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STRONG LIFTINGS COMMUTING WITH MINIMAL DISTAL FLOWS

RUSSELL A. JOHNSON

In this paper, we treat an aspect of the following problem. If a compact Hausdorff space X is given, and if T is a group of homeomorphisms of X which preserves a measure μ, then find conditions under which $M^\infty(X, \mu)$ admits a strong lifting (or strong linear lifting) which commutes with T. We will prove the following results.

Introduction. (1) Let (X, T) be a minimal distal flow. Then there exists an invariant measure μ such that $M^\infty(X, \mu)$ admits a strong linear lifting ρ commuting with T. The linear lifting ρ is "quasi-multiplicative" in the sense that $\rho(f \cdot g) = \rho(f) \cdot \rho(g)$ if $f \in C(X)$ and $g \in M^\infty(X, \mu)$. In particular, if (X, T) admits a unique invariant measure μ, then $M^\infty(X, \mu)$ admits ρ as above. This result may be viewed as a generalization of "Theorem LCG" of A. and C. Ionescu-Tulcea [7]; see 1.7. If T is *abelian*, then $M^\infty(X, \mu)$ admits a strong *lifting*.

(2) Let G be a compact group with Haar measure μ. Then $M^\infty(G, \mu)$ admits a strong linear lifting ρ (which is quasi-multiplicative), which commutes with both left and right multiplications on G.

The author would like to thank the referee for correcting and improving Corollary 3.10.

Preliminalies.

NOTATION 1.1. Let X be a compact Hausdorff space. If μ is a positive Radon measure on X, let $M^\infty(X, \mu)$ be the set of bounded, μ-measurable, complex-valued functions on X. Let $L^\infty(X, \mu)$ be the set of equivalence classes in $M^\infty(X, \mu)$ under the (usual) equivalence relation: $f \sim g \Leftrightarrow f - g = 0$ $\mu -$ a.e. If E is a Banach space, let $M^\infty(X, E, \mu) = \{f: X \to E | f$ is weakly μ-measurable, and Range (f) is precompact$\}$. (Recall $f: X \to E|$ is *weakly μ-measurable* if $x \to \langle f(x), e' \rangle$ is μ-measurable for all $e' = E' =$ topological dual of E.)

DEFINITIONS 1.2. Let X, μ be as in 1.1. A map ρ of $M^\infty(M, \mu)$ to itself is a *linear lifting* of $M^\infty(X, \mu)$ if: (i) $\rho(f) = f$ $\mu -$ a.e.; (ii) $f = g$ $\mu -$ a.e. $\Rightarrow \rho(f) = \rho(g)$ everywhere; (iii) $\rho(1) = 1$; (iv) $f \geqq 0 \Rightarrow \rho(f) \geqq 0$; (v) $\rho(af + bg) = a\rho(f) + b\rho(g)$ $(f, g \in M^\infty(X, \mu)$; $a, b \in C)$. If, in addition, (vi) $\rho(f \cdot g) = \rho(f) \cdot \rho(g)$ for all $f, g \in M^\infty(M, \mu)$, then ρ

is a *lifting* of $M^\infty(X, \mu)$. If (i)-(v) hold (if (i)-(vi) hold), and, in addition, (vii) $\rho(f) = f$ for all $f \in C(X)$, then ρ is a *strong linear lifting* (*strong lifting*). See [10, p. 34].

DEFINITION 1.3. Let ρ be a linear lifting of $M^\infty(X, \mu)$, and let E be a Banach space. We "extend ρ to $M^\infty(X, E, \mu)$" as follows: $\langle e', \rho(\phi)(x)\rangle = \rho\langle e', \phi\rangle(x)$ ($\phi \in M^\infty(X, E, \mu)$, $e' \in E'$, $x \in X$).

DEFINITION 1.4. Let ρ be a linear lifting of $M^\infty(X, \mu)$. Suppose that $\rho(f \cdot g) = \rho(f) \cdot \rho(g)$ whenever $f \in C(X)$ and $g \in M^\infty(X, \mu)$. Then ρ is a *quasi-multiplicative* linear lifting of $M^\infty(M, \mu)$.

DEFINITIONS 1.5. Let G be a topological group. The pair (G, X) is a *left transformation group* (t.g.) or *flow* if there is a continuous map $\Phi: G \times X \to X: (g, x) \to g \cdot x$ such that (i) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$; (ii) $idy \cdot x = x (g_1, g_2 \in G$; $idy =$ identity of G; $x \in X)$. One defines a *right transformation group* in the obvious way. Say that (G, X) is *free* (or, *G acts freely*) if, whenever $g \cdot x = x$, one has $g = idy$ $(g \in G,\ x \in X)$.

DEFINITIONS 1.6. Let G be a *compact* topological group, and let T be a locally compact topological group. The triple (G, X, T) is a *bitransformation graup* if (i) (G, X) and (X, T) are (left and right, respectively) t.g.s; (ii) $(g \cdot x) \cdot t = g \cdot (x \cdot t)$ $(g \in G,\ x \in X,\ t \in T)$. In our considerations, the topology of T will play no role, so we will assume T is discrete. If (G, X, T) is a bitransformation group, and $f \in M^\infty(X, \mu)$, we let $(f \cdot g)(x) = f(g \cdot x)$, and $(t \cdot f)(x) = f(x \cdot t) (g \in G$, $x \in X,\ t \in T)$.

DEFINITION 1.7. Let (X, T) be a right t.g. with T a topological group. Say that (X, T) is *distal* [2, 4] if whenever x and y are distinct elements of X, there is no net $(t_\alpha) \subset T$ such that $\lim_\alpha x \cdot t_\alpha = \lim_\alpha y \cdot t_\alpha$. If $X = T = G$ where G is a compact group, then the t.g. (G, G) defined by multiplication on G is distal. Say that (X, T) is *minimal* if, for each $x \in X$, the orbit $\{x \cdot t \mid t \in T\}$ is dense in X.

DEFINITION 1.8. Let Y be another compact Hausdorff space, and let $\tau: X \to Y$ be a continuous surjection. Again let μ be a positive Radon measure on X, and define $\nu = \tau(\mu)$. Then $M^\infty(Y, \nu)$ may be embedded in $M^\infty(X, \mu)$ via $f \to f \circ \tau$. Suppose ρ is a linear lifting of $M^\infty(X, \mu)$, and ρ_0 is a linear lifting of $M^\infty(Y, \nu)$. Say ρ *extends* ρ_0 if $\rho|_{M^\infty(Y,\nu)} = \rho_0$.

We will need several simple results concerning quasi-multiplicative, strong linear liftings. We include them in the following lemma.

LEMMA 1.9. *Let X be a compact Hausdorff space, μ a positive Radon measure on X with Support $(\mu) = X$. Let ρ be a quasi-multiplicative, strong linear lifting of $M^\infty(X, \mu)$. Let E be a Banach space.*

(a) *Let $\phi \in M^\infty(X, E, \mu)$. Let $f \in C(X)$. Then $\rho(f \cdot \phi)(x) = f(x) \cdot \rho(\phi)(x)(x \in X)$.*

(b) *Let $f: X \to E$ be weakly continuous. Let $\phi \in M^\infty(X, \mu)$. Then $\rho(\phi \cdot f)(x) = \rho(\phi)(x) \cdot f(x)(x \in X)$.*

(c) *Let $\phi \in M^\infty(X, E, \mu)$. Suppose $K \subset U \subset X$, where K is compact and U is open. If $\phi(x) = 0$ for $\mu -$ a.a. $x \in U$, then $\rho(\phi)(x) = 0$ for all $x \in K$.*

Proof. Using the definition of $\rho(f \cdot \phi)$ (1.2), we have $\langle e', \rho(f \cdot \phi)(x)\rangle = \rho\langle e', f \cdot \phi\rangle(x) = \rho(f \cdot \langle e', \phi\rangle)(x) = \rho(f)(x) \cdot \rho\langle e', \phi\rangle(x) = f(x)\langle e', \rho(\phi)(x)\rangle$ $(e' \in E', x \in X)$. Part (a) follows. Part (b) is proved in a similar way. To prove (c), let $f \in C(X)$ be equal to zero on K and 1 on $X \sim U$. Then $f(x)\phi(x) = \phi(x)$ for μ-a.a.x. It follows that $\rho(f \cdot \phi)(x) = \rho(\phi)(x)$ for all $x \in X$. By 1.7(a), $\rho(\phi)(x) = 0$ if $x \in K$.

We remark that, in 1.7(c), one need only assume that $\phi(x) = 0$ *weakly* a.e. on U; i.e., that $\langle e', \phi(x)\rangle = 0$ for μ-a.a. $x \in U$ $(e' \in E')$. Also note that E may very well be C, in which case $M^\infty(X, E, \mu) = M^\infty(X, \mu)$.

2. A reduction.

We will prove a preliminary result (2.2), which will then be used in proving the main Theorems 3.1 and 3.7.

Assumptions, Notation 2.1. Let X be a compact Hausdorff space with Radon measure μ such that (i) $\mu(X) = 1$; (ii) Support $(\mu) = X$. Let (G, X, T) be a bitransformation group (1.5), where G is compact and T is any (discrete) group. Suppose μ is both G- and T- invariant (thus $\mu(f \cdot g) = \mu(f)$ and $\mu(t \cdot f) = \mu(f)$ for all $f \in C(X)$, $t \in T$, and $g \in G$). Also suppose G acts *freely* (1.5). Let $Y = X/G$ (the space of G-orbits with the quotient topology), with $\pi: X \to Y$ the canonical projection. Since G and T commute (1.5), there is a natural transformation group (Y, T). If ρ is a linear lifting of $M^\infty(X, \mu)$, say that ρ *commutes with* G (and T) if $\rho(f \cdot g) = \rho(f) \cdot g$ (and $\rho(t \cdot f) = t \cdot \rho(f)$) for all $f \in M^\infty(X, \mu)$ and $g \in G$ (and $t \in T$).

PROPOSITION 2.2. *With assumptions and notation as in* 2.1, *let $\nu = \pi(\mu)$. Suppose ρ_0 is a quasi-multiplicative, strong linear lifting of $M^\infty(Y, \nu)$ which commutes with T. Then there is a quasi-multiplicative, strong linear lifting ρ of $M^\infty(X, \mu)$ which extends ρ_0 and commutes with G and T.*

The proof is modeled on the proof of a similar proposition in [9]. We first show that 2.2 is implied by a seemingly weaker result. More terminology is needed.

Notation 2.3. Let H be a closed, normal subgroup of G. Let $\pi_H: X \to X/H$ be the projection, and let $\nu_H = \pi_H(\mu)$. Then $(G/H, X/H)$ is a free t.g. Each $t \in T$ induces a homeomorphism (again called t) of X/H onto X/H, and $(G/H, X/H, T)$ is a bitransformation group.

THEOREM 2.4. *With the notation of* 2.3, *let H be Lie. Write $Z = X/H$. Suppose there is a strong, quasi-multiplicative, linear lifting δ of $M^\infty(Z, \nu_H)$ which commutes with G/H and T. Then there is a strong, quasi-multiplicative, linear lifting ρ of $M^\infty(X, \mu)$ which extends δ and commutes with G and T.*

Proof of 2.2, *using* 2.4. Let J be the set of all pairs (P, β), where P is a closed normal subgroup of G, and β is a quasi-multiplicative, strong linear lifting of $M^\infty(X/P, \nu_P)$ which extends ρ_0 and commutes with G/P and T. Then $(G, \rho_0) \in J$. Order J as follows: $(H_1, \beta_1) \leqq (H_2, \beta_2) \Leftrightarrow H_1 \supset H_2$ and β_2 extends β_1. We first show (*) J is inductive under $\leqq$.

To prove (*), we use methods of [8, pp. 29-33]. Let $J_0 = \{(P_\alpha, \beta_\alpha) | \alpha \in A\}$ be a totally ordered subset of J, and let $P = \bigcap_{\alpha \in A} P_\alpha$. Suppose first that A has no countable cofinal set. In this case, $M^\infty(X/P, \nu_P) = \bigcup_{\alpha \in A} M^\infty(X/P_\alpha, \nu_{P_\alpha})$. Thus if $f \in M^\infty(X/P, \nu_P)$, we may well-defined $\beta(f) = \beta_\alpha(f)$ for appropriate α. It is easily seen that (P, β) is in J, and that it is an upper bound for J_0.

Now assume that A contains a countable cofinal subset. We assume that $J_0 = \{(P_n, \beta_n) | n \geqq 1\}$, and let $P = \bigcap_{n \geqq 1} P_n$. Let Q_n be the projection of $M^\infty(X/P, \nu_P)$ onto $M^\infty(X/P_n, \nu_{P_n})$ [8, Theorem 3, p. 32]. As in [8, Theorem 2, p. 46], we let $\mathscr{U}$ be an ultrafilter on $\{n | n \geqq 1\}$ finer than the Fréchet filter. Define $\beta(f)(x) = \lim_{\mathscr{U}} \beta_n(Q_n f)(x) (f \in M^\infty(X/P, \nu_P);\ x \in X/P)$. As in [8, Theorem 2, p. 46], one checks that β is a linear lifting. We must show that β is (i) strong; (ii) quasi-multiplicative.

To do this, fix n momentarily. We will give a formula for Q_n. Let $L = P_n/P$. Then $X/P_n \approx (X/P)/L$. If $f \in L^2(X/P, \nu_P) \supset L^\infty(X/P, \nu_P)$, let $(\tilde{Q}_n f)(x) = \int_L f(l \cdot x)dl$ ($x \in X/P$; dl = normalized Haar measure on L). The right-hand side is defined ν_P-a.e., and may be viewed as an element of $L^2(X/P_n, \nu_{P_n}) \supset L^\infty(X/P_n, \nu_{P_n})$. Simple manipulations, plus uniqueness in [8, Prop. 7, p. 29], show that $\tilde{Q}_n = Q_n$.

Let $f \in C(X/P)$. From the formula just given, we see that $Q_n f \to f$ *uniformly*. It is now easy to check that β is strong. To see that β is quasi-multiplicative, let $f \in C(X, P)$, $g \in M^\infty(X/P, \nu_P)$. Let $f_n = Q_n f$. Observe that $|\beta_n(Q_n(f \cdot g))(x) - \beta_n(Q_n(f_n \cdot g))(x)| \leqq \|Q_n(f \cdot g) - f_n \cdot g)\|_\infty$, the norm being that of $L^\infty(X/P, \nu_P)$. By [8, Prop. 7(2), p. 29], this is $\leqq \|f \cdot g - f_n \cdot g\|_\infty \leqq \|f - f_n\|_\infty \|g\|_\infty \to 0$ as $n \to \infty$. So, if $x \in X/P$, then $\beta(f \cdot g)(x) = \lim_{\mathscr{U}} \beta_n(Q_n(f \cdot g))(x) = \lim_{\mathscr{U}} \beta_n(Q_n(f_n \cdot g))(x) =$ (by Prop. 7(4), p. 29, of [8]) $\lim_{\mathscr{U}} \beta_n(f_n \cdot Q_n g)(x) = \lim_{\mathscr{U}} f_n(x) \cdot \beta_n(Q_n g)(x) = f(x) \cdot \beta(g)(x)$. So β is quasi-multiplicative. It is easy to check that β commutes with G/P (this uses 28.72e of [5]), and T. Hence (P, β) majorizes J_0.

Now let (K, ρ) be a maximal element of J. If $K \neq \{idy\}$, we may use the technique of [7] to find a closed normal subgroup P of G such that $P \neq K$ and K/P is a Lie group. Applying 2.4 (with $G \leftrightarrow G/P$, $H \leftrightarrow K/P$), we find an element $(\bar{K}, \bar{\rho})$ of J which strictly majorizes (K, ρ). This contradicts maximality, so $K = \{idy\}$. Hence 2.2 is true if 2.4 is true.

We turn now to the proof of 2.4. Basically, it is a rehash of the proof of Theorem 2.7 in [9], with modifications due to the fact that we now assume δ to be a strong *linear* lifting. We indicate the modifications; it is assumed that the reader has [9, §3] before him. Notation is as in 2.3.

Proof of 2.4. Let $f \in M^\infty(X, \mu)$. Recall $Z = X/H$. For the moment, we forget about T, and consider only that part of 2.4 which refers to G and H. For $z_0 \in Z$, define $R^f(z_0)$ as in [9, 3.5]. The first modification must be made in the proof of [9, 3.7]. Note that [9, O1] need not be true, since δ is not a lifting. We avoid this problem by replacing [9, O1] with 1.8(c) (with $E = C$), and by letting L resp. $\tilde{L}$ be compact subsets of $\mathscr{O}$ resp. $\tilde{\mathscr{O}}$ such that $z_0 \in L \subset \tilde{L}$. The argument of the fifth paragraph on [9, p. 75] now proves that $\tilde{B}(z) = A_z(B(z))$ for all $z \in L \subset \tilde{L}$; in particular for $z = z_0$.

The second modification must be made in (*) of the proof of [9, 3.8(b)]. We can no longer state that, if $w \in M^\infty(Z, \nu_H)$ and $b \in M^\infty(Z, L^P(H, \lambda))$, then $\delta(w \cdot b)(z) = \delta(w)(z) \cdot \delta(b)(z)$. However, note that $b_p \colon Z \to L^P(H, \lambda)$ (defined in [9, 3.3)] is weakly continuous if $f \in C(X)$. So, we may replace [9, (*) and (O1)] by 1.8(b) and 1.8(c).

In the proof of [9, 3.8(c)], we again replace [9, (*) and (O1)] by 1.8(b) and 1.8(c).

In 3.10 and 3.11, we make the change discussed in [10]. Namely, let (W_n) be a D'-sequence in H such that $g^{-1} W_n g = W_n (g \in G)$. As in [9], define $T_n^f(x_0) = 1/\lambda(W_n) \int_H R^f(z_0)(hx_0)\psi_{W_n}(h)d\lambda(h)$ $(x_0 \in X,\ z_0 = \pi(x_0)$, $\psi \leftrightarrow$ characteristic function). Then, let $\rho(f)(x) = \lim_{\mathscr{U}} T_n^f(x_0)$, where $\mathscr{U}$ is an ultrafilter finer than the Fréchet filter. It turns out (use the Case I portions of [9, 3.11–3.14, and also 3.15]) that ρ is a strong linear lifting of $M^\infty(X, \mu)$ which extends δ and commutes with G. We will show that ρ is also quasi-multiplicative. To do this, suppose $f \in C(X)$ and $g \in M^\infty(X, \mu)$. Then $\lim_{\mathscr{}} T_n^f(x) = f(x)$ for *all* $x \in X$ [9, 3.14(b)]. Also, $R^f(z) =$ the equivalence class of $f|_{\pi_H^{-1}(z_0)}$ in $L^\infty(X, \lambda_{z_0})$ for *all* $z_0 \in Z$ (see [9, 2.6 and 3.8(b)]). Finally, $||R^g(z_0)||_\infty \leqq ||g||_\infty$ [6, 3.4(c)]. So,

$$||T_n^{f\cdot g}(x_0) - f(x_0)T_n^g(x_0)|| = \frac{1}{\lambda(W_n)}\left|\int_H (f(hx_0) - f(x_0))R^g(z_0)(hx_0)\psi_{W_n}(h)d\lambda(h)\right|$$

$$\leqq ||g||_\infty \frac{1}{\lambda(W_n)}\int_H |f(hx_0) - f(x_0)|\psi_{W_n}(h)d\lambda(h) \longrightarrow 0$$

as $n \to \infty$ since f is continuous. Hence $\rho(f \cdot g)(x_0) = f(x_0) \cdot \rho(g)(x)$, and ρ is quasi-multiplicative.

So far, we have shown that $M^\infty(X, \mu)$ admits a strong, quasi-multiplicative, linear lifting ρ which extends δ and commutes with G. To complete the proof of 2.4, we must show that ρ commutes with T. To see this, it suffices to prove

$$(\dagger) \quad R^{t\cdot f}(z_0)(hx_0) = R^f(z_0 \cdot t)(hx_0 \cdot t)(x_0 \in X,\ z_0 = \pi_H(x_0),\ h \in H,\ t \in T)\ .$$

But (for notation see [9, 3.3]), one has $b_p^{t\cdot f}(z) = b_p^f(z \cdot t)$ for ν_H-a.a.z (because the map $z \to z \cdot t$ preserves ν_H). Let σ be a linear functional on $L^p(H, \lambda)$. Then (for notation see [9, 3.4(c)]), one has $\langle B^{t\cdot f}(z_0), \sigma\rangle = \delta\langle b_p^{t\cdot f}, \sigma\rangle(z_0) = \delta\langle b_p^f(z \cdot t), \sigma\rangle(z_0) =$ (since δ commutes with T) $= \langle B^f(z_0 \cdot t), \sigma\rangle$. By [9, 3.5], we see that (†) is true. This completes the proof of 2.4.

REMARK 2.5. Prof. D. Johnson has shown (unpublished) how that part of the proof of 2.4 involving a D'-sequence may be simplified using an approximate identity on $L^1(H, \lambda)$.

3. Main results.

THEOREM 3.1. *Let G be a compact topological group with Haar measure γ. Then $M^\infty(G, \gamma)$ admits a strong, quasi-multiplicative, linear lifting ρ which commutes with both left and right translations on G.*

Proof. Apply 2.2 with $X = T = G$.

Let us now consider minimal distal flows (1.7). From [2, 3, 4, 5] we have the definition and theorem given in 3.2 and 3.3 below.

DEFINITION 3.2. Let (X, T) and (Y, T) be transformation groups.

Say (X, T) is an *almost-periodic* (a.p.) *extension* of (Y, T) if there is a bitransformation group (G, Z, T) and a closed subgroup H of G (not normal, in general) such that (i) $(Z/G, T) \simeq (Y, T)$ (i.e., there is a homeomorphism $h: Y \to Z/G$ such that $h(y \cdot t) = h(y) \cdot t$ for all $y \in Y$, $t \in T$); (ii) $(Z/H, T) \simeq (X, T)$.

Furstenberg Structure Theorem 3.3. Let (X, T) be a minimal distal flow. There is an ordinal α and a collection $\{(X_\beta, T) \mid \beta \leqq \alpha\}$ of flows such that (i) X_0 contains just one point; (ii) (X_β, T) in an a.p. extension of $(X_{\beta-1}, T)$ if β is a successor ordinal; (iii) if β is a limit ordinal, then (X_β, T) is an inverse limit of $\{(X_\omega, T) \mid \omega < \beta\}$ ([3]; thus $C(X_\beta) = \operatorname{clos} \bigcup_{\omega<\beta} C(X_\omega)$, where $C(X_\omega)$ is injected into $C(X_\beta)$ in the natural way); (iv) $(X_\alpha, T) \simeq (X, T)$.

Notation 3.4. Let (X, T) be a minimal distal flow, and let $\{(X_\beta, T) \mid \beta \leqq \alpha\}$ be as in 3.3. If β is a successor ordinal, let (G_β, Z_β, T) be a bitransformation group and $H_\beta \subset G$ a closed subgroup such that (i) $(Z_\beta/G_\beta, T) \simeq (X_{\beta-1}, T)$; (ii) $(Z_\beta/H_\beta, T) \simeq (X_\beta, T)$. If $\beta \leqq \omega \leqq \alpha$, there is a homomorphism (i.e., a map which commutes with the flows) $\Pi_{\gamma\beta}: (X_\gamma, T) \to (X_\beta, T)$. We write Π_β for the homomorphism taking (X, T) to $(X_\beta, T)(\beta < \alpha)$. If μ is a Radon measure on X, let $\mu_\alpha = \Pi_\alpha(\mu)$.

DEFINITION 3.5. Consider some left t.g. (L, W) with L and W compact. Let $Y = W/L$, and let ν be a Radon measure on Y. Let γ be normalized Haar measure on L. The *L-Haar lift* μ of ν is defined as follows:

$$\mu(f) = \int_Y \left(\int_G f(g \cdot x) d\gamma(g)\right) d\nu(y) \qquad (f \in C(W)) .$$

PROPOSITION 3.6. *There is a T-invariant probability measure μ on X such that* (i) *if β is any ordinal $\leqq \alpha$, if $\omega < \beta$, and if $f \in C(X_\omega)$, then $\mu_\beta(f) = \mu_\omega(f)$;* (ii) *if β is a successor ordinal, and if $\eta_\beta: (Z_\beta, T) \to (Z_\beta/H_\beta, T) \simeq (H_\beta, T)$ (see* 3.4), *then $\mu_\beta = \eta_\beta(\nu)$, where ν is the G_β-Haar lift of $\mu_{\beta-1}$.*

The proof of 3.6 is an easy application of 3.3 and transfinite induction.

THEOREM 3.7. *Let (X, T) be a minimal distal. There is an invariant measure μ on X such that $M^\infty(X, \mu)$ admits a strong, quasi-multiplicative, linear lifting ρ which commutes with T.*

Proof. Let μ be as in 3.6. Let J be the set of ordinals $\beta \leqq \alpha$

for which $M^{\infty}(X_{\beta}, \mu_{\beta})$ admits a quasi-multiplicative, strong linear lifting ρ_{β} which commutes with T. Clearly $0 \in J$. Suppose $\gamma \in J$, and let $\beta = \gamma + 1$. Let ν be the G_{β}-Haar lift of ν_{γ}. By 2.2, $M^{\infty}(Z_{\beta}, \nu)$ admits a quasi-multiplicative, strong linear lifting $\tilde{\rho}_{\beta}$, which extends ρ_{γ} and commutes with G_{β} and T. Then $\tilde{\rho}_{\beta}$ commutes with H_{β}, and so the formula $\rho_{\beta}(f) = \tilde{\rho}_{\beta}(f)(f \in M^{\infty}(X_{\beta}, \mu_{\beta}) \subset M^{\infty}(Z_{\beta}, \nu))$ defines a quasi-multiplicative, strong linear lifting of $M^{\infty}(X_{\beta}, \mu_{\beta})$ which commutes with T. If β is a limit ordinal, and if $\{\gamma \mid \gamma < \beta\} \subset J$, then the methods used in the proof of 2.2 may be applied again to show that $\beta \in J$. Hence $\alpha \in J$, and ρ_{α} satisfies the conditions of 3.7.

COROLLARY 3.8. *If (X, T) is minimal distal with unique invariant measure μ, then $M^{\infty}(X, \mu)$ admits a quasi-multiplicative, strong linear lifting which commutes with T.*

COROLLARY 3.9. *If T is abelian and (X, T) is minimal distal, then there is an invariant measure μ on X for which $M^{\infty}(X, \mu)$ admits a strong lifting which commutes with T.*

Proof. Let $x_0 \in X$, and suppose $x_0 \cdot t_0 = x_0$ for some $t_0 \in T$. We claim that, in this case, $xt_0 = x$ for *all* $x \in X$. For, minimality of (X, T) implies that there is a net $(t_{\alpha}) \subset T$ such that $x_0 \cdot t_{\alpha} \rightarrow x$. Then $x \cdot t_0 = \lim_{\alpha}(x_0 \cdot t_{\alpha}) \cdot t_0 = \lim_{\alpha}(x_0 \cdot t_0) \cdot t_{\alpha} = x$. Hence if $S = \{t \in T \mid t$ fixes some $x \in X\}$, then $S = \{t \in T \mid t = idy$ on $X\}$. We may therefore (replacing T by T/S) assume that T acts *freely* (1.5) on X. Now, by 3.7, there is a strong *linear* lifting of $M^{\infty}(X, \mu)$ which commutes with T. By [11, Remark 2 following Theorem 1], there is a *lifting* ρ of $M^{\infty}(X, \mu)$ commuting with T. By [10, Theorem 2, p. 105], ρ is strong.

COROLLARY 3.10. *If (X, T) is a regular* [1] *minimal flow, then there is an invariant measure μ on X such that $M^{\infty}(X, \mu)$ admits a strong lifting ρ which commutes with T. In particular, (X, T) may be the universal minimal distal flow* [3].

Proof. We begin as in 3.9. Let $x_0 \in X$, and suppose $x_0 \cdot t_0 = x_0$ for some $t_0 \in T$. Let $x \in X$. By [1, Theorem 3], there is a homeomorphism $\varphi: X \rightarrow X$ such that (i) φ commutes with T; (ii) $\varphi(x_0) = x$. Then $x \cdot t_0 = \varphi(x_0) \cdot t_0 = \varphi(x_0 \cdot t_0) = \varphi(x_0) = x$. Now proceed as in 3.9.

REFERENCES

1. J. Auslander, *Regular minimal sets*, Trans. Amer. Math. Soc., **123** (1966), 469-479.
2. R. Ellis, *The Furstenberg structure theorem*, to appear in Pacific J. Math.
3. ———, *Lectures on Topological Dynamics*, Benjamin, New York, 1967.

4. R. Ellis, S. Glasner, and L. Shapiro, *PI flows*, Advances in Math., **17** (1975), 213-260.
5. H. Furstenberg, *The structure of distal flows*, Amer. J. Math., **85** (1963), 477-515.
6. E. Hewitt and K. Ross, *Abstract Harmonic Analysis II*, Springer-Verlag, New York-Heidelberg-Berlin, 1970.
7. A. and C. Ionescu-Tulcea, *On the existence of a lifting ... locally compact group*, Proc. Fifth Berekeley Symp. Math. Stat. and Prob., vol. 2, part 1, 63-97.
8. ———, *Topics in the Theory of Lifting*, Spring-Verlag, New York, 1969.
9. R. Johnson, *Existence of a strong lifting commuting with a compact groups of transformations*, Pacific J. Math., **76** (1978), 69-81.
10. ———, *Existence of a strong lifting commuting with a compact group of transformations II*, Pacific. J. Math., **82** (1979), 457-461.
11. A. Tulcea, *On the lifting property (V)*, Annals of Math. Stat., **36** (1965), 819-828.

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University of Southern California
Los Angeles, CA 90007