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ASYMPTOTIC PROPERTIES OF NONOSCILLATORY SOLUTIONS OF HIGHER ORDER DIFFERENTIAL EQUATIONS

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A classification of the nonoscillatory solutions based on their asymptotic properties of the differential equation $y^{(n)} + py = 0$ is discussed. In particular, the number of solutions belonging to the Kiguradze class A_j is determined.

We investigate asymptotic properties of the nonoscillatory solutions of the differential equation

$$y^{(n)} + py = 0\,, \tag{E}$$

where p is a continuous function of one sign on an interval $[a, \infty)$. Various aspects of Eq. (E) have been investigated by a number of authors [1-15]; in most cases, under the condition that the integral

$$I(r) \equiv \int_a^\infty x^r\,|p(x)|\,dx \tag{1}$$

is either finite or infinite for some constant r. For instance, Eq. (E) is oscillatory on $[a, \infty)$ if the integral (1) is infinite with $r = n - 1 - \varepsilon$ for some $\varepsilon > 0$ [4, 8]. On the other hand, if $I(n - 1)$ is finite, (E) is nonoscillatory; in fact, it is eventually disconjugate [9, 14, 15]. Under the same condition, results on the existence of a fundamental system of solutions possessing certain asymptotic properties have also been obtained [5, 13]. Of particular interest to the present work, however, is the notion of class A_p introduced by Kiguradze [4] with the help of inequalities in Lemma 1.

A solution of (E) is said to be *nonoscillatory* on $[a, \infty)$ if it does not have an infinite number of zeros on $[a, \infty)$. (Unless the contrary is stated, the word "solution" is used as an abbreviation for "nontrivial solution.") Eq. (E) is said to be nonoscillatory on $[a, \infty)$ if every solution of (E) is nonoscillatory on $[a, \infty)$. If there exists a point $b \geqq a$ such that no solution of (E) has more than $n - 1$ zeros on $[b, \infty)$, Eq. (E) is said to be *eventually disconjugate* on $[a, \infty)$.

As previous studies of Eq. (E) indicate, asymptotic properties of the solutions strongly depend on the parity of n and the sign of p. For this reason, it is convenient to classify Eq. (E) into the following four distinct classes:

$$n \text{ even}, \qquad p \geqq 0\,, \tag{i}$$

$$\text{(ii)} \qquad n \text{ odd}, \qquad p \geqq 0 ,$$

$$\text{(iii)} \qquad n \text{ even}, \qquad p \leqq 0 ,$$

$$\text{(iv)} \qquad n \text{ odd}, \qquad p \leqq 0 .$$

Eq. (E) satisfying condition (i), for example, is denoted by $(\mathrm{E_i})$; $(\mathrm{E_{ii}})$, $(\mathrm{E_{iii}})$, and $(\mathrm{E_{iv}})$ are similarly defined.

We state important inequalities which will be used in defining the class A_p and also in some proofs.

LEMMA 1. *Let y be a nonoscillatory solution of* (E) *such that $y \geqq 0$ on $[b, \infty)$ for some $b \geqq a$, and let $p \not\equiv 0$ on $[b_1, \infty)$ for every $b_1 \geqq a$. Define $[C]$ to be the greatest integer less than or equal to C.*

If y is a solution of $(\mathrm{E_i})$ or $(\mathrm{E_{iv}})$, there exists an integer j, $0 \leqq j \leqq [(n-1)/2]$, such that

$$y^{(i)} > 0 , \qquad i = 0, 1, \cdots, 2j , \tag{2}$$

on $[b_2, \infty)$ for some $b_2 \geqq b$, and

$$(-1)^{i+1} y^{(i)} > 0 , \qquad i = 2j+1, \cdots, n-1 , \tag{2$'$}$$

on $[b, \infty)$.

If y is a solution of $(\mathrm{E_{ii}})$ or $(\mathrm{E_{iii}})$, there exists an integer j, $0 \leqq j \leqq [n/2]$, such that

$$y^{(i)} > 0 , \qquad i = 0, 1, \cdots, 2j-1 , \tag{3}$$

on $[b_2, \infty)$ for some $b_2 \geqq b$, and

$$(-1)^{i} y^{(i)} > 0 , \qquad i = 2j, \cdots, n-1 , \tag{3$'$}$$

on $[b, \infty)$.

Various versions of Lemma 1 appear in the literature [2, 5, 6, 12]. However, the important features of the present version are that the inequalities in Lemma 1 are strict and that the inequalities (2′) and (3′) hold on $[b, \infty)$—rather than on $[b_2, \infty)$ for some $b_2 \geqq b$—if $y \geqq 0$ on $[b, \infty)$. Following Kiguradze [4], we shall say that a nonoscillatory solution y of $(\mathrm{E_i})$ or $(\mathrm{E_{iv}})$ belongs to *class* A_j if y or $-y$ satisfies the inequalities (2) and (2′) for $0 \leqq j \leqq [(n-1)/2]$. Similarly, a nonoscillatory solution y of $(\mathrm{E_{ii}})$ or $(\mathrm{E_{iii}})$ is said to belong to *class* A_j if y or $-y$ satisfies the inequalities (3) and (3′) for $0 \leqq j \leqq [n/2]$. In view of the above definition, we may restate Lemma 1 as follows: The family $\{A_0, A_1, \cdots, A_{[(n-1)/2]}\}$ forms a partition of the nonoscillatory solutions of $(\mathrm{E_i})$ and $(\mathrm{E_{iv}})$, and the family $\{A_0, A_1, \cdots, A_{[n/2]}\}$ forms a partition of the nonoscillatory solutions of

(E_{ii}) and (E_{iii}).

LEMMA 2. *If the class A_k contains three solutions v_1, v_2, and v_3 of which every nontrivial linear combination again belongs to A_k, where $0 \leqq k \leqq [(n-2)/2]$ for (E_i) and (E_{iv}) and $1 \leqq k \leqq [(n-1)/2]$ for (E_{ii}) and (E_{iii}), then A_k contains three solutions y_1, y_2, and y_3, each a linear combination of v_1, v_2, and v_3, such that*

$$\lim_{x\to\infty} \frac{y_j(x)}{y_i(x)} = \infty\ , \qquad 1 \leqq i < j \leqq 3\ .$$

Proof. Without loss of generality, we may assume that $v_3 > v_2 > v_1 > 0$ on $[c, \infty)$ for some $c \geqq a$. The quotient v_j/v_i, $1 \leqq i < j \leqq 3$, cannot assume a fixed value γ an infinite number of times on $[c, \infty)$, for otherwise $v_j - \gamma v_i$ would be an oscillatory solution contrary to the hypothesis. Therefore,

$$\limsup_{x\to\infty} \frac{v_j(x)}{v_i(x)} = \liminf_{x\to\infty} \frac{v_j(x)}{v_i(x)} = \lim_{x\to\infty} \frac{v_j(x)}{v_i(x)} = K_{ij}\ ,$$

$1 \leqq K_{ij} \leqq \infty$, $1 \leqq i < j \leqq 3$. At first there appear to be eight different possibilities we must consider, depending on $K_{ij} = \infty$ or $K_{ij} < \infty$, $1 \leqq i < j \leqq 3$. But note that if two of the constants K_{ij}, $1 \leqq i < j \leqq 3$, are finite, the third also must be finite. Furthermore, it is impossible to have $K_{12} = K_{23} = \infty$ and $K_{13} < \infty$. Hence we need only to consider the following four cases.

(a) $K_{ij} = \infty$, $1 \leqq i < j \leqq 3$. Put $y_i = v_i$, $i = 1, 2, 3$.

(b) $K_{12} < \infty$, $K_{13} = K_{23} = \infty$. In this case

$$\lim_{x\to\infty} \frac{v_2(x) - K_{12}v_1(x)}{v_1(x)} = 0\ , \quad \text{i.e., } \lim_{x\to\infty} \left| \frac{v_1(x)}{v_2(x) - K_{12}v_1(x)} \right| = \infty\ .$$

Put $y_1 = v_2 - K_{12}v_1$, $y_2 = v_1$, and $y_3 = v_3$.

(c) $K_{12} = K_{13} = \infty$, $K_{23} < \infty$. Here we have

$$\lim_{x\to\infty} \frac{v_3(x) - K_{23}v_2(x)}{v_2(x)} = 0\ .$$

Suppose that

$$\lim_{x\to\infty} \frac{v_3(x) - K_{23}v_2(x)}{v_1(x)} = K\ .$$

If $|K| = \infty$, put $y_1 = v_1$, $y_2 = v_3 - K_{23}v_2$, and $y_3 = v_2$. On the other hand, if $|K| < \infty$, then

$$\lim_{x\to\infty} \frac{v_3(x) - K_{23}v_2(x) - Kv_1(x)}{v_1(x)} = 0$$

and we put $y_1 = v_3 - K_{23}v_2 - Kv_1$, $y_2 = v_1$, and $y_3 = v_2$.

(d) $K_{ij} < \infty$, $1 \leqq i < j \leqq 3$. For this case

$$\lim_{x\to\infty} \frac{v_2(x) - K_{12}v_1(x)}{v_1(x)} = \lim_{x\to\infty} \frac{v_3(x) - K_{13}v_1(x)}{v_1(x)} = 0 .$$

Suppose that

$$\lim_{x\to\infty} \frac{v_2(x) - K_{12}v_1(x)}{v_3(x) - K_{13}v_1(x)} = K .$$

If $|K| = \infty$, let $y_1 = v_3 - K_{13}v_1$, $y_2 = v_2 - K_{12}v_1$, and $y_3 = v_1$. If $|K| < \infty$, then

$$\lim_{x\to\infty} \frac{v_2(x) - K_{12}v_1(x) - K(v_3(x) - K_{13}v_1(x))}{v_3(x) - K_{13}v_1(x)} = 0$$

and we put $y_1 = v_2 - (K_{12} - KK_{13})v_1 - Kv_3$, $y_2 = v_3 - K_{13}v_1$, and $y_3 = v_1$. The solutions y_i, $i = 1, 2, 3$, defined in (a)-(d) belong to A_k and satisfies

$$\lim_{x\to\infty} \left| \frac{y_j(x)}{y_i(x)} \right| = \infty , \qquad 1 \leqq i < j \leqq 3 .$$

Since we may take $-y_i$ if y_i is eventually negative as $x \to \infty$, the proof is complete.

LEMMA 3. *Suppose that* Eq. (E) *has there nonoscillatory solutions* y_1, y_2, *and* y_3 *such that*

$$\lim_{x\to\infty} \frac{y_j(x)}{y_i(x)} = \infty , \qquad 1 \leqq i < j \leqq 3 , \tag{4}$$

and $y_3 > y_2 > y_1 > 0$ *on* $[\xi, \infty)$. *If* η *is an arbitrary point on* $[\xi, \infty)$, *there exists a solution* $v \equiv \sum_{k=1}^{3} \alpha_k y_k$ *such that* $v \geqq 0$ *on* $[\xi, \infty)$ *and* $v(\zeta) = v'(\zeta) = 0$ *for some point* ζ *on* $[\eta, \infty)$.

Proof. Choose a constant $K > 0$ such that $u \equiv y_2 - Ky_1 < 0$ on $[\xi, \eta]$. Since $u < 0$ on $[\xi, \eta]$ and eventually $u(x) > 0$ as $x \to \infty$, u vanishes at some point of (η, ∞). Let σ be the first zero of u on (η, ∞). Define $K_1 = \sup G$, where G is the set of real numbers $\beta \geqq 1$ such that $y_3 - \beta u \geqq 0$ on $[\sigma, \infty)$. Evidently, G is bounded above and it is nonempty because $y_3 - u > 0$, i.e., $1 \in G$. Let $\beta \in G$ and $\tau \in [\sigma, \infty)$. If $u(\tau) \leqq 0$, then $y_3(\tau) - K_1u(\tau) \geqq y_3(\tau) > 0$. On the other hand, if $u(\tau) > 0$, then $y_3(\tau)/u(\tau) \geqq \beta$ for all $\beta \in G$, and thus $y_3(\tau)/u(\tau) \geqq K_1$. Since τ is arbitrary, the solution $v \equiv y_3 - K_1u \geqq 0$ on $[\sigma, \infty)$. Therefore, if $v(\zeta) = 0$ for some $\zeta \in (\sigma, \infty)$, then $v'(\zeta) = 0$. Hence, the proof is complete if we can show that $v(\zeta) = 0$ for some $\zeta \in (\sigma, \infty)$. Assume to the contrary that $v > 0$ on (σ, ∞). Let $\varepsilon_1 > 0$ be given. There exists $\rho > \sigma$ such that $u > 0$ on $[\rho, \infty)$ and $v(x)/u(x) > \varepsilon_1$, $x \in [\rho, \infty)$, since

$$\lim_{x\to\infty} \frac{v(x)}{u(x)} = \infty$$

by (4). Choose an $\varepsilon_2 > 0$ so that $v(x) > \varepsilon_2 u(x)$, $x \in [\sigma, \rho]$. Put $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$. Then $v - \varepsilon u > 0$ on $[\sigma, \infty)$, i.e., $y_3 - (K_1+\varepsilon)u > 0$ on $[\sigma, \infty)$, contradicting the choice of K_1. Thus, $v(\zeta) = 0$ for some $\zeta \in (\sigma, \infty)$. Finally, it is evident that $v > 0$ on $[\xi, \sigma]$ and $v \geqq 0$ on $[\xi, \infty)$.

We are ready to consider the problem of determining the number of solutions belonging to class A_j. Let $q(A_j)$ be the maximum number of linearly independent solutions belonging to A_j with the property that every nontrivial linear combination of them again belongs to class A_j.

THEOREM. *Assume that* Eq. (E) *is nonoscillatory on* $[a, \infty)$ *and that* $p \not\equiv 0$ *on* $[a_1, \infty)$ *for every* $a_1 \geqq a$. *Then*
$q(A_j) = 2$, $j = 0, 1, \cdots, (n-2)/2$, *for* $(\mathrm{E_i})$;
$q(A_0) = 1$, $q(A_j) = 2$, $j = 1, 2, \cdots, (n-1)/2$, *for* $(\mathrm{E_{ii}})$;
$q(A_0) = 1$, $q(A_j) = 2$, $j = 1, 2, \cdots, (n-2)/2$, $q(A_{n/2}) = 1$, *for* $(\mathrm{E_{iii}})$;
$q(A_j) = 2$, $j = 0, 1, \cdots, (n-3)/2$, $q(A_{(n-1)/2}) = 1$ *for* $(\mathrm{E_{iv}})$.

Proof. We shall prove the theorem for $(\mathrm{E_{iii}})$: $q(A_0) = 1$, $q(A_j) = 2$, $j = 1, 2, \cdots, (n-2)/2$, and $q(A_{n/2}) = 1$. Suppose that the class A_j contains a set B_j of $q(A_j)$ solutions of which every nontrivial linear combination again belongs to A_j, $j = 0, 1, \cdots, n/2$. Using Lemmas 1 and 2, we can easily deduce that the set $B = \bigcup_{j=0}^{n/2} B_j$ containing $\sum_{j=0}^{n/2} q(A_j)$ solutions is a fundamental system for $(\mathrm{E_{iii}})$. Thus, $\sum_{j=0}^{n/2} q(A_j) = n$. For this reason, it suffices to prove that

$$(5) \qquad q(A_0) \leqq 1\ , \quad q(A_j) \leqq 2\ , \quad j = 1, 2, \cdots, (n-2)/2,\ q(A_{n/2}) \leqq 1\ .$$

If $q(A_0) > 1$, then there exist two solutions y_1 and y_2 belonging to A_0 and a constant K such that $w \equiv y_1 - Ky_2 \in A_0$, $w(c) = 0$, and $w \geqq 0$ on $[c, \infty)$ for some $c \geqq a$. But this contradicts Lemma 1 (see also Kiguradze [5, Lemma 7]) and proves that $q(A_0) \leqq 1$. Suppose that $q(A_k) > 2$ for some k, $1 \leqq k \leqq (n-2)/2$. Then the class A_k contains at least three solutions y_1, y_2, and y_3, of which every nontrivial linear combination again belongs to A_k. By Lemma 2, we may assume that

$$\lim_{x\to\infty} \frac{y_j(x)}{y_i(x)} = \infty\ , \qquad 1 \leqq i < j \leqq 3\ ,$$

and $y_3 > y_2 > y_1 > 0$ on $[\xi, \infty)$ for some $\xi \geqq a$. Let $\{\eta_i\}$ be an increasing sequence of numbers such that $\eta_i \geqq \xi$ and $\eta_i \to \infty$ as $i \to \infty$. By Lemma 3 there exists for each i, a solution

$$v_i \equiv \alpha_i y_1 + \beta_i y_2 + \gamma_i y_3 \, , \qquad \alpha_i^2 + \beta_i^2 + \gamma_i^2 = 1 \, ,$$

such that $v_i \geqq 0$ on $[\xi, \infty)$ and $v_i(\zeta_i) = v_i'(\zeta_i) = 0$ for some $\zeta_i \in (\eta_i, \infty)$. Obviously, there are convergent subsequences $\{\alpha_{i_k}\}$, $\{\beta_{i_k}\}$, and $\{\gamma_{i_k}\}$, which will be again denoted by $\{\alpha_i\}$, $\{\beta_i\}$, and $\{\gamma_i\}$, respectively, for notational simplicity. Put

$$\lim_{i\to\infty} \alpha_i = \alpha \, , \qquad \lim_{i\to\infty} \beta_i = \beta \, , \qquad \lim_{i\to\infty} \gamma_i = \gamma \, .$$

Then $w(x) \equiv \alpha y_1(x) + \beta y_2(x) + \gamma y_3(x)$ is a nonoscillatory solution belonging to the class A_k. Since $w \geqq 0$ on $[\xi, \infty)$, we have

$$w > 0, w' > 0, \cdots, w^{(2k-1)} > 0 \, , \tag{6}$$

on $[b_2, \infty)$ for some $b_2 \geqq \xi$, and

$$w^{(2k)} > 0 \, , \quad w^{(2k+1)} < 0 \, , \quad w^{(2k+2)} > 0, \cdots, w^{(n-1)} < 0 \, , \tag{7}$$

on $[\xi, \infty)$ by Lemma 1. We now use a line of reasoning due to Kondrat'ev [7]. Since $\lim_{i\to\infty} v_i^{(j)} = w^{(j)}$, $j = 0, 1, \cdots, n$, uniformly on any finite subinterval of $[a, \infty)$, there exists a number N such that

$$v_i^{(j)}(b_2) > \frac{w^{(j)}(b_2)}{2} > 0 \, , \qquad j = 0, 1, \cdots, 2k-1 \, , \tag{8}$$

if $i > N$. We may assume that $\eta_i > b_2$ for $i > N$. Since $v_i \in A_k$ and $v_i \geqq 0$ on $[\xi, \infty)$ for all i, $v_i^{(2k)} > 0$ on $[\xi, \infty)$ by Lemma 1. Thus,

$$v_i^{(2k-1)}(b_2) \leqq v_i^{(2k-1)}(\tau) \, , \qquad \tau \in [b_2, \infty) \, . \tag{9}$$

Substituting (9) in (8) with $j = 2k-1$, we obtain

$$v_i^{(2k-1)}(\tau) > \frac{w^{(2k-1)}(b_2)}{2} \, , \qquad \tau \in [b_2, \infty) \, . \tag{10}$$

Integrating the above inequality from b_2 to $x \in [b_2, \infty)$ and substituting in the resulting expression the inequality (8) with $j = 2k-2$, we get

$$v_i^{(2k-2)}(x) > \frac{w^{(2k-1)}(b_2)}{2}(x - b_2) + \frac{w^{(2k-2)}(b_2)}{2} \, .$$

Repeating a similar procedure $2k-2$ times, we arrive at the inequality

$$v_i(x) > \frac{w^{(2k-1)}(b_2)}{2(2k-1)!}(x-b_2)^{2k-1} + \frac{w^{(2k-2)}(b_2)}{2(2k-2)!}(x-b_2)^{2k-2} + \cdots + \frac{w(b_2)}{2}, \qquad x \in [b_2, \infty) . \tag{11}$$

This inequality, however, cannot hold throughout the interval $[b_2, \infty)$. Indeed, for $x = \zeta_i > \eta_i > b_2 (i > N)$, the left-hand side $v_i(\zeta_i) = 0$, while the right-hand side is positive by (6). This contradiction proves that $q(A_j) \leqq 2$, $j = 1, 2, \cdots, (n-2)/2$. The proof that $q(A_{n/2}) \leqq 1$ is more or less similar to the preceding case. Suppose that $A_{n/2}$ contains two solutions y_1 and y_2 of which every nontrivial linear combination belongs to $A_{n/2}$. Assume that $y_2 > y_1 > 0$ on $[\xi, \infty)$, and let $\{\eta_i\}$ be defined as before. Put

$$v_i \equiv \alpha_i y_1 + \beta_i y_2 , \qquad \alpha_i^2 + \beta_i^2 = 1 ,$$

such that $v_i(\eta_i) = 0$. If

$$\lim_{i\to\infty} \alpha_i = \alpha , \qquad \lim_{i\to\infty} \beta_i = \beta$$

(take subsequences, if necessary), define $w \equiv \alpha y_1 + \beta y_2$. Then $w \in A_{n/2}$ and we may assume that $w \geqq 0$ on $[b, \infty)$ for some b. Hence, by Lemma 1,

$$w > 0, w' > 0, \cdots, w^{(n-1)} > 0 , \tag{12}$$

on $[b_2, \infty)$ for some $b_2 \geqq b$, and the inequality (8) holds for $i > N_1$, for some N_1, and for $j = 0, 1, \cdots, n-1$. Assume that $\eta_i > b_2$ for $i > N_1$. For each $i > N_1$, there exists $c_i \in (b_2, \eta_i]$ such that $v_i(c_i) = 0$ and $v_i > 0$ on $[b_2, c_i)$, since $v_i(\eta_i) = 0$. On the interval $[b_2, c_i]$, we have $v_i^{(n)}(x) = -p(x)v_i(x) \geqq 0$. Therefore, $v_i^{(n-1)}(b_2) \leqq v_i^{(n-1)}(\tau)$, $\tau \in [b_2, c_i]$, and when this inequality is substituted in (8) with $j = n-1$, we get

$$v_i^{(n-1)}(\tau) > \frac{w^{(n-1)}(b_2)}{2} , \qquad \tau \in [b_2, c_i] . \tag{13}$$

Following the procedure employed to get from (10) to (11), we alternately integrate (13) from b_2 to $x \in [b_2, c_i]$ and substitute in the resulting expression a suitable inequality from (8) (which holds for $j = 0, 1, \cdots, n-1$, in this case). When this process is repeated $n-1$ times, we arrive at

$$v_i(x) > \frac{w^{(n-1)}(b_2)}{2(n-1)!}(x-b_2)^{n-1} + \frac{w^{(n-2)}(b_2)}{2(n-2)!}(x-b_2)^{n-2} + \cdots + \frac{w(b_2)}{2}, \qquad x \in [b_2, c_i] .$$

However, this inequality cannot hold at $x = c_i$ because $v_i(c_i) = 0$

while the right-hand side is positive by virtue of (12). Consequently, $q(A_{n/2}) \leqq 1$, and the proof is complete for $(\mathrm{E}_{\mathrm{iii}})$. Proofs for $(\mathrm{E}_{\mathrm{i}})$, $(\mathrm{E}_{\mathrm{ii}})$, and $(\mathrm{E}_{\mathrm{iv}})$ are similar.

This theorem generalizes a main result of Etgen and Taylor [3].

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