PACIFIC JOURNAL OF MATHEMATICS
Vol. 91, No. 1, 1980

ON A CHARACTERIZATION USING RANDOM SUMS

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Let X_1, X_2, and X_3 be independent random variables and let $Z_1 = X_1 + X_3$ and $Z_2 = X_2 + X_3$. It is known that if the characteristic functions of X_k, $k = 1, 2, 3$, do not vanish then the distribution of (Z_1, Z_2) determines the distributions of X_1, X_2, and X_3 up to a shift. The aim of this paper is to prove a result of a similar nature using sums of a random number of random variables. We shall use $\sim$ for "has the same distribution as," r. v. for "random variable, " ch. f. for "characteristic function," and p. g. f. for "probability generating function."

THEOREM 1. *Let $N, X_1, X_2, \cdots, Y_1, Y_2, \cdots$ be independent* r.v.'s *where $X_n \sim X$, $Y_n \sim Y$, $n = 1, 2, \cdots$, and X and Y are nondegenerate real-valued* r.v.'s *having* ch.f.'s *φ and ψ, respectively, which are of bounded variation on every finite interval. Let N be a nonnegative integer-valued* r.v. *with* p.g.f.

$$Q(s) = p_0 + \sum_{n=1}^{\infty} p_n s^n , \qquad |s| \leqq 1 , \qquad p_n = P(N = n)$$

and $0 < EN = m < \infty$. Assume that there is a neighborhood of 1 relative to the unit disk such that Q^{-1} exists in this neighborhood. Denote

$$U = 0 \text{ for } N = 0,\ U = X_1 + X_2 + \cdots + X_N \text{ for } N > 0 \text{ , and}$$
$$V = 0 \text{ for } N = 0,\ V = Y_1 + Y_2 + \cdots + Y_N \text{ for } N > 0 \text{ .}$$

Then the distribution of (U, V) uniquely determines the distribution of N.

Proof. Since $N, X_1, X_2, \cdots, Y_1, Y_2, \cdots$ are independent r.v.'s, the ch.f. of (U, V), $\varphi_{(U,V)}$, satisfies the following:

$$\begin{aligned}
\varphi_{(U,V)}(r, t) &= E(e^{irU+itV}) \\
&= E(e^{irU+itV} \mid N = 0) \cdot P(N = 0) + \sum_{n=1}^{\infty} E(e^{irU+itV} \mid N = n) \cdot P(N = n) \\
&= E(1) \cdot p_0 + \sum_{n=1}^{\infty} E(e^{ir(X_1+\cdots+X_n)+it(Y_1+\cdots+Y_n)}) \cdot p_n \\
&= p_0 + \sum_{n=1}^{\infty} [E(e^{irX}) \cdot E(e^{itY})]^n \cdot p_n \\
&= p_0 + \sum_{n=1}^{\infty} [\varphi(r) \cdot \psi(t)]^n \cdot p_n \\
&= Q(\varphi(r) \cdot \psi(t)) , \qquad r, t \in R .
\end{aligned}$$

Suppose there are other r.v.'s $N^*, X_1^*, X_2^*, \cdots, Y_1^*, Y_2^*, \cdots,$ satisfying the assumptions. By repeating the above procedure denoting U^* and V^* similarly we obtain

$$(1) \qquad \varphi_{(U^*,V^*)}(r, t) = Q^*(\varphi^*(r) \cdot \psi^*(t)) , \qquad r, t \in R .$$

Since (U^*, V^*) has the same distribution as (U, V), their ch.f.'s are identical; thus,

$$(2) \qquad Q^*(\varphi^*(r) \cdot \psi^*(t)) = Q(\varphi(r) \cdot \psi(t)) , \qquad r, t \in R .$$

Relation (2) is a functional equation and from this equation it will be shown that $Q^* = Q$.

The function Q is analytic inside the disk, thus the image of a domain under Q is a domain. There is a neighborhood of 1 relative to the unit disk such that Q^{*-1} exists and is analytic in this neighborhood. Thus there exists a neighborhood A of 1 relative to the unit disk such that Q^{*-1} exists and is analytic in $Q(A)$. Define

$$(3) \qquad q(s) = Q^{*-1}(Q(s)) \qquad s \in A .$$

Note that q is analytic in A and maps A into the unit disk. It can be assumed without loss of generality that $0 \notin A$.

Using relations (2) and (3),

$$(4) \qquad q(\varphi(r) \cdot \psi(t)) = \varphi^*(r) \cdot \psi^*(t) \qquad r, t \in R, \varphi(r) \cdot \psi(t) \in A .$$

By alternately allowing $r = 0$ and $t = 0$ it is found that $q(\varphi(r)) = \varphi^*(r)$ and $q(\psi(t)) = \psi^*(t)$. Substituting these into relation (4)

$$(5) \qquad q(\varphi(r) \cdot \psi(t)) = q(\varphi(r)) \cdot q(\psi(t)) \qquad r, t \in R, \ \varphi(r) \cdot \psi(t) \in A .$$

Since $0 \notin A$, there exist continuous functions φ_0 and ψ_0 such that $\varphi(r) = e^{\varphi_0(r)}$ and $\psi(t) = e^{\psi_0(t)}$ and $\varphi_0(0) = \psi_0(0) = 0$ where $\varphi(r) \cdot \psi(t) \in A$. Since φ and ψ are of bounded variation on finite intervals, φ_0 and ψ_0 are of bounded variation on finite intervals. Define

$$(6) \qquad q_0(b) = \ln q(e^b) \qquad e^b \in A ,$$

where we take the branch for which $\ln 1 = 0$. Then from relation (6)

$$(7) \qquad \begin{aligned} q_0(\varphi_0(r) + \psi_0(t)) &= \ln q(e^{\varphi_0(r) + \psi_0(t)}) \\ &= \ln q(\varphi(r) \cdot \psi(t)) \\ &= \ln [q(\varphi(r)) \cdot q(\psi(t))] \\ &= \ln q(\varphi(r)) + \ln q(\psi(t)) \\ &= \ln q(e^{\varphi_0(r)}) + \ln q(e^{\psi_0(t)}) \\ &= q_0(\varphi_0(r)) + q_0(\psi_0(t)) , \qquad \varphi(r) \cdot \psi(t) \in A . \end{aligned}$$

Consider the following integrals obtained by using equation (7)

$$\begin{aligned}\int_0^\beta q_0(\varphi_0(\alpha)+\psi_0(t))d\psi_0(t) &= \int_0^\beta [q_0(\varphi_0(\alpha))+q_0(\psi_0(t))]d\psi_0(t) \\ &= q_0(\varphi_0(\alpha))\cdot\psi_0(\beta)+\int_0^\beta q_0(\psi_0(t))d\psi_0(t)\end{aligned} \tag{8}$$

and

$$\begin{aligned}\int_0^\alpha q_0(\varphi_0(r)+\psi_0(\beta))d\varphi_0(r) &= \int_0^\alpha [q_0(\varphi_0(r))+q_0(\psi_0(\beta))]d\varphi_0(r) \\ &= \int_0^\alpha q_0(\varphi_0(r))d\varphi_0(r)+q_0(\psi_0(\beta))\cdot\varphi_0(\alpha)\end{aligned} \tag{9}$$

where α and β are fixed real numbers such that $\varphi(r)\cdot\psi(t)\in A$ for $0\leqq r\leqq\alpha$ and $0\leqq t\leqq\beta$. These integrals exist because φ_0 and ψ_0 are of bounded variation on finite intervals and q_0 is analytic. Using a change of variables on relations (8) and (9), the following integrals are obtained.

$$\int_{\psi_0(\alpha)}^{\varphi_0(\alpha)+\psi_0(\beta)} q_0(v)dv = q_0(\varphi_0(\alpha))\cdot\psi_0(\beta)+\int_0^{\psi_0(\beta)} q_0(v)dv\ . \tag{10}$$

$$\int_{\psi_0(\beta)}^{\varphi_0(\alpha)+\psi_0(\beta)} q_0(v)dv = q_0(\psi_0(\beta))\cdot\varphi_0(\alpha)+\int_0^{\varphi_0(\alpha)} q_0(v)dv\ . \tag{11}$$

By adding equations (10) and (11) right sides to left sides the following equation is obtained,

$$\begin{aligned}&\int_0^{\varphi_0(\alpha)+\psi_0(\beta)} q_0(v)dv + q_0(\psi_0(\beta))\cdot\varphi_0(\alpha) \\ &\quad = \int_0^{\varphi_0(\alpha)+\psi_0(\beta)} q_0(v)dv + q_0(\varphi_0(\alpha))\cdot\psi_0(\beta)\ .\end{aligned} \tag{12}$$

From this it is seen that

$$q_0(\psi_0(\beta))\cdot\varphi_0(\alpha) = q_0(\varphi_0(\alpha))\cdot\psi_0(\beta)\ . \tag{13}$$

Since X and Y are nondegenerate, $|\varphi(r)|<1$ and $|\psi(t)|<1$ almost everywhere. Thus $\varphi_0(\alpha)$ and $\psi_0(\beta)$ are different from zero almost everywhere and

$$\frac{q_0(\psi_0(\beta))}{\psi_0(\beta)} = \frac{q_0(\varphi_0(\alpha))}{\varphi_0(\alpha)}\ . \tag{14}$$

Since the choice of α is independent of β

$$q_0(\varphi_0(\alpha)) = c\varphi_0(\alpha) \qquad \text{where } c \text{ is a complex number}\ . \tag{15}$$

Since $q_0(b)=\ln q(e^b)$, $q(s)=s^c$ for $s\in A$.

Since c is complex, $c=a+ib$ where $a,b\in R$. Thus $Q^{*-1}(Q(s))=$

s^{a+ib} for $s \in A$ since $q(s) = Q^{*-1}(Q(s))$. Since A is a relative neighborhood of 1, there is a segment of the real line $[\delta, 1] \subset A$ where $0 < \delta < 1$. The function Q maps the unit disk into the unit disk, and Q^{*-1} maps $Q(A)$ into the unit disk. For $s \in [\delta, 1]$, $s^c = e^{c \ln s} = e^{a \ln s + ib \ln s} = e^{a \ln s} \cdot e^{ib \ln s}$. Since $|s^c| \leqq 1$, $a \ln s \leqq 0$ for $s \in [\delta, 1]$. Thus $a \geqq 0$ since $\ln s \leqq 0$. The function $Q(s)$ is real for s a real number and $Q^{*-1}(Q(s))$ is real for $Q(s)$ a real number. Thus for $s \in [\delta, 1]$, s^c is a real number and $b \ln s = 0 \bmod (2\pi)$. Thus $b = 0$ and $c = a \geqq 0$.

Since $Q^{*-1}(Q(s)) = s^c$ for $s \in A$, then $Q(s) = Q^*(s^c)$ for $s \in A$. The functions Q, Q^*, s^c are analytic for $0 < |s| < 1$, thus $Q(s) = Q^*(s^c)$ for $0 < |s| < 1$. Suppose $c = 0$. Then $Q(s) = Q^*(1) = 1$ for $0 < |s| < 1$. This implies that $EN = 0$ which is a contradiction. Thus $c \neq 0$. Since the expectation of N and N^* exist $\lim_{s \to 1} Q'(s) = \lim_{s \to 1} cs^{c-1} Q^{*\prime}(s^c)$ or $m = cm$. Thus $c = 1$ and $Q(s) = Q^*(s)$ for all $|s| \leqq 1$. $\square$

REMARK. A characterization for the distribution of N has been found using the assumptions of Theorem 1. The following shows that the assumption "that there is a neighborhood of 1 relative to the unit disk such that Q^{-1} exists in this neighborhood" is redundant.

THEOREM 2. *Let N be a nonnegative integer-valued* r.v. *with* p.g.f.

$$Q(s) = p_0 + \sum_{n=1}^{\infty} p_n s^n , \qquad |s| \leqq 1 , \qquad p_n = P(N = n) .$$

If $0 < EN < +\infty$, then Q is one-to-one in a relative neighborhood of 1.

Proof. Let $D = \{s: |s| < 1, s \in C\}$ and $Q(s) = u(s) + iv(s)$ where u and v are real-valued functions.

Let

$$G(x_1, y_1, x_2, y_2) = \begin{pmatrix} u_x(x_1, y_1), & u_y(x_1, y_1) \\ v_x(x_2, y_2), & v_y(x_2, y_2) \end{pmatrix}$$

where $x_1 + iy_1$, $x_2 + iy_2$ are in $\bar{D}$. The function Q is analytic in D if and only if u and v are differentiable in D and satisfy the Cauchy-Riemann equations.

Let

$$f(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} \qquad x + iy \in C .$$

Thus $f(x, y)$ is differentiable in D, f' may be represented by the Jacobian matrix of f,

$$f'(x, y) = \begin{pmatrix} u_x(x, y), u_y(x, y) \\ v_x(x, y), v_y(x, y) \end{pmatrix} \qquad x + iy \in C ,$$

and $f'(x, y)$ has continuous existension to $\bar{D}$.

The mapping $\det[G(x_1, y_1, x_2, y_2)]\colon R^4 \to R$ is continuous on $\bar{D} \times \bar{D} \subset R^4$. But

$$\det[G(x_1, y_1, x_2, y_2)] = u_x(x_1, y_1) \cdot v_y(x_2, y_2) - u_y(x_1, y_1) \cdot v_x(x_2, y_2) .$$

Since $\det[G(1, 0, 1, 0)] = |Q'(1)|^2 \neq 0$, there exists a convex neighborhood of 1 such that $\det[G(x_1, y_1, x_2, y_2] \neq 0$ is this convex (closed) neighborhood. Without loss of generality, we assume $\det[G(x_1, y_1, x_2, y_2)] \neq 0$ for all $x_1 + iy_1, x_2 + iy_2 \in \bar{D}$.

Let $\vec{c}, \vec{d} \in \bar{D}$. By the Mean Value Theorem for vector-valued functions

$$f(\vec{c}) - f(\vec{d}) = G(\vec{c}_1, \vec{c}_2)(\vec{c} - \vec{d})$$

where $\vec{c}_j = (1 - t_j)\vec{c} + t_j\vec{d}$, $j = 1, 2$, for some $t_j \in (0, 1)$. Note that $\vec{c}_j \in D$, $j = 1, 2$.

Since $\det G[(x_1, y_1, x_2, y_2)] \neq 0$, the matrix $G(x_1, y_1, x_2, y_2)$ represents a one-to-one linear map. Thus, if $\vec{c} \neq \vec{d}$, then $f(\vec{c}) \neq f(\vec{d})$. Thus, Q is one-to-one in a relative neighborhood of 1. $\square$

Note that in Theorem 1 nothing is said about the distributions of X and Y. The following example will show that more assumptions are needed in order to determine the distributions of X and Y.

EXAMPLE 1. Let N and N^* be distributed according to the p.g.f. $Q(s) = s^2$, $|s| \leqq 1$. Let X be distributed according to the characteristic function $\varphi(r) = 1 - 2|r|/\pi$ for $|r| \leqq \pi$ and $\varphi(r)$ is periodic with period 2π, and let $X^* \sim |\varphi(r)|$. Let $Y \sim Y^* \sim \psi(t)$ where $\psi(t)$ is any nonvanishing real-valued ch.f. $(U, V) \sim (U^*, V^*)$ since

$$Q^*(\varphi^*(r) \cdot \psi^*(t)) = Q(\varphi(r) \cdot \psi(t)) , \qquad r, t \in R \quad \text{although} \quad \varphi^*(r) \neq \varphi(r) .$$

Thus more conditions must be imposed in order to prove Theorem 3.

THEOREM 3. *Let $N, X_1, X_2, \cdots, Y_1, Y_2, \cdots$ be* r.v.'s *satisfying the assumptions of Theorem* 1, *and U and V be defined as in Theorem* 1. *Then the distribution of (U, V) uniquely determines the distributions of X and Y if one of the following conditions holds*:

(i) *The characteristic functions* φ *and* ψ *are analytic at zero.*

(ii) *There is a relative neighborhood* B *of* 1 *such that* $\varphi(r)\cdot\psi(t)\in B$, $r, t\in R$, *and* Q *is one-to-one on* B.

Proof. From the proof of Theorem 1 $Q^* = Q$ and

$$(1)\qquad Q(\varphi^*(r)\cdot\psi^*(t)) = Q(\varphi(r)\cdot\psi(t)) \qquad r, t\in R\,.$$

Thus by alternately letting $r = 0$ and $t = 0$

$$(2)\qquad Q(\varphi^*(r)) = Q(\varphi(r)) \quad\text{and}\quad Q(\psi^*(t)) = Q(\psi(t)) \qquad r, t\in R\,.$$

If condition (ii) is assumed, then it is clear that $\varphi^* = \varphi$ and $\psi^* = \psi$.

If conition (i) is assumed, then as before, Q has a local inverse at one and $\varphi^*(r) = \varphi(r)$ and $\psi^*(t) = \psi(t)$ for r, t in some neighborhood of zero. But since the functions are analytic ch.f.'s, $\varphi^* = \varphi$ and $\psi^* = \psi$.

Thus the distributions of X and Y are determined uniquely.

The following theorem has a proof very similar to that of Theorem 1.

THEOREM 4. *Let* $N, X_1, X_2, \cdots, Y_1, Y_2, \cdots$ *be independent* r.v.'s *with* $X_n\sim X$, $Y_n\sim Y$, $n = 1, 2, \cdots$, *where* X *and* Y *are symmetric real-valued nondegenerate* r.v.'s *having* ch.f.'s φ *and* ψ, *respectively, with* $0\leqq\varphi(r)\leqq 1$ *and* $0\leqq\psi(t)\leqq 1$, $r, t\in R$. *Let* N *be a nonnegative integer-valued* r.v. *with* p.g.f.

$$Q(s) = p_0 + \sum_{n=1}^{\infty} p_n s^n\,, \qquad |s|\leqq 1\,, \qquad p_n = P(N = n)$$

where $0 < EN = m < \infty$.

Denote U and V as in Theorem 1.

Then the distribution of (U, V) uniquely determines the distributions of X, Y, and N.

Proof. The proof of this theorem is the same as the proof of Theorem 1 up to relation (2). At this point the fact that φ and ψ are nonnegative real-valued functions can be used to simplify the proof. Since $EN > 0$ and $EN^* > 0$, Q and Q^* are strictly increasing on the interval $[0, 1]$. Thus the inverse of Q and Q^* exist as functions from $[p_0, 1]$ and $[p_0^*, 1]$, respectively, onto $[0, 1]$. Without loss of generality $p_0^*\leqq p_0$. By letting

$$(1)\qquad q(s) = Q^{*-1}(Q(s)) \qquad s\in[0, 1]$$

and using relation (2) in Theorem 1

$$q(\varphi(r)\cdot\psi(t)) = \varphi^*(r)\cdot\psi^*(t) \qquad r, t \in R\ . \tag{2}$$

Note that q is continuous since Q^* and Q are continuous. Taking alternately $r = 0$ and $t = 0$ and substituting in equation (2) gives

$$q(\varphi(r)\cdot\psi(t)) = q(\varphi(r))\cdot q(\psi(t)) \qquad r, t \in R\ . \tag{3}$$

Denote $A = \{a\colon a = \varphi(r),\ r \in R\}$ and $B = \{b\colon b = \psi(t),\ t \in R\}$. Since X and Y are nondegenerate, φ and ψ are not identically equal to 1. Since φ and ψ are real-valued, continuous, and $\varphi(0) = \psi(0) = 1$, there is an interval $[c, 1]$, $0 < c < 1$, such that $[c, 1] \subset A \cap B$. Thus

$$q(ab) = q(a)\cdot q(b) \qquad \text{for} \quad a, b, ab \in [c, 1]\ . \tag{4}$$

From [1], $q(s) = s^k$ for $s \in [c, 1]$ and k some real number. Using the same argument as in Theorem 1, $k = 1$ and $Q^*(s) = Q(s)$, $|s| \leqq 1$. Thus the distribution of N is uniquely determined.

Using relation (1), $q(s) = s$, and relation (2) yields $\varphi^*(r) = \varphi(r)$, $r \in R$, and $\psi^*(t) = \psi(t)$, $t \in R$. Thus the distributions of X and Y are uniquely determined.

REMARKS. In each of the theorems we have assumed $0 < EN = m < +\infty$. This assumption can be replaced by the assumption, "There exists a fixed smallest positive index j_0, such that $p_{j_0} > 0$." The theorems can be generalized if X and Y are random variables taking values in a locally compact Abelian group or taking values in a locally convex topological vector space if appropriate assumptions are made.

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Received December 20, 1976 and in revised form May 30, 1980.

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