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DIFFERENTIABLY k-NORMAL ANALYTIC SPACES AND EXTENSIONS OF HOLOMORPHIC DIFFERENTIAL FORMS

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In this paper the concept of normality for a complex analytic space X is strengthened to the requirement that every local holomorphic p-form, for all $0 \leq p \leq$ some integer k, defined on the regular points of X extend across the singular variety. A condition for when this occurs is given in terms of a notion of independence, in the exterior algebra $\Omega^*_{\Delta^N}$, of the differentials $dF_1, \cdots, dF_r$ of local generating functions F_i of the ideal of X in some ambient polydisc $\Delta^N \subset C^N$. One result is that for a complete intersection, "k-independent implies $(k-2)$-normal" (precise definitions are given below), which extends some ideas of Oka, Abhyankar, Thimm, and Markoe on criteria for normality.

Recall that a complex space $(X, \mathcal{O}_X)$ is *normal* at a point $x \in X$ if every bounded holomorphic function defined on the regular points in a punctured neighborhood of x extends analytically to the full neighborhood. This is equivalent to the condition that the ring $\mathcal{O}_{X,x}$ be integrally closed in its field of quotients, and except for regular points x in dimension 1 the boundedness requirement is irrelevant: if $\dim X > 1$, $x \in X$ is normal $\Leftrightarrow$ for all sufficiently small neighborhoods U of x the restriction of sections $\Gamma(U, \mathcal{O}_X) \to \Gamma(U - \sum, \mathcal{O}_X)$ is an isomorphism, for $\sum$ the set of singular points of X. In 1974 A. Markoe [6] observed that the basic modern ideas in the topic of cohomology with supports gives a very simple criterion of normality in terms of the homological codimension of the structure sheaf:

THEOREM (*Markoe*). *Let $(X, \mathcal{O}_X)$ be a reduced complex space with singular set $\sum$. Then $\forall x \in X$, if* $\operatorname{codh}_x \mathcal{O}_X > \dim_x \sum + 1$, *then X is normal at x.*

Here $\operatorname{codh}_x \mathcal{O}_X = \max\{k \mid \exists$ germs $f_1, \cdots, f_k$ in the maximal ideal of $\mathcal{O}_{X,x}$ such that $\forall i \leqq k$, the coset $f_i + \sum_{j<i} f_j \mathcal{O}_{X,x}$ is not a zero divisor in the ring $\mathcal{O}_{X,x}/\sum_{j<i} f_j \mathcal{O}_{X,x}\}$. For the standard concepts of sheaf cohomology with supports and their relation to the algebraic properties of the stalks the reader may consult [5], [8], [9] or [11]. This generalizes earlier results of Oka [7], Abhyankar [1], and Thimm [10] for hypersurfaces and complete intersections.

At about the same time the present writer became interested in the question of extending holomorphic differential forms across subvarieties of analytic spaces in an effort to understand the local contribution of singular points to the groups $H^q(X, \Omega_X^p)$, especially for compact spaces where the dimensions of these groups are important numerical invariants (see [2] and [3] for some results of this sort for hypersurfaces). Since in particular a 0-form is just an analytic function it seems natural to consider spaces with a higher degree of "normality" and to extend and relate Markoe's result to statements about higher order differential forms. For instance we will see below (Proposition 6) that if X is a complete intersection at each point, then X is normal if and only if there are no local holomorphic 1-forms supported on the singular set.

DEFINITION 1. Let $(X, \mathscr{O}_X)$ be a reduced complex subspace of a domain $D \subset C^N$, with ideal sheaf $\mathscr{I}_X \subset \mathscr{O}_D$. *By the sheaf of germs of local holomorphic p-forms on* X we mean the sheaf on X

$$\Omega_X^p = \Omega_D^p / (\mathscr{I}_X \Omega_D^p + d\mathscr{I}_X \wedge \Omega_D^{p-1}) ,$$

where $d\mathscr{I}_X \wedge \Omega_D^{p-1}$ is the subsheaf of Ω_D^p consisting of those germs of the form $df \wedge \varphi^{p-1}$, $f \in \mathscr{I}_X$. For $U \subset X$ an open set, by a *holomorphic p-form on* U we shall mean a section on U of this sheaf.

DEFINITION 2. For $X \subset D$ as above and k a non-negative integer, a point $x \in X$ is said to be *differentiably k-normal* if for any integer $p \leqq k$ and any sufficiently small neighborhood U of x, every holomorphic p-form ω^p defined on the regular points of U extends to a holomorphic p-form $\tilde{\omega}^p$ on all of U. X itself *is differentiably k-normal* if each of its points is differentiably k-normal. That is, X is differentiably k-normal if $\forall p \leqq k$ the restriction of sections $\Gamma(U, \Omega_X^p) \to \Gamma(U - \sum, \Omega_X^p)$ is surjective for all open sets U, where $\sum$ is the singular set of X.

REMARKS. It is clear that Ω_X^p is coherent and (hence) that $\forall k$ the set $\sum_k$ of points of X that are not differentiably k-normal is a subvariety of X. If $\dim X > 1$, then differentiably 0-normal is the same as normal, and $\sum \supseteqq \sum_N \supseteqq \sum_{N-1} \supseteqq \cdots \supseteqq \sum_0$. The adverb "differentiably" is used here to distinguish the concept under view from that of the "k-normality" of Andreotti and Siu [2]. There a space is *k-normal* if the kth gap sheaf $\mathscr{O}_X^{[k]}$ is equal to the structure sheaf $\mathscr{O}_X$—that is, if holomorphic functions always extend across subvarieties of dimension $\leqq k$. I thank the referee for drawing my attention to this terminology.

The main result of [3] has the consequence that if X is locally a hypersurface, then X is differentiably k-normal but not differentiably

$(k+1)$-normal for $k=(\operatorname{codim} \sum)-2$. To give a concrete example, put

$$F(z_1, \cdots, z_{n+1}) = (z_1)^m + \cdots + (z_{n+1})^m$$

for some integer $m \geqq 2$, and let $X \subset C^{n+1}$ be the Fermat cone defined by $F = 0$. X has one singular point, the origin in C^{n+1}, and it is easy to establish (by Corollary 5 below, for instance) that X is differentiably $(n-2)$-normal.

To show that X is not, however, differentiably $(n-1)$-normal, denote by U_i, $i = 1, \cdots, n+1$, the affine set $\{z_i \neq 0\} \subset C^{n+1}$, and define a holomorphic $(n-1)$-form ω_i^{n-1} on U_i by

$$\omega_i^{n-1} = (z_i)^{1-m} \sum_{l \neq i} (-1)^{\sigma_{il}} z_l dz_1 \wedge \cdots \widehat{dz_i} \cdots \widehat{dz_l} \cdots \wedge dz_{n+1} .$$

Here $\widehat{\ }$ means "omit this factor", and $\sigma_{il} = 0$ if $(i, l, 1, 2, \cdots \hat{i} \cdots \hat{l} \cdots, n+1)$ is an even permutation of $(1, 2, \cdots, n+1)$, 1 if an odd permutation. Direct computation verifies that on $U_i \cap U_j (i < j)$,

$$\omega_i^{n-1} - \omega_j^{n-1} = F\psi_{ij}^{n-1} + dF \wedge \varphi_{ij}^{n-2}$$

for

$$\psi_{ij}^{n-1} = (-1)^{\sigma_{ij}} (z_i z_j)^{1-m} dz_1 \wedge \cdots \widehat{dz_i} \cdots \widehat{dz_j} \cdots \wedge dz_{n+1}$$

and for

$$\varphi_{ij}^{n-2} = (m-1)^{-1} (z_i z_j)^{1-m} \sum_{l \neq i,j} (-1)^{\tau_{ilj}} z_l dz_1 \wedge \cdots \widehat{dz_i} \cdots \widehat{dz_j} \cdots \widehat{dz_l} \cdots \wedge dz_{n+1} ,$$

where $\tau_{ilj} = 0$ if $1 < l < j$, 1 otherwise. Thus the ω_i^{n-1} together comprise a well-defined section ω^{n-1} of Ω_X^{n-1} on $X - \{0\}$.

But ω^{n-1} does not extend across 0. For if it did, then (since C^{n+1} is Stein) there would exist a globally defined $(n-1)$-form $\tilde{\omega}^{n-1}$ on C^{n+1} satisfying, for all i,

$$(*) \qquad \tilde{\omega}^{n-1} = \omega_i^{n-1} + F\psi_i^{n-1} + dF \wedge \varphi_i^{n-2}$$

for some $(n-1)$-, $(n-2)$-forms ψ_i^{n-1}, φ_i^{n-2} on U_i. Put

$$\tilde{\omega}^{n-1} = \sum_{1 \leqq k < l \leqq n+1} f_{kl} dz_1 \wedge \cdots \widehat{dz_k} \cdots \widehat{dz_l} \cdots \wedge dz_{n+1}$$

$$\psi_i^{n-1} = \sum_{1 \leqq k < l \leqq n+1} g_{ikl} dz_1 \wedge \cdots \widehat{dz_k} \cdots \widehat{dz_l} \cdots \wedge dz_{n+1}$$

$$\varphi_i^{n-2} = \sum_{1 \leqq p < q < r \leqq n+1} h_{ipqr} dz_1 \wedge \cdots \widehat{dz_p} \cdots \widehat{dz_q} \cdots \widehat{dz_r} \cdots \wedge dz_{n+1} ,$$

where the f's are entire holomorphic functions on C^{n+1} and the g's and h's are defined and homomorphic on U_i. Then for $i < j$, equating coefficients of $dz_1 \wedge \cdots \widehat{dz_i} \cdots \widehat{dz_j} \cdots \wedge dz_{n+1}$ in (*) gives, on U_i,

$$f_{ij} = (-1)^{\sigma_{ij}} z_j (z_i)^{1-m} + F g_{iij} + (m-1) \sum_{l \neq i,j} z_l \bar{h}_{iijl} \tag{**}$$

for

$$\bar{h}_{iijl} = \begin{cases} (-1)^{l+1} h_{ilij} & \text{for} \quad l < i \\ (-1)^{l} h_{iilj} & \text{for} \quad i < l < j \, . \\ (-1)^{l+1} h_{iijl} & \text{for} \quad l > j \end{cases}$$

Now let α be an mth root of -1, and evaluate (**) along the punctured line $L_\alpha - \{0\}$, for L_α defined by $z_j = \alpha z_i$, $z_l = 0$ for $l \neq i, j$, to conclude

$$f_{ij} = (-1)^{\sigma_{ij}} \alpha z^{2-m} \tag{***}$$

on $L_\alpha - \{0\}$. Thus if $m > 2$, f_{ij} cannot be defined at the origin, a contradiction. If $m = 2$, then $f_{ij} = (-1)^{\sigma_{ij}} \alpha$ on $L_\alpha - \{0\}$, while if $\beta \neq \alpha$ is the other square root of -1, then similarly $f_{ij} = (-1)^{\sigma_{ij}} \beta$ on $L_\beta - \{0\}$. Thus in this case also f_{ij} cannot be defined at 0. This contradiction shows that ω^{n-1} cannot be extended from $X - \{0\}$ to all of X, and hence that X is not differentiably $(n-1)$-normal.

Actually, much more can be said about $(n-1)$-forms on this space X. For $m_1, \cdots, m_{n+1}$ integers, with $0 \leqq m_l \leqq m - 2$ for all l, define a holomorphic $(n-1)$-form on $X - \{0\}$ by

$$\omega^{n-1}_{m_1, \cdots, m_{n+1}} = \left(\prod_{i=1}^{m} (z_l)^{m_l} \right) \omega^{n-1} \, .$$

By the same argument as above for ω^{n-1}, $\omega^{n-1}_{m_1, \cdots, m_{n+1}}$ does not extend across 0. In fact, the set $\{\omega^{n-1}_{m_1, \cdots, m_{n+1}}\}$ for all such indices $m_1, \cdots, m_{n-1}$ forms a basis over $\boldsymbol{C}$ for the quotient of stalks $\Omega^{n-1}_{X-\{0\},0} / \Omega^{n-1}_{X,0}$. That is, if U is any neighborhood of 0 in X, then every holomorphic $(n-1)$-form ξ^{n-1} on $U - \{0\}$ can be written uniquely

$$\xi^{n-1} = p \omega^{n-1} + \eta^{n-1}$$

where p is a polynomial in $z_1, \cdots, z_{n+1}$ with constant coefficients and of degree at most $m-2$ in each variable z_i, and where η^{n-1} is a holomorphic $(n-1)$-form on all of U. This example quite easily generalizes to the Brieskorn varieties $(z_1)^{p_1} + \cdots + (z_{n+1})^{p_{n+1}} = 0$.

We want next to introduce a notion of independence of differential forms, which is our main tool in studying differentiable k-normality.

DEFINITION 3. Let R be a commutative ring, let M be the free R-module on generators $dx^1, \cdots, dx^N$, and denote by $\Lambda^* M$ the total exterior algebra of M. A sequence $\Phi_1, \cdots, \Phi_r \in M$ is called *k-inde-*

pendent over R if $\forall p \leqq k$ and $\forall i \leqq r$, if

$$\omega_i^p \wedge \Phi_i = \sum_{j<i} \omega_j^p \wedge \Phi_j$$

for some $\omega_j^p \in \Lambda^p M$, $j = 1, \cdots, i$, then $\exists \varphi_j^{p-1} \in \Lambda^{p-1} M$, $j = 1, \cdots, i$, such that

$$\omega_i^p = \sum_{j \leqq i} \varphi_j^{p-1} \wedge \Phi_j .$$

THEOREM 4. *Let $(X, \mathscr{O}_X)$ be a reduced subspace of a domain D in C^N. Denote by $\mathscr{I}_X \subset \mathscr{O}_D$ the ideal sheaf defining X and by $\sum$ the set of singular points of X. Let $x \in X$ and suppose that for some integer $k \geqq 0$,*

(i) $\operatorname{codh}_x \mathscr{O}_X > \dim_x \sum + k + 1$, *and*

(ii) *there exist generators $F_1, \cdots, F_r$ of $\mathscr{I}_{X,x}$ such that $dF_1, \cdots, dF_r$ are k-independent over $\mathscr{O}_{X,x}$.*

Then for all integers p, q with $p + q \leqq k + 1$,

$$(\mathscr{H}_\Sigma^q \Omega_X^p)_x = 0 .$$

Proof. Without loss of generality assume that the functions F_i are defined throughout D and generate $\mathscr{I}_X$ at each point of D. Put $X_0 = D$, $X_1 = V(F_1)$, $X_2 = V(F_1, F_2)$, $\cdots$, $X_r = X$, where $V(F_1, \cdots, F_i)$ means the variety of $F_1, \cdots, F_i$ with ideal sheaf $\sum_{j \leqq i} F_j \mathscr{O}_D$. Fix $k' \leqq k + 1$. We will prove by induction on i that $\forall i$

$$(*) \qquad \mathscr{H}_\Sigma^q (\Omega_{X_i}^p \otimes \mathscr{O}_X)_x = 0 \quad \forall q + p = k' .$$

The case $i = r$, then, is the desired result.

If $i = 0$, $\Omega_{X_i}^p \otimes \mathscr{O}_X = \Omega_D^p \otimes \mathscr{O}_X$ is free, so (*) holds by the condition $q \leqq k' < \operatorname{codh}_x \mathscr{O}_X - \dim_x \sum$ ([9], Theorem 1.14). Now let $i > 0$ and assume inductively that $\mathscr{H}_\Sigma^q (\Omega_X^p \otimes \mathscr{O}_X)_x = 0$ $\forall q + p = k'$. We have the complex

$$(**) \quad 0 \longrightarrow \mathscr{O}_X \xrightarrow{\rho_{dF_i}^{(0)}} \Omega_{X_{i-1}}^1 \otimes \mathscr{O}_X \xrightarrow{\rho_{dF_i}^{(1)}} \Omega_{X_{i-1}}^2 \otimes \mathscr{O}_X \longrightarrow \cdots$$
$$\longrightarrow \Omega_{X_{i-1}}^{p-1} \otimes \mathscr{O}_X \xrightarrow{\rho_{dF_i}^{(p-1)}} \Omega_{X_{i-1}}^p \otimes \mathscr{O}_X \xrightarrow{\pi_i^{(p)}} \Omega_X^p \otimes \mathscr{O}_X \longrightarrow 0 ,$$

where $\rho_{dF_i}^{(j)} \colon \Omega_{X_{i-1}}^j \otimes \mathscr{O}_X \to \Omega_{X_{i-1}}^{j+1} \otimes \mathscr{O}_X$ is induced by right wedge multiplication by dF_i and where $\pi_i^{(p)}$ is the natural projection. Since $dF_1, \cdots, dF_i$ are at least $(p-1)$-independent over $\mathscr{O}_{X,x}$ at x, (**) is exact at x. Hence for $j = 1, \cdots, p-1$, the sequences

$$0 \longrightarrow \operatorname{im} \rho_{dF_i}^{(j-1)} \longrightarrow \Omega_{X_{i-1}}^j \otimes \mathscr{O}_X \longrightarrow \operatorname{im} \rho_{dF_i}^{(j)} \longrightarrow 0$$

are exact at x, and at the last stage, so also is

$$0 \longrightarrow \operatorname{im} \rho_{dF_i}^{(p-1)} \longrightarrow \Omega_{X_{i-1}}^p \otimes \mathcal{O}_X \longrightarrow \Omega_{X_i}^p \otimes \mathcal{O}_X \longrightarrow 0 .$$

Taking $\mathscr{H}_\Sigma^q$ at x, this yields

$$\cdots \longrightarrow \mathscr{H}_\Sigma^{p+q-j}(\Omega_{X_{i-1}}^j \otimes \mathcal{O}_X)_x \longrightarrow \mathscr{H}_\Sigma^{p+q-j}(\operatorname{im} \rho_{dF_i}^{(j)})_x \longrightarrow \mathscr{H}_\Sigma^{p+q-j+1}(\operatorname{im} \rho_{dF_i}^{(j-1)})_x \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \mathscr{H}_\Sigma^q(\Omega_{X_{i-1}}^p \otimes \mathcal{O}_X)_x \longrightarrow \mathscr{H}_\Sigma^q(\Omega_{X_i}^p \otimes \mathcal{O}_X)_x \longrightarrow \mathscr{H}_\Sigma^{q+1}(\operatorname{im} \rho_{dF_i}^{(p-1)})_x \longrightarrow \cdots .$$

By the inductive hypothesis, the first group in each of these triples vanishes, so the induced maps

$$\mathscr{H}_\Sigma^q(\Omega_{X_i}^p \otimes \mathcal{O}_X) \longrightarrow \mathscr{H}_\Sigma^{q+1}(\operatorname{im} \rho_{dF_i}^{(p-1)})_x \longrightarrow \mathscr{H}_\Sigma^{q+2}(\operatorname{im} \rho_{dF_i}^{(p+2)})_x \longrightarrow \cdots \longrightarrow \mathscr{H}_\Sigma^{p+q-1}(\operatorname{im} \rho_{dF_i}^{(1)})_x \longrightarrow \mathscr{H}_\Sigma^{p+q}(\operatorname{im} \rho_{dF_i}^{(0)})_x \cong (\mathscr{H}_\Sigma^{p+q}\mathcal{O}_X)_x = 0 ,$$

are all injective. Hence in particular $\mathscr{H}_\Sigma^q(\Omega_{X_i}^p \otimes \mathcal{O}_X)_x = 0$. □

COROLLARY 5. *Let $X \subset D \subset C^N$ be a complete intersection of dimension n. Suppose that $\mathscr{I}_X \subset \mathcal{O}_D$ has generators $F_1, \cdots, F_{N-n}$ whose varieties meet transversally and are such that $dF_1, \cdots, dF_{N-n}$ are k-independent in any order over $\mathcal{O}_X$ at each point of X. Then X is differentiably $(k-2)$-normal.*

Proof. It is shown in [3] (Lemmas 1 and 2 and Remark 6) that the single function F_i is k-independent over $\mathcal{O}_X$ at $x \Leftrightarrow$ for some choice of local co-ordinates $z^1, \cdots, z^N$ in D the derivatives $\partial F_i/\partial z^1, \cdots, \partial F_i/\partial z^k$ form a regular $\mathcal{O}_{X,x}$-sequence $\Leftrightarrow \operatorname{codim}_X(\sum_i \cap X) \geqq k$ at x, for $\sum_i$ the singular set of the variety X_i of F_i. Since the $V(F_i)$ meet transversally, $\sum = \bigcup_{i=1}^{N-n}(\sum_i \cap X)$. Thus $\dim_x \sum = \max\{\dim_x(\sum_i \cap X)\} \leqq n - k = \operatorname{codh}{}_x\mathcal{O}_X - k$, at each point $x \in X$. By the theorem, then, $\mathscr{H}_\Sigma^q \Omega_X^p = 0$ $\forall p + q \geqq k - 1$. Taking $q = 1$ we conclude that $\mathscr{H}_\Sigma^1 \Omega_X^p = 0$ $\forall p \leqq k - 2$. That is, for every open set U, $H_\Sigma^1(U, \Omega_X^p) = 0$. The conclusion now follows from the sequence

$$H^0(U, \Omega_X^p) \longrightarrow H^0(U - \textstyle\sum, \Omega_X^p) \longrightarrow H_\Sigma^1(U, \Omega_X^p) .$$

REMARK. Taking $q = 0$ in the conclusion to Theorem 4 (respectively, in its application in Corollary 5) shows that such spaces have no local holomorphic p-forms supported on $\sum$ for $p = 0, 1, \cdots, k+1$ (respectively, for $p = 0, 1, \cdots, k-1$). This observation suggests the characterization of normality mentioned in the introduction:

PROPOSITION 6. *Let x be a point of a reduced analytic space*

$(X, \mathcal{O}_X)$ such that X is a complete intersection at x. Then X is normal at $x \Leftrightarrow H^0_\Sigma(U, \Omega^1_X) = 0$ $\forall$ sufficiently small neighborhoods U of x, for Σ the set of singular points of X.

Proof. Complete intersections have 0-independent generators. Namely, if locally X is an n-dimensional subvariety of N-dimensional polydisc Δ^N, and if $F_1, \cdots, F_{N-n}$ generate the ideal of X in Δ^N, then at the regular points of X near x the differentials $dF_1, \cdots, dF_{N-n}$ are independent over $\boldsymbol{C}$. That is, if $g_i dF_i = \sum_{j<i} g_j dF_j$, then the g's are identically 0 most places in a neighborhood of x, hence everywhere.

Now Theorem 4 applies. X is normal at $x \Rightarrow \operatorname{codim}_x \Sigma > 1 \Rightarrow (\mathscr{H}^0_\Sigma \Omega^1_X)_x = 0 \Rightarrow H^0_\Sigma(U, \Omega^1_X) = 0 \forall$ sufficiently small neighborhoods U of x.

Conversely, $(\mathscr{H}^0_\Sigma \Omega^1_X)_x = 0 \Rightarrow dh_x \Omega^1_X < \operatorname{codim}_x \Sigma$ ([9], Theorem 1.14). If $\operatorname{codim}_x \Sigma$ were equal to 1, this would mean that $dh_x \Omega^1_X = 0$ and $\Omega^1_{X,x}$ is free. But then x is a regular point of X, contradicting $\operatorname{codim}_x \Sigma = 1$. (If $\dim X = 1$, we should look at $\mathscr{H}^0_{\Sigma \cup \{x\}} \Omega^1_X$ throughout, and at this point achieve not a contradiction but the assertion that x is regular, hence normal.) The alternative is $\operatorname{codim}_x \Sigma > 1$, which implies normality by the Oka-Abhyankar-Thimm-Markoe criterion, or by Theorem 4.

Added in proof. It has recently come to the author's attention that similar results have been obtained by G.-M. Greuel, *Der Gauss-Manin-Zusammenhang isolierter Singularitäten von vollstandigen Durchschnitten* Math. Ann., 214 (1975), 235-266. For isolated singularities of hypersurfaces the topic was first considered from the present point of view by Brieskorn, *Die Monodromie der isolierten Singularitäten von Hyperflachen,* Manuscripta Math., 2 (1970), 103-161.

REFERENCES

1. S. Abhyankar, *Concepts of order and rank on a complex space and a condition for normality,* Math. Ann., **141** (1960), 171-192.
2. A. Andreotti and Y.-T. Siu, *Projective embedding of pseudoconcave spaces,* Annali di Pisa Scuola Norm. Sup., (3) **24** (1970), 231-278.
3. L. Brenton, *Complex singularities: a Riemann-Roch theorem for complex surfaces and extensions of holomorphic differential forms,* Thesis, University of Washington, Seattle, 1974.
4. ———, *On the homological dimension of sheaves of differential forms on a complex hypersurface,* (1978), (to appear).
5. R. Hartshorne, *Local cohomology,* Lecture notes no. 41, Springer-Verlag, Berlin-Heidelberg-New York, 1967.
6. A. Markoe, *A characterization of normal analytic spaces by the homological codimension of the structure sheaf,* Pacific J. Math., **52** (1974), 485-489.
7. K. Oka, *Lemma Fondamental,* J. Math. Soc. Japan, **3** (1951), 204-214.
8. G. Scheja, *Riemannsche Hebbarkeitssätze für cohomologieklassen,* Math. Ann., **144**

(1962), 345-360.
9. Y.-T. Siu and G. Trautmann, *Gap sheaves and extensions of coherent analytic subsheaves,* Lecture notes no. 172, Springer-Verlag, Berlin-Heidelberg-New York, 1971.
10. W. Thimm, *Über Moduln und Ideal von holomorphic Funktionen mehrerer Variablen,* Math. Ann., **139** (1959), 1-13.

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