional commutative semi-simple Banach algebra and hence there exist idempotents $E_1, \cdots, E_m$ in $\mathscr{N}$ which form a basis for $\mathscr{N}$ and which satisfy $E_i E_j = \delta_{ij} E_j$. We note that since $E_i \in \mathscr{N}$, $i = 1, \cdots, m$, E_i must be an eigenvector of T with $TE_i = E_i$.

(ii) Suppose $E = \sum_{k=1}^{m} E_k$. Then $1 = \sum_{k=1}^{m} E_k + (1-E)$ and so for each $f \in B$, $f = \sum_{k=1}^{m} fE_k + f(1-E)$. Thus $B = BE_1 + \cdots + BE_m + B(1-E)$. Further, since $E_i E_j = \delta_{ij} E_j$ and $E_i(1-E) = 0$ for all i, it is easy to verify that f can be uniquely represented in this form. Therefore $B = BE_1 \oplus \cdots \oplus BE_m \oplus B(1-E)$.

(iii) In view of Lemma 1.2 all that remains to be demonstrated here is that all the eigenvectors of T in BE_j corresponding to 1 have the form cE_j, c complex. Now, if $T(fE_j) = fE_j \in BE_j$, then $fE_j \in \mathscr{N}$ so that $fE_j = \sum_{i=1}^{m} a_i E_i$. Therefore $fE_j = fE_j^2 = (\sum_{i=1}^{m} a_i E_i)E_j = a_j E_j$ as claimed.

(iv) $B(1-E)$ is a closed subalgebra of B which is invariant under T since $TE = E$. Also, since $\sigma(T) = \{0, 1\}$, in order to prove that T is nilpotent on $B(1-E)$, it suffices to show that $T(f(1-E)) = f(1-E)$ implies $f(1-E) = 0$. But, if $T(f(1-E)) = f(1-E)$, then $f(1-E) \in \mathscr{N} \cap B(1-E) = (0)$. Hence T is nilpotent on $B(1-E)$ and so there exists a positive integer M such that $T^M(1-E) = 0$ or, equivalently, $T^M 1 = T^M E = E$.

(v) If $T1 = 1$, then $1 = T^M 1 = E$ from (iv). Therefore $1 - E = 0$ and $B = BE_1 \oplus \cdots \oplus BE_m$.

REMARK. The decomposition $B = BE_1 \oplus \cdots \oplus BE_m \oplus B(1-E)$ leads to a splitting of the maximal ideal space X of B into disjoint open and closed subsets $Y_1, \cdots, Y_m, Y$, of X where $Y_k = \{x \mid \hat{E}_k(x) = 1\}$, $k = 1, \cdots, m$ and $Y = X \backslash \bigcup_{k=1}^{m} Y_k$. Further, Y_k is the maximal ideal space of BE_k and Y is the maximal ideal space of $B(1-E)$. If φ is the map on $X \cup \{\theta\}$ induced by T, then $\varphi(Y_k) \subset Y_k$, $k = 1, \cdots, m$, and $\varphi^{-1}(Y_k) = Y_k$. The last equality holds since $TE_k = E_k$.

The next lemma describes the behavior of T^* on each Y_k.

LEMMA 1.6. *Suppose B is a commutative semi-simple Banach algebra with unit 1 and maximal ideal space X. Let T be a compact endomorphism of B with the property that $\sigma(T) = \{0, 1\}$ and the only eigenvectors corresponding to 1 are the constants. If φ is the map on X induced by T, then φ maps X into itself and there exists a unique element $\bar{x} \in X$ such that $\varphi(\bar{x}) = \bar{x}$. Furthermore, $\lim_{n\to\infty} \hat{f}(\varphi_n(y)) = \hat{f}(\bar{x})$ for all $y \in X$ and $f \in B$, and $\cap \varphi_n(X) = \{\bar{x}\}$.*

Proof. The map φ takes X into itself since $T1 = 1$.

Since T is a compact operator and the space of eigenvectors corresponding to 1 is one-dimensional by hypothesis, B can be written $B = R_1 \oplus N_1$ where $R_1 = \{(T-I)f \mid f \in B\}$ and $N_1 = \{f \mid Tf = f\} = (c)$. The closed subspaces R_1 and N_1 are invariant under T [4].

Further, T is quasinilpotent on R_1. For, if $g \in R_1$ and $Tg = g$, then $g \in N_1 \cap R_1 = (0)$. Therefore 1 is not an eigenvalue of T on R_1. Also there are no other eigenvalues of T on R_1 since each eigenvalue of T on R_1 is an eigenvalue of T on B and $\sigma(T) = \{0, 1\}$ by hypothesis. Thus T is quasinilpotent on R_1 and so

$$\lim_{n\to\infty}\left(\sup_{Tf\neq f}\frac{\|T^n(T-I)f\|}{\|(T-I)f\|}\right)^{1/n} = \lim_{n\to\infty}\|T^n\|_{R_1}^{1/n} = 0 .$$

Therefore for each $\varepsilon > 0$ there exists $P^* > 0$ such that $\|T^n(T-I)f\| < P^*\varepsilon^n\|(T-I)f\|$ for all positive integers n and all $f \in B$. Then letting $P = P^*\|T - I\|$ we have $\|T^n(T-I)f\| < P\varepsilon^n\|f\|$ for all positive integers n and all $f \in B$.

Now fix $x \in X$. For each $f \in B$, $|\hat{f}(\varphi_{n+1}(x)) - \hat{f}(\varphi_n(x))| = |[(T - I)f]^\wedge(\varphi_n(x))| = |[T^n(T-I)f]^\wedge(x)| \leqq \|T^n(T-I)f\| < P\varepsilon^n\|f\|$ for all positive integers n. Therefore $\{\hat{f}(\varphi_n(x))\}$ is a Cauchy sequence of complex numbers and so $\lim_{n\to\infty}\hat{f}(\varphi_n(x))$ exists for each $f \in B$. Let $l(f) = \lim_{n\to\infty}\hat{f}(\varphi_n(x))$. Then it is easy to verify that l is a linear multiplicative functional on B. Also $l \neq \theta$ since $T1 = 1$ implies $l(1) = 1 \neq 0$. Consequently there exists $\bar{x} \in X$ defined by $\hat{f}(\bar{x}) = l(f)$ for all $f \in B$ and thus $\lim_{n\to\infty}(T^n f)^\wedge(x) = \lim_{n\to\infty}\hat{f}(\varphi_n(x)) = \hat{f}(\bar{x})$ for all $f \in B$. Also $Tf \in B$, and so $\lim_{n\to\infty}(Tf)^\wedge(\varphi_n(x)) = (Tf)^\wedge(\bar{x})$ for all $f \in B$; this implies $\lim_{n\to\infty}\hat{f}(\varphi(\varphi_n(x)) = \hat{f}(\varphi(\bar{x}))$. However,

$$\lim_{n\to\infty}\hat{f}(\varphi(\varphi_n(x))) = \lim_{n\to\infty}\hat{f}(\varphi_{n+1}(x)) = \hat{f}(\bar{x})$$

for all $f \in B$. Therefore $\hat{f}(\varphi(\bar{x})) = \hat{f}(\bar{x})$ for all $f \in B$ which proves that $\bar{x}$ is a fixed point of φ.

We next show that $\cap \varphi_n(X) = \{\bar{x}\}$. To this end, let $M_{\bar{x}} = \{f \mid \hat{f}(\bar{x}) = 0\}$. Since $\varphi(\bar{x}) = \bar{x}$, the closed maximal ideal $M_{\bar{x}}$ is invariant under T. Also 1 is not an eigenvalue of $T|_{M_{\bar{x}}}$. For, if there exists $f \in M_{\bar{x}}$ with $Tf = f$, then f is an eigenvector of T which must equal a constant c, say, by hypothesis. But $c = 0$ since the only constant in $M_{\bar{x}}$ is 0. Since $\sigma(T) = \{0, 1\}$, T is quasinilpotent on $M_{\bar{x}}$.

Now let y be an arbitrary element in X. Since $f - \hat{f}(\bar{x})1 \in M\bar{x}$ and T is quasinilpotent on $M\bar{x}$ we have that

$$\lim_{n\to\infty} |\hat{f}(\varphi_n(y)) - \hat{f}(\bar{x})|^{1/n} = \lim_{n\to\infty} |T^n(f - \hat{f}(\bar{x})1)\hat{\ }(y)|^{1/n} = 0 .$$

Using an argument similar to one used in the first part of this proof, it can be shown that for each $\varepsilon > 0$ there exists $P_1 > 0$ such that $|\hat{f}(\varphi_n(y)) - \hat{f}(\bar{x})| < P_1\varepsilon^n \|f\|$ for all $f \in B$, $n > 0$ and $y \in X$. This implies that if $\mathscr{U}$ is an open subset of X with $\bar{x} \in \mathscr{U}$, then $\varphi_n(X) \subset \mathscr{U}$ for large n. Therefore $\bigcap \varphi_n(X) = \{\bar{x}\}$. It now follows easily that $\bar{x}$ is the only fixed point of φ.

(The uniqueness of $\bar{x}$ also follows from the fact that the dimensions of $\{f \mid Tf = f\}$ and $\{l \in B^* \mid T^*l = l\}$ are equal. Since $\{f \mid Tf = f\}$ is one dimensional, once we have shown that $\bar{x}$ is a fixed point of φ in X, then it must be unique.)

We now combine these lemmas to prove the following.

THEOREM 1.7. *Suppose B is a commutative semi-simple Banach algebra with unit* 1 *and maximal ideal space X and T is a nonzero compact endomorphism of B. If φ is the map on $X' = X \cup \{\theta\}$ induced by T, then $\bigcap \varphi_n(X')$ is finite. If X is connected, then $\bigcap \varphi_n(X)$ is a singleton.*

Proof. If T is nilpotent, then $\bigcap \varphi_n(X') = \{\theta\}$ and there is nothing further to prove.

Assume T is not nilpotent. From Lemma 1.3 there exists a smallest positive integer M and a nonzero idempotent $E = T^M 1$ with the property that $TE = E$, $T\colon BE \to BE$ and $B = BE \oplus B(1 - E)$. Also $Z = \{x \in X \mid \hat{E}(x) = 1\}$ is the maximal ideal space of BE, $\varphi(Z) \subset Z$ and $\varphi_M\colon X \backslash Z \to \{\theta\}$. Let $\mathscr{S} = \bigcap \varphi_n(Z)$. Since $\bigcap \varphi_n(X') = \mathscr{S} \cup \{\theta\}$ it suffices to prove that $\mathscr{S}$ is finite.

Consider T on BE. Since E is a unit in BE and $TE = E$, Lemma 1.4 implies that T induces a compact endomorphism $\bar{T}$ on $B_1 = BE/\ker(\mathscr{S})$ which satisfies $\bar{T}\bar{E} = \bar{E}$ and $\sigma(\bar{T}) \subset \{\lambda \mid |\lambda| = 1\} \cup \{0\}$. Letting X_1 denote the maximal ideal space of B_1 and $\bar{\varphi}$ the map on X_1 induced by $\bar{T}$, Lemma 1.4 also implies $\mathscr{S} = \bigcap \bar{\varphi}_n(X_1)$.

Since $\bar{T}$ is a compact endomorphism on B_1 and $\sigma(\bar{T}) \subset \{\lambda \mid |\lambda| = 1\} \cup \{0\}$, each nonzero eigenvalue of $\bar{T}$ is a root of unity and so there exists a positive integer N for which $\sigma(\bar{T}^N) = \{0, 1\}$. Also $\bar{T}\bar{E} = \bar{E}$ implies $\bar{T}^N\bar{E} = \bar{E}$. Therefore $\bar{T}^N$ is a compact endomorphism of B_1 with $\sigma(\bar{T}^N) = \{0, 1\}$ and by Lemma 1.5, B_1 can be written $B_1 = B_1\bar{E}_1 \oplus \cdots \oplus B_1\bar{E}_m$ where $\bar{E} = \sum_{k=1}^m \bar{E}_k$, $\bar{E}_k$ are idempotents in B_1, $\bar{T}^N \bar{E}_k = \bar{E}_k$ and all the eigenvectors of $\bar{T}^N$ on $B_1\bar{E}_k$ corresponding to 1 have the form $c\bar{E}_k$, c complex. We also have that $X_1 = Y_1 \cup \cdots \cup Y_m$ where Y_k is the maximal ideal space of $B_1\bar{E}_k$. It is clear that $\bar{\varphi}_N$ is the map on X_1 induced by $\bar{T}^N$ and so we have that $\bar{\varphi}_N(Y_k) \subset Y_k$, $k = 1, \cdots, m$. Thus $\cap \bar{\varphi}_{Nn}(X_1) = \cap \bar{\varphi}_{Nn}(Y_1) \cup \cdots \cup \cap \bar{\varphi}_{Nn}(Y_m)$.

Now using the fact that all the eigenvectors of $\bar{T}^N$ on $B_1\bar{E}_k$ have the form $c\bar{E}_k$, c complex, it follows from Lemma 1.6 that there exist $\bar{x}_k \in Y_k$ with $\cap \bar{\varphi}_{Nn}(Y_k) = \{\bar{x}_k\}$, $k = 1, \cdots, m$. Therefore

$$\begin{aligned} \mathscr{S} = \cap \varphi_n(Z) = \cap \bar{\varphi}_n(X_1) &= \cap \bar{\varphi}_{Nn}(X_1) \\ &= \cap \bar{\varphi}_{Nn}(Y_1) \cup \cdots \cup \cap \bar{\varphi}_{Nn}(Y_m) \\ &= \{\bar{x}_1, \cdots, \bar{x}_m\} . \end{aligned}$$

Thus $\mathscr{S}$ is finite and hence $\cap \varphi_n(X')$ is finite, as needed.

Finally, if X is connected, then the only nonzero idempotent in B is 1. In this case $T1 = 1$ and therefore φ maps X into itself. Hence $S = \cap \varphi_n(X)$ is connected and since S is finite, S must be a singleton.

2. We conclude with several miscellaneous theorems and examples relating to compact endomorphisms.

It was noted in the introduction that if a is a specific point in the maximal ideal space of a commutative semi-simple Banach algebra with unit 1, then the map $T: f \to \hat{f}(a)1$ is a compact endomorphism of B. We will show that if X is a compact connected Hausdorff space, then every nonzero compact endomorphism of $C(X)$ has this form. We also show that the same is true for C^1, the algebra of functions on [0, 1] with continuous first derivatives. We will begin this section with a theorem about compact endomorphisms of function algebras.

Recall that a function algebra is a sup-norm closed subalgebra of continuous functions on a compact set X which separates points of X and contains the constants. A peak set of a function algebra is a closed subset E of X for which there exists a function f in the algebra with $\|f\| = f(x) = 1$ for $x \in E$ and $|f(x)| < 1$ for $x \in X \backslash E$. A generalized peak point is a point x_0 in X such that $\{x_0\}$ is an intersection of peak sets, and the strong boundary of a function algebra is the collection of generalized peak points. Further, if W is a G_δ subset of X containing a generalized peak point x_0, then there exists a peak set E with $x_0 \in E \subset W$ [1].

THEOREM 2.1. *Let X be a compact connected Hausdorff space and suppose B is a function algebra on X whose maximal ideal* *space is X. Further, assume $0 \neq T$ is a compact endomorphism of B with φ the continuous function on X induced by T. If $\varphi(x_0)$ is a generalized peak point of B for some $x_0 \in X$, then $Tf = f(\varphi(x_0))1$ for all $f \in B$.*

Proof. Assume $\varphi(x_0)$ is a generalized peak point of B. The claim is that $\varphi(x) = \varphi(x_0)$ for all $x \in X$. Suppose the contrary that there exists $y \in X$ with $\varphi(y) \neq \varphi(x_0)$. Since $\varphi(x_0)$ is a generalized peak point, there exists a peak set E such that $\varphi(x_0) \in E$ and $\varphi(y) \notin E$. For this set E, let $f \in B$ satisfy $||f|| = f(x) = 1$ for all $x \in E$ and $|f(x)| < 1$ for $x \in X \backslash E$. Further, let $f_n = (\frac{1}{2}(1+f))^n$. Then $||f_n|| = 1$ and since T is a compact operator, there exist a subsequence $\{f_{n_k}\}$ and a function $g \in B$ with $Tf_{n_k} \to g$ uniformly. Clearly $\lim_{n\to\infty}(\frac{1}{2}(1 + f(x)))^n = 1$ if $f(x) = 1$ and $\lim_{n\to\infty}(\frac{1}{2}(1 + f(x)))^n = 0$ if $f(x) \neq 1$. Since $g(x) = \lim_{k\to\infty}(\frac{1}{2}(1 + f(\varphi(x))))^{n_k}$ for $x \in X$, the continuous function g can assume at most two values, 0 and 1. However, the domain of g is connected. Hence g must be constant. This leads to a contradiction since if $\varphi(y) \notin E$, then $g(y) = \lim_{k\to\infty}(\frac{1}{2}(1 + f(\varphi(y))))^{n_k} = 0$ while $g(x_0) = \lim_{k\to\infty}(\frac{1}{2}(1 + f(\varphi(x_0))))^{n_k} = 1$. Therefore $\varphi(x) = \varphi(x_0)$ for all $x \in X$ as claimed.

If X is a compact Hausdorff space, then every $x \in X$ is a generalized peak point of $C(X)$. Consequently, we have the following immediate corollary of Theorem 2.1.

COROLLARY 2.2. *If X is a compact connected Hausdorff space, then every nonzero compact endomorphism T of $C(X)$ has the form $Tf = f(x_0)1$ for some $x_0 \in X$.*

THEOREM 2.3. *Let C^1 be the algebra of functions on $[0, 1]$ with continuous first derivatives, pointwise arithmetic operations and $||f|| = ||f||_\infty + ||f'||_\infty$. Then every nonzero compact endomorphism T on C^1 has the form $Tf = f(c)1$ for some $c \in [0, 1]$.*

Proof. Let T be a compact endomorphism of C^1 and φ the map on $[0, 1]$ induced by T. Then $\varphi \in C^1$. We claim that φ is a constant function. Suppose φ is not constant. Then there exists $a \in (0, 1)$ with $\varphi'(a) \neq 0$. Let $b = \varphi(a)$. Then $b \in (0, 1)$. For each positive integer n, let $f_n(x) = \int_0^x e^{-n(b-t)^2} dt$. Then $f_n \in C^1$, $\sup_{0\le x\le 1}|f_n(x)| = \int_0^1 e^{-n(b-t)^2} dt < 1$ and $\sup_{0\le x\le 1}|f_0'(x)| = \sup_{0\le x\le 1} e^{-n(b-x)^2} = 1$. Therefore $||f_n|| < 2$ for all n. Since $\{f_n\}$ is a bounded set in C^1 and T is a compact endomorphism, there exist $g \in C^1$ and $\{f_{n_k}\}$ with $Tf_{n_k} \to g$. In particular g' is continuous and $(Tf_{n_k})' \to g'$ uniformly. Now

$(Tf_{n_k})'(x) = f_{n_k}'(\varphi(x))\varphi'(x) = e^{-n_k(b-\varphi(x))^2}\varphi'(x)$, and hence

$$g'(a) = \lim_{k\to\infty} e^{-n_k(b-\varphi(a))^2}\varphi'(a) = \varphi'(a) .$$

Since $\varphi(a) = b$, $g'(a) \neq 0$. However, since $\varphi'(a) \neq 0$, $\varphi(x) \neq b$ in some deleted interval about a, and so it follows that

$$g'(x) = \lim_{k\to\infty} e^{-n_k(b-\varphi(x))^2}\varphi'(x) = 0$$

in that deleted interval. This is a contradiction to the continuity of g'. Hence $\varphi' = 0$ and φ is a constant function. Therefore $(Tf)(x) = f(\varphi(x)) = f(c)$ for some $c \in [0, 1]$ and so $Tf = f(c)1$.

Modifications of the statements and proofs of Theorem 2.1 and Corollary 2.2 for disconnected X are straightforward. For example, if X is an arbitrary compact Hausdorff space and T is a compact endomorphism of $C(X)$, then there exist a finite number of idempotents $E_1, \cdots, E_m$ in $C(X)$ and points $t_1, \cdots, t_m \in X$ with $Tf = \sum_{k=1}^m f(t_k)E_{\pi(k)}$ where π is a permutation of the set of integers $\{1, \cdots, m\}$.

There is a similarity between Theorem 2.1 and the example of the disc algebra, namely, that in both cases the range of a nonconstant φ does not intersect the strong boundary. However, it is not possible to extend this by replacing strong boundary with Silov boundary as the following example shows. ($\boldsymbol{C}$ and $\boldsymbol{R}$ denote the complex and real numbers, respectively.)

EXAMPLE. Let X be the subset of $\boldsymbol{C} \times \boldsymbol{R}$ defined by $X = \{(z, 0) \mid |z| \leqq 1\} \cup \{(0, t) \mid 0 \leqq t \leqq 1\}$ and let $B = \{f \in C(X) \mid z \to f(z, 0)$ is analytic$\}$. Then B is a function algebra whose Silov boundary is $\{(z, 0) \mid |z| = 1\} \cup \{(0, t) \mid 0 \leqq t \leqq 1\}$. The point $(0, 0)$ is in the Silov boundary, but is not a generalized peak point. Define φ on X by $\varphi(z, 0) = (z/2, 0)$ and $\varphi(0, t) = (0, 0)$. Then it is easy to verify that T: $Tf = f\circ\varphi$ is a compact endomorphism of B and $\varphi(0, 0) = (0, 0)$ is in the Silov boundary. However T does not have the form $Tf = f(0, 0)1$. Note, though, that $\cap\, \varphi_n(X) = \{(0, 0)\}$.

Another reasonable conjecture from the example of the disc algebra might be that if T is an endomorphism of a function algebra B on X for which $\varphi(X)$ does not intersect the Silov boundary, then T is compact. This, too, is not true.

EXAMPLE. Let $X = \{(z, t) \mid |z| \leqq 1$ and $0 \leqq t \leqq 1\}$ and let $B = \{f \in C(X) \mid z \to f(z, t)$ is analytic for each $t\}$. The Silov boundary of B is $\{(z, t) \mid |z| = 1, 0 \leqq t \leqq 1\}$. Define φ by $\varphi(z, t) = (z/2, t)$. Then

$\varphi(X) = \{(z, t) \mid |z| \leqq 1/2,\ 0 \leqq t \leqq 1\}$ does not intersect the Silov boundary, yet $Tf = f \circ \varphi$ is not compact since, for instance, $\cap \varphi_n(X) = \{(0, t) \mid 0 \leqq t \leqq 1\}$ is not a singleton.

As a final example along these lines, we note that even if $\cap \varphi_n(X)$ is a singleton, the endomorphism $Tf = f \circ \varphi$ need not be compact. For, let $B = C(\bar{D})$, the algebra of continuous functions on the closed unit disc $\bar{D}$ and let $\varphi(z) = z/2$. Then $\cap \varphi_n(\bar{D}) = \{0\}$, while $Tf = f \circ \varphi$ is not compact because, as we have seen, each compact endomorphism on $C(\bar{D})$ has the form $Tf = f(a)1$ for some $a \in \bar{D}$.

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