PACIFIC JOURNAL OF MATHEMATICS
Vol. 92, No. 1, 1981

APPROXIMATION BY RATIONAL MODULES ON BOUNDARY SETS

JAMES LI-MING WANG

Let X be a compact subset of the complex plane. Let the module $\mathscr{R}(X)\bar{\mathscr{P}}_m$ be the space of all functions of the form

$$r_0(z)+r_1(z)\bar{z}+\cdots+r_m(z)\bar{z}^m$$

where each r_i is a rational function with poles off X. We prove that $\mathscr{R}(X)\bar{\mathscr{P}}_1$ is dense in $L^p(\partial X)$ for all $1 \leqq p < \infty$.

1. Introduction. Let X be a compact subset of the complex plane. Let the module $\mathscr{R}(X)\bar{\mathscr{P}}_m$ be the space $\mathscr{R} + \mathscr{R}\bar{z} + \cdots \mathscr{R}\bar{z}^m$

$$=\{r_0(z) + r_1(z)\bar{z}+\cdots+r_m(z)\bar{z}^m\}\ ,$$

where each r_i is a rational function with poles off X.

The concept of rational modules arises in a natural fashion when one attempts to study rational approximation in Lipschitz norms. In [5] and [6], O'Farrell studied the relation of the problems of approximation by rational modules in different Lipschitz norms, and in the uniform norms, etc., to one another. Not long ago the author proved in [9] that $\mathscr{R}(X)\bar{\mathscr{P}}_1$ is dense in $L^p(X)$ for all $1 \leqq p < \infty$ and $\mathscr{R}(X)\bar{\mathscr{P}}_2$ is dense in $C(X)$ if X has no interior.

It is apparent that if X has interior, then $\mathscr{R}(X)\bar{\mathscr{P}}_m$ can not be dense in $C(X)$ or $L^p(X, dm)$, $1 \leqq p < \infty$, where dm denotes the 2-dimensional Lebesgue measure. Also it is clear that if X has interior, the $\mathscr{R}(X)\bar{\mathscr{P}}_m$ can not be dense in $C(\partial X)$, where ∂X is the topological boundary set of X. In this note, however, we prove that $\mathscr{R}(X)\bar{\mathscr{P}}_1$ is dense in $L^p(\partial X, dm)$ for all $1 \leqq p < \infty$.

2. Theorem and corollary. Throughout this note, $L^p(\partial X)$ stands for $L^p(\partial X, dm)$.

Let μ be a (finite Borel) measure on X. The Cauchy transform $\hat{\mu}$ is defined by

$$\hat{\mu}(z) = \int\frac{d\mu(\zeta)}{\zeta - z}\ .$$

Some basic properties for $\hat{\mu}$ can be found in [4]. If g is a function on X, we will write $\hat{g}$ for $\widehat{gdm}$.

We use the symbol $\bar{\partial}$ for the operator $\partial/\partial x + i(\partial/\partial y)$ and write $g \perp V$ if $\int fgdm = 0$ for all f in V.

The following lemmas play important roles in this theory. Lemma 1 is a special case of the key lemma in [5], and Lemma 2 is used by the author in [9].

LEMMA 1. *Let* μ *be a measure on* X. *Then* $\mu \perp \mathscr{R}(X)\bar{\mathscr{P}}_1$ *if and any only if* $\hat{\mu} \perp \mathscr{R}(X)$.

Proof. Because $\int f d\mu = -\pi^{-1}\int (\bar{\partial} f)\hat{\mu}\, dm$ for all f in $\mathscr{R}(X)\bar{\mathscr{P}}_1$ (cf. [4, p. 38]).

LEMMA 2. *If* $g \in L^p(X)$, *then* $\hat{g}$ *is continuous when* $p > 2$ *and* $\hat{\hat{g}}$ *is continuous when* $1 < p \leqq 2$.

Proof. The Cauchy transform is essentially the convolution of a function (or a measure) and the function ζ^{-1} which belongs to L^r_{loc} for all $1 \leqq r < 2$. So Lemma 2 is classical when $p > 2$. An application of the Young's inequality [7, p. 271] takes care of the rest.

THEOREM. *Let* X *be a compact set. Then* $\mathscr{R}(X)\bar{\mathscr{P}}_1$ *is dense in* $L^p(\partial X)$ *for all* $1 \leqq p < \infty$.

Proof. Let g be any function in $L^q(\partial X)$, $1 < q \leqq \infty$, $p^{-1} + q^{-1} = 1$, such that $g \perp \mathscr{R}(X)\bar{\mathscr{P}}_1$. Lemma 1 implies $\hat{g} \perp \mathscr{R}(X)$ and therefore $\hat{\hat{g}} = 0$ off X. Also $\hat{\hat{g}}$ is continuous by Lemma 2. It follows that $\hat{\hat{g}} = 0$ everywhere on ∂X. Now $\hat{g} \in L^s$ for some $s > 2$, and so it follows from the theory of singular integrals [2] that $\hat{\hat{g}}$ is absolutely continuous on (almost) every line parallel to each of the coordinate axes and that the partial derivatives $\partial(\hat{\hat{g}})/\partial x$ and $\partial(\hat{\hat{g}})/\partial y$ exist almost everywhere (dm) in the usual sense. By a lemma of Schwartz [8] (I owe this idea to James Brennan, who has shown me his work in [1]) these derivatives coincide with the corresponding distribution derivatives and so

$$\hat{g} = -\pi^{-1}\bar{\partial}(\hat{\hat{g}})$$

almost everywhere in the usual sense. By Fubini's theorem, almost every point of ∂X is a point of linear density (and hence a point of accumulation) for ∂X in the direction of both coordinate axes and so $\partial(\hat{\hat{g}})/\partial x = \partial(\hat{\hat{g}})/\partial y = 0$ almost everywhere on ∂X. It follows that $\hat{g} = 0$ almost everywhere on ∂X. Applying a similar argument to $\hat{g}$ we conclude that $g = 0$ almost everywhere on ∂X and the theorem is proved.

As a corollary, we have the following results in [9].

COROLLARY. *Let X be a compact set with no interior. Then*
(i) $\mathscr{R}(X)\bar{\mathscr{P}}_1$ *is dense in* $L^p(X)$ *for all* $1 \leqq p < \infty$.
(ii) $\mathscr{R}(X)\bar{\mathscr{P}}_2$ *is dense in* $C(X)$.
(iii) $\mathscr{R}(X)\bar{\mathscr{P}}_2$ *is dense in* $\mathrm{lip}(\alpha, X)$ *for all* $0 < \alpha < 1$.
(iv) $\mathscr{R}(X)\bar{\mathscr{P}}_3$ *is dense in* $D^1(X)$.

Davie's theorem in [3] asserts that for any compact set Y with boundary $X = \partial Y$, we have

$$[\mathscr{R}(X) + A(Y)]_u = C(X) ,$$

where $A(Y)$ denotes the algebra of all continuous functions on Y which are analytic on $\overset{\circ}{Y}$ and $[\ ]_u$ denotes the uniform closure. Corollary obviously strengthens this result, since $\mathscr{R}(X)\bar{\mathscr{P}}_2$ and $\mathscr{R}(X) + (\mathscr{R}(X)\bar{\mathscr{P}}_1)^\wedge$ have the same closure on X, where $(\mathscr{R}(X)\bar{\mathscr{P}}_1)^\wedge = \{(f|X)^\wedge : f \in \mathscr{R}(X)\bar{\mathscr{P}}_1\} \subseteqq A(Y)$. For other extensions of Davie's theorem, we refer the reader to the paper of O'Farrell [5].

REFERENCES

1. J. Brennan, *Invariant subspaces and weighted polynomial approximation*, Ark. mat., **11** (1973), 167-189.
2. A. P. Calderon and A. Zygmund, *On the existence of certain singular integrals*, Acta Math., **88** (1952), 85-139.
3. A. M. Davie, *Bounded approximation and Dirichlet sets*, J. Functional Analysis, **6** (1970), 460-467.
4. J. Garnett, *Analytic capacity and measure*, Springer Lecture Notes, No. 297, 1973.
5. A. O'Farrell, *Annihilators of rational modules*, J. Functional Analysis, **19** (1975), 373-389.
6. ———, *Hausdorff content and rational approximation in fractional Lipschitz norms*, Trans. Amer. Math. Soc., **228** (1977), 187-206.
7. E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Univ. Press, Princeton, N. J., 1970.
8. L. Schwartz, *Theorie des distributions*, Hermann, Paris, 1966.
9. J. Wang, *Approximation by rational modules on nowhere dense sets*, Pacific J. Math., **80** (1979), 293-295.

Received February 13, 1979 and in revised form February 15, 1980. Research supported in part by a NSF grant.

UNIVERSITY OF ALABAMA
UNIVERSITY, AL 35486