PACIFIC JOURNAL OF MATHEMATICS
Vol. 101, No. 1, 1982

EMBEDDING HOMOLOGY 3-SPHERES IN S^5

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The purpose of this note is to give a proof independent of high-dimensional surgery theory of the following embedding result:

THEOREM. Let Σ^3 be the homology 3-sphere resulting from a Dehn surgery of type $1/2a$ on a knot in S^3. Then Σ^3 smoothly embeds in S^5 with complement a homotopy circle.

This theorem illustrates the connection between two major areas of ignorance in low-dimensional topology. For instance, if the homology sphere Σ^3 bounds a contractible 4-manifold V^4, then, using the 5-dimensional Poincaré conjecture, we see that $\Sigma^3 \times 0 \hookrightarrow \Sigma^3 \times D^2 \cup V^4 \times S^1$ is a smooth embedding of Σ into S^5 with complement homotopy equivalent to a circle. Conversely, if Σ smoothly embeds in S^5 with $S^5 - \Sigma \simeq S^1$, and if the Browder-Levine fibering theorem [1] holds in dimension 5, then $S^5 - \Sigma^3 \times \mathring{D}^2$ fibers over S^1, and the fiber is necessarily contractible.

High dimensional surgery theory can be used to completely solve this problem. Given Σ^3, convert $\Sigma^3 \times T^2$ to $K \simeq S^3 \times T^2$ via surgery, with $\Sigma^3 \subset K$ (see [6]). By work of Kirby-Siebenmann, K is homeomorphic to $S^3 \times T^2$. Lifting to the universal cover, we get $\Sigma \subset S^3 \times \boldsymbol{R}^2 \subset S^5$, and we see that every homology 3-sphere topologically embeds in S^5 with complement a homotopy circle. However, if Σ has nontrivial Rochlin invariant, a standard argument shows that the embedding cannot be smooth or PL. (If it were smooth (PL), make the homotopy equivalence $f: S^5 - \Sigma^3 \times \mathring{D}^2 \to S^1$ transverse to a point $p \in S^1$. Then $f^{-1}(p)$ would be a smooth (PL) spin manifold V^4 with zero signature and $\partial V = \Sigma$, contradicting the fact that Σ has nontrivial Rochlin invariant.) If Σ has trivial Rochlin invariant, the argument in [8] shows that the embedding can be taken to be smooth or PL. (See [7] for a much deeper analysis of knotting of homology 3-spheres in S^5.) Nevertheless, it seems desirable to give a more elementary construction for these embeddings when possible. It would be nice if these methods, together with the Kirby-Rolfsen calculus for links in S^3, could provide the desired embeddings for all Σ^3 with zero Rochlin invariant.

This proof grew out of studying Fintushel and Pao's attempt [3] to show that Scharlemann's possibly exotic $S^3 \times S^1 \# S^2 \times S^2$ is standard [6]. The basic construction is from [3] and will be described below.

I would like to thank the referee for very useful and constructive comments.

Proof of the theorem. Let $K \subset S^3$ be a smooth knot, and let Σ^3 be the homology 3-sphere resulting from a Dehn surgery on K of type $1/2a$. Let m and ℓ be a meridian and preferred longitude of K. It is not hard to see that surgery on the curve $\ell \times \{*\}$ in the 4-manifold $\Sigma^3 \times S^1$ produces a manifold homotopy equivalent to $S^3 \times S^1 \# S^2 \times S^2$ or $S^3 \times S^1 \# S^2 \tilde{\times} S^2$, depending on the framing used, where $S^2 \tilde{\times} S^2$ is the nontrivial S^2 bundle over S^2. We will sketch the proof ([3]) that the manifold is in fact diffeomorphic to $S^3 \times S^1 \# S^2 \times S^2$, assuming we use the framing which produces an even 4-manifold, and we will also keep track of homology generators for future use.

Think of surgery on $\ell \times \{*\}$ in $\Sigma^3 \times S^1$ as follows: First remove a tubular neighborhood $T \approx S^1 \times D^2 \times S^1$ of $\ell \times S^1$ in $\Sigma^3 \times S^1$, leaving $(S^3 - K \times \mathring{D}^2) \times S^1$. Let $X \approx S^1 \times D^3$ be a tubular neighborhood of $\ell \times \{*\}$, where X sits in T in the obvious fashion, so that $\overline{T - X} = S^1 \times D^2 \times I$. To surger ℓ, replace X by $D^2 \times S^2$, identifying $S^1 \times \{\text{polar caps}\} \subset D^2 \times S^2$ with $S^1 \times D^2 \times \{\pm 1\} \subset S^1 \times D^2 \times I$.

The identification $D^2 \times S^2 \bigcup_{S^1 \times D^2 \times \{\pm 1\}} S^1 \times D^2 \times I$ produces a 4-manifold P^4 which can be identified as the result of plumbing two copies of $S^2 \times D^2$ at two points. The boundary of P^4 is T^3 with homology generators e_1, e_2, e_3 as follows: e_1 is a meridian of $S^1 \times D^2 \times I$, e_2 is that longitude of $S^1 \times D^2 \times I$ which, after being isotoped across a plumbing point, becomes a meridian to $D^2 \times \text{equator} \subset D^2 \times S^2$, and e_3 generates $H_1(P^4) \cong \boldsymbol{Z}$. Actually, e_3 is defined only modulo multiples of e_1 and e_2, but P^4 admits self-diffeomorphisms taking any generator of $H_1(P)$ to any other generator (see [2], Lemma 3.3), so we can ignore this point.

If we let N denote the result of surgery on $\ell \times \{*\}$ in $\Sigma^3 \times S^1$ (using the framing induced from the zero framing of ℓ in Σ^3), we see that N is the union of P^4 and $(S^3 - K \times \mathring{D}^2) \times S^1$ defined by the matrix

$$\begin{array}{c|c} & \begin{matrix} e_1 & e_2 & e_3 \end{matrix} \\ \begin{matrix} m \\ \ell \\ h \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array},$$

where m is a meridian to K in S^3, and h generates the circle factor in $(S^3 - K \times D^2) \times S^1$.

Notice that there are two natural 2-spheres in P^4, the cores of the two copies of $S^2 \times D^2$. We have $H_2(P) \cong \boldsymbol{Z}^2$, generated by the cores, which we denote A and B, where A corresponds to the S^2 added in the surgery, and B is

$$D^2 \times \{\text{north and south poles}\} \cup S^1 \times \{0\} \times I$$

in the decomposition $P = D^2 \times S^2 \cup S^1 \times D^2 \times I$. Also, $H_2(T^3 = \partial P^4) \cong \boldsymbol{Z}^3$, generated by $e_1 \wedge e_2$, $e_1 \wedge e_3$, and $e_2 \wedge e_3$, which we write as e_{12}, e_{13}, e_{23}. The inclusion $T^3 \hookrightarrow P^4$ induces $e_{12} \mapsto 0$, $e_{13} \mapsto A$, $e_{23} \mapsto B$. Finally,

$$H_2((S^3 - K \times \mathring{D}^2) \times S^1) \cong \boldsymbol{Z} ,$$

generated by $m \wedge h$.

Examination of the Mayer-Vietoris sequence for N yields $H_2(N) \cong \boldsymbol{Z}^2$, with explicit generators. The 2-sphere A is one generator. Since e_2 bounds a disk in P, and is glued to ℓ, which bounds a Seifert surface in $S^3 - K \times \mathring{D}^2$, we may glue the disk to the surface to produce the other generator, which we refer to as the generator arising from e_2. Notice that B is trivial in $H_2(N)$.

Now create W^5 by adding a 2-handle to $\Sigma^3 \times S^1 \times I$ along $\ell \times \{*\} \times \{1\}$, producing a cobordism from $\Sigma^3 \times S^1$ to N. The class of A in $H_2(N)$ dies in $H_2(W)$, while the class arising from e_2 lives in $H_2(W)$. In fact, it is easy to see that

$$H_i(W) = \begin{cases} \boldsymbol{Z} , & i = 0, 1, 2, 3, 4 \\ 0 , & i = 5 \end{cases}$$

with all of $H_*(W)$ coming from $H_*(N)$.

Now, as first observed by Pao [5], P^4 admits the following self-diffeomorphism: remove one copy of $S^2 \times D^2$ and replace it by an element in the kernel of $\pi_1 SO(2) \to \pi_1 SO(3)$. This idea can easily be used to produce a self-diffeomorphism f which fixes e_3 and one of e_1, e_2 (say e_2), and takes e_1 to e_1 plus an even multiple of e_2. (To do this we remove and replace B.) This gives the following diagram:

$$\begin{tikzcd}
P \arrow[d, "f"] & \partial P \arrow[l, hook'] \arrow[rr, "\begin{pmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}"] & & (S^3 - K \times \mathring{D}^2) \times S^1 \arrow[d, equal] \\
P & \partial P \arrow[l, hook'] \arrow[rr, "\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}"'] & & (S^3 - K \times \mathring{D}^2) \times S^1
\end{tikzcd}$$

The top row gives N, the bottom $S^3 \times S^1 \# S^2 \times S^2$, yielding

$$N \xrightarrow[\approx]{f} S^3 \times S^1 \# S^2 \times S^2 \,.$$

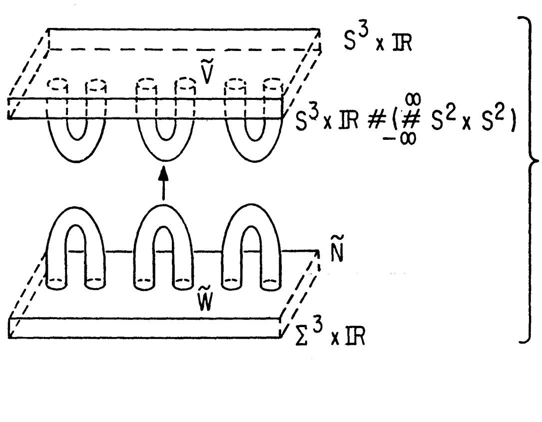
We can also create V^5, a cobordism from $S^3 \times S^1$ to $S^3 \times S^1 \# S^2 \times S^2$, by attaching a 2-handle along $/ \times \{*\} \times \{0\}$ in $S^3 \times S^1 \times I$. Glue W to V using the diffeomorphism f, creating a cobordism X^5 from $\Sigma^3 \times S^1$ to $S^3 \times S^1$.

The point is this: The class of A in $H_2(N)$ is taken to the corresponding class in $H_2(S^3 \times S^1 \# S^2 \times S^2)$, which dies in $H_2(V)$. This is certainly not true geometrically, since f takes A to $A + 2aB$ (essentially, A is altered by the "belt trick"), but B is homologically trivial. The class in $H_2(N)$ arising from e_2 is geometrically taken to the corresponding class in $H_2(S^3 \times S^1 \# S^2 \times S^2)$.

Now examine $H_*(X)$. Since A bounds D^3 in W, and $A + 2aB$ bounds a 3-chain in V, we produce a generator in $H_3(X)$. This 3-cycle has intersection number ± 1 with the generator of $H_2(W)$ arising from e_2, and the generator of $H_2(W)$ arising from e_2 is identified with a class in $H_2(V)$ which we can represent by an embedded 2-sphere (with trivial normal bundle), since $/$ bounds a singular disk in S^3.

Now surger the generator of $H_2(X)$. Standard sequences for this surgery show that this simultaneously kills the generator of $H_2(X)$ and its dual in $H_3(X)$. The result is a homology product, Y, from $\Sigma^3 \times S^1$ to $S^3 \times S^1$, and $\pi_1 Y \cong \boldsymbol{Z}$, coming from the circle factor in either boundary component. If we now glue $D^4 \times S^1$ to Y along $S^3 \times S^1$, and glue $\Sigma^3 \times D^2$ along $\Sigma^3 \times S^1$, we have a simply-connected homology 5-sphere, hence S^5. Thus, we have a smooth embedding of Σ^3 in S^5 with $\pi_1(S^5 - \Sigma^3 \times \mathring{D}^2) \cong \boldsymbol{Z}$.

Actually, it follows from [6] that for every homology 3-sphere Σ, $\Sigma \times S^1$ is homology-cobordant to $S^3 \times S^1$. The argument is as follows: embed Σ^3 in S^5 and remove a tubular neighborhood $\Sigma^3 \times D^2$



of Σ and a tubular neighborhood $S^1 \times D^4$ of a meridian to the knotted Σ^3. The result is a homology-cobordism Y^5 from $\Sigma^3 \times S^1$ to $S^3 \times S^1$, and $\pi_1(\Sigma) \to \pi_1(Y)$ is trivial. In general, $\pi_1 Y$ will be mysterious.

Consider the universal cover $\widetilde{X}$: We have $H_2(\widetilde{X}) \cong \boldsymbol{Z}(\boldsymbol{Z})$ and $H_3(\widetilde{X}) \cong \boldsymbol{Z} \oplus \boldsymbol{Z}(\boldsymbol{Z})$. If we do $\boldsymbol{Z}$ surgeries equivariantly, killing the $\boldsymbol{Z}(\boldsymbol{Z})$ factors, the result is $\widetilde{Y}$. To create $\widetilde{S^5 - \Sigma \times D^2}$, attach $D^4 \times \boldsymbol{R}$ to $\widetilde{Y}$ along $S^3 \times \boldsymbol{R}$. This kills $H_3(\widetilde{Y})$, and thus $\widetilde{S^5 - \Sigma \times D^2}$ is contractible, so that $S^5 - \Sigma \times \mathring{D}^2$ is a $K(\boldsymbol{Z}, 1)$. This proves the theorem.

REMARKS. (1) Surgery of type $1/2a$ on a knot in S^3 results in a homology 3-sphere with zero Rochlin invariant, by [4].

(2) The proof is equally valid for (a) knots in homology spheres which bound contractible 4-manifolds, or (b) surgeries of type $1/2a_i$, $i = 1, \cdots, n$, on a link of n components, provided the components are algebraically unlinked (by doing n times as many surgeries). In particular, the theorem is valid for connected sums of Σ's as above.

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Received June 1, 1980.

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