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BEST APPROXIMATION PROBLEMS IN TENSOR-PRODUCT SPACES

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This paper concerns an existence problem for best approximations of bivariate functions. The approximating functions are taken from infinite-dimensional subspaces having tensor product form. Problems of this type arise, for example, in approximating the kernel of an integral equation by a degenerate ("separable") kernel. A sample of our results is this: let G and H be finite-dimensional subspaces in continuous function spaces $C(S)$ and $C(T)$ respectively. If one of these subspaces has a continuous proximity map and the other a Lipschitzian proximity map, then $G \otimes C(T) + C(S) \otimes H$ is proximinal in $C(S \times T)$; i.e., best approximations exist in this subspace.

Practical problems in numerical analysis, especially in solving two-point boundary value problems or integral equations, often require the approximation of a bivariate function by a combination of univariate functions. For example, if $f(s, t)$ is defined for $s \in S$ and $t \in T$, an approximation to f of the following form may be required:

$$f(s, t) \approx \sum_{i=1}^{m} x_i(s)h_i(t) + \sum_{i=1}^{n} y_i(t)g_i(s) . \tag{1}$$

Here the *base* functions g_i and h_i are prescribed, and the *coefficient* functions x_i and y_i are at our disposal.

The problem of finding a best uniform approximation of the form (1) when all the functions involved are continuous is a difficult one, the difficulties being both theoretical and algorithmic. In the special case $n = m = 1$, with $g_1(s) = h_1(t) = 1$, the problem reduces to finding $x \in C(S)$ and $y \in C(T)$ which minimize the expression

$$\|f - x - y\| = \sup \sup_{t} |f(s, t) - x(s) - y(t)| . \tag{2}$$

The existence of minimizing pairs (x, y) and an efficient algorithm for determining one of them were established by Diliberto and Straus [3]. See also [1, 8, 5, 10, 9] for later work.

The general case of best approximation in (1) with uniform norm remains open. In this paper, the existence of optimal solutions to problem (1) is studied. Ideally, we would like to have *constructive* proofs of existence, but in general the available proofs are nonconstructive.

If U is a linear subspace of a normed space X, the distance from x to U is

$$\operatorname{dist}(x, U) = \inf_{u \in U} \|x - u\| . \tag{3}$$

If the infimum in (3) is attained for each $x \in X$, then the subspace U is said to be *proximinal*. A mapping $A: X \to U$ such that $\|x - Ax\| = \operatorname{dist}(x, U)$ for all x is called a *proximity* map for U. Every proximinal subspace has a proximity map, but not necessarily a continuous one.

The following result from [13, p. 130] will be useful:

THEOREM 1. *If U, V and $U + V$, are closed subspaces of a Banach space, then there is a constant c such that each element of $U + V$ is expressible as $u + v$ with $u \in U$, $v \in V$, and $\|u\| + \|v\| \leqq c\|u + v\|$.*

THEOREM 2. *For a pair of closed subspaces U and V in a Banach space the following properties are equivalent:*

(1) *$U + V$ is closed*

(2) *$U^\perp + V^\perp$ is closed*

(3) *$U^\perp + V^\perp$ is weak*-closed*

(4) *$U^\perp + V^\perp$ is proximinal.*

Proof. The implication $1 \Rightarrow 3$ is proved as follows. H. Reiter showed in [12] that if U, V, and $U + V$ are closed subspaces in a Banach space, then $U^\perp + V^\perp = (U \cap V)^\perp$. Since the annihilator of a subspace is weak*-closed, [13, p. 91], $1 \Rightarrow 3$. The implication $3 \Rightarrow 4$ is an observation made by Phelps [11]. The implication $4 \Rightarrow 2$ is trivial, since every proximinal set is closed. The implication $2 \Rightarrow 1$ is another result of Reiter [12]. $\square$

THEOREM 3. *Let U and V be weak*-closed subspaces in a conjugate Banach space X^*. If $U + V$ is norm closed, then it is weak*-closed and proximinal.*

Proof. Since U and V are weak*-closed, they satisfy $U = (U_\perp)^\perp$ and $V = (V_\perp)^\perp$, where $U_\perp = \{x \in X: \langle x, u\rangle = 0 \text{ for all } u \in U\}$. By Theorem 2 (in particular the implications $2 \Rightarrow 3 \Rightarrow 4$), our conclusion follows. $\square$

THEOREM 4. *Let U and V be subspaces in a normed space X. Assume that U is proximinal, and that for each $x \in X$ there corresponds a weakly compact set $K(x) \subset V$ with the property*

$$\inf_{v \in K(x)} \operatorname{dist}(x - v, U) = \inf_{v \in V} \operatorname{dist}(x - v, U) .$$

Then $U + V$ is proximinal.

Proof. By the Hahn-Banach theorem,

$$\operatorname{dist}(x, U) = \sup\{\langle \phi, x\rangle : \phi \in U^{\perp}, \|\phi\| = 1\} .$$

This shows that the function $x \mapsto \operatorname{dist}(x, U)$ is weakly lower-semi-continuous, since it is the supremum of a family of weakly continuous functions. Therefore, if x is fixed, the expression $\operatorname{dist}(x - v, U)$ will attain its infimum at some point $v' \in K(x)$. Select $u' \in U$ as a best approximation to $x - v'$. Then for any $v \in V$ and $u \in U$ we have

$$\|x - u' - v'\| = \operatorname{dist}(x - v', U) \leqq \operatorname{dist}(x - v, U) \leqq \|x - v - u\| . \square$$

THEOREM 5. *Let U and V be proximinal subspaces in a Banach space X. Assume that $U + V$ is closed, and that V has a proximity map A such that for each $c \in X$, the map $u \mapsto A(c - u)$ is weakly compact from U into V. Then $U + V$ is proximinal.*

Proof. Let c be any element of X, and select $z_n \in U + V$ so that $\|c - z_n\| \to \operatorname{dist}(c, U + V)$. The sequence $\{z_n\}$ is bounded. Since $U + V$ is closed, Theorem 1 implies that z_n can be expressed as $u_n + v_n$ with $u_n \in U$, $v_n \in V$, and $\{u_n\}$ bounded. Put $v'_n = A(c - u_n)$. Since $\{u_n\}$ is bounded, $\{v'_n\}$ lies in a weakly compact subset K of V. Then for each n,

$$\begin{aligned}\inf_{v \in K} \operatorname{dist}(c - v, U) &\leqq \operatorname{dist}(c - v'_n, U) \leqq \|c - u_n - v'_n\| \\ &\leqq \|c - u_n - v_n\| .\end{aligned}$$

Hence

$$\inf_{v \in K} \operatorname{dist}(c - v, U) \leqq \operatorname{dist}(c, U + V) = \inf_{v \in V} \operatorname{dist}(c - v, U) .$$

Thus Theorem 4 is applicable, and $U + V$ is proximinal. $\square$

The *uncompleted tensor product* of two normed spaces X and Y is the set of all finite sums of the form $\sum x_i \otimes y_i$ with $x_i \in X$ and $y_i \in Y$. An equivalence relation is introduced by stipulating that $\sum x_i \otimes y_i$ is (equivalent to) 0 when $\sum \langle f, x_i\rangle y_i = 0$ for all $f \in X^*$.

A norm α on $X \otimes Y$ is termed a *cross-norm* if $\alpha(x \otimes y) = \|x\|\|y\|$ for all $x \in X$ and all $y \in Y$. A cross-norm α is said to be a *uniform* cross-norm if

$$\alpha(\sum Ax_i \otimes By_i) \leqq \|A\|\|B\|\alpha(\sum x_i \otimes y_i)$$

for any bounded linear operators A and B.

The completion of the normed linear space $X \otimes Y$ with a cross-norm α is denoted here by $X \otimes_\alpha Y$. For other matters concerning tensor products, see Schatten [14], Gilbert and Leih [9], or Diestel and Uhl [2]. In particular, we use the isometric identification $\mathscr{L}(X, Y^*) = (X \otimes_\gamma Y)^*$ [14, p. 47].

The following theorem resulted from discussions with Professor John E. Gilbert, to whom we are indebted.

THEOREM 6. *Let G and H be complemented subspaces in Banach spaces X and Y respectively. For any uniform cross-norm α, $(G \otimes_\alpha Y) + (X \otimes_\alpha H)$ is complemented (and therefore closed) in $X \otimes_\alpha Y$.*

Proof. Let P be a (bounded linear) projection of X onto G. Define P' on the uncompleted tensor product $X \otimes Y$ by putting $P'(\sum x_i \otimes y_i) = \sum Px_i \otimes y_i$. By the uniform property of the cross-norm α, we have $\alpha[P'(\sum x_i \otimes y_i)] \leqq \|P\|\alpha(\sum x_i \otimes y_i)$. Thus P' is uniformly continuous on a dense subset of $X \otimes_\alpha Y$ and has therefore a unique continuous extension to $X \otimes_\alpha Y$. Thus extended, P' is a projection of $X \otimes_\alpha Y$ onto $G \otimes_\alpha Y$. In the same way, starting with a projection Q of Y onto H we define a projection Q' of $X \otimes_\alpha Y$ onto $X \otimes_\alpha H$. One verifies easily that P' commutes with Q'. Hence [see 4, p. 481] $P' + Q' - P'Q'$ is a projection of $X \otimes_\alpha Y$ onto $(G \otimes_\alpha Y) + (X \otimes_\alpha H)$. □

In the following discussion, T will denote an arbitrary compact Hausdorff space. Then $C(T)$ is the usual space of continuous real-valued functions on T.

The special cross-norm λ is defined by the equation

$$\lambda(\sum x_i \otimes y_i) = \sup_f \|\sum \langle f, x_i \rangle y_i\|$$

where f ranges over the unit cell in X^*.

The next theorem has been given in [6]; the proof is included because it is brief.

THEOREM 7. *If there exists a continuous proximity map from the Banach space X onto a subspace G, then $C(T) \otimes_\lambda G$ is proximinal in $C(T) \otimes_\lambda X$.*

Proof. By a theorem of Grothendieck, [15, p. 357], $C(T) \otimes_\lambda X$ is isometric with $C(T, X)$. The latter is the Banach space of all continuous maps f from T into X, normed by putting $\|f\| = \sup_t \|f(t)\|_X$. If A is a continuous proximity map from X onto G then let A' be defined from $C(T, X)$ onto $C(T, G)$ by the equation $A'f = A \circ f$. It is elementary to prove that A' is a continuous proximity map. □

THEOREM 8. *If G is a subspace of $C(S)$ such that $G \otimes_\lambda C(T)$ is proximinal in $C(S \times T)$, then G is proximinal.*

Proof. Assume that $G \otimes_\lambda C(T)$ is proximinal. Let x be any element of $C(S)$. Put $x'(x, t) = x(s)$ for all $(s, t) \in S \times T$. Note that for any $g \in G$,

$$\operatorname{dist}(x', G \otimes_\lambda C(T)) \leqq \|x' - g \otimes 1\| = \|x - g\|$$

whence $\operatorname{dist}(x', G \otimes_\lambda C(T)) \leqq \operatorname{dist}(x, G)$. Let z be a best approximation to x' from $G \otimes_\lambda C(T)$. Select $\tau \in T$ so that $\|x' - z\| = \sup_s |x'(s, \tau) - z(s, \tau)|$. Put $g(s) = z(s, \tau)$. Then $g \in G$, and g is a best approximation to x since

$$\begin{aligned}\|x - g\| &= \sup_s |x(s) - g(s)| = \sup_s |x'(s, \tau) - z(s, \tau)| = \|x' - z\| \\ &= \operatorname{dist}(x', G \otimes_\lambda C(T)) \leqq \operatorname{dist}(x, G)\ .\end{aligned}$$ □

The following result is called "The Sitting-Duck Theorem" because it is thought to be true under weaker hypotheses on H, and is therefore vulnerable to generalization.

THEOREM 10 ("*Sitting Duck*"). *Let G be a finite-dimensional subspace of $C(S)$ with a continuous proximity map. Let H be a finite-dimensional subspace of $C(T)$ with a Lipschitzian proximity map. Then $G \otimes C(T) + C(S) \otimes H$ is complemented and proximinal in $C(S \times T)$.*

Proof. By Theorem 7, the subspaces $U = G \otimes C(T)$ and $V = C(S) \otimes H$ are proximinal. By Theorem 6, $U + V$ is complemented and closed. Let A be a Lipschitzian proximity map of $C(T)$ onto H, and put $(A'f)(s, t) = (Af_s)(t)$. Then A' is a proximity map of $C(S \times T)$ onto V. Define $\Gamma: U \to V$ by $\Gamma u = A'(f - u)$, where f is now fixed. By the following lemma, Γ is compact. By Theorem 5, $U + V$ is proximinal. (Note: $f_s(t) = f^t(s) = f(s, t)$.) □

REMARK. Instead of assuming that G has a continuous proximity map, we can assume that $G \otimes C(T)$ is proximinal in $C(S \times T)$.

LEMMA. *The map $\Gamma: U \to V$ defined in the proof of Theorem* 10 *is compact.*

Proof. Let $B = \{u \in U: \|u\| \leqq k\}$. We will show that $\Gamma(B)$ has compact closure in V. By the Ascoli theorem, it suffices to show that $\Gamma(B)$ is bounded and equicontinuous.

If $u \in U$ then $\|\Gamma(u)\| = \|A'(f-u)\| \leqq 2\|f-u\| \leqq 2\|f\| + 2k$. Hence $\Gamma(B)$ is bounded. The remainder of the proof addresses the equicontinuity. Assume that $\|Ax - Ay\| \leqq \lambda\|x-y\|$ for $x, y \in C(T)$. Let n denote the dimension of G.

Select $\{g_1, \cdots, g_n\} \subset G$ and $\{\phi_1, \cdots, \phi_n\} \subset C(S)^*$ so that $\langle \phi_i, g_j \rangle = \delta_{ij}$, $\|g_i\| = \|\phi_i\| = 1$ ("biorthonormality"). If $u(s, t) = \sum_{i=1}^n x_i(t) g_i(s)$ then

$$|x_i(t)| = |\langle \phi_i, u^t \rangle| \leqq \|u^t\| \leqq k .$$

Let (s_0, t_0) be a point of $S \times T$ at which equicontinuity is to be proved. Let $\varepsilon > 0$. By the equicontinuity of the unit cell in G there is a neighborhood N_1 of s_0 such that for all $s \in N_1$ and for all $g \in G$, $|g(s) - g(s_0)| \leqq \varepsilon\|g\|$. Similarly, there is a neighborhood N_2 of t_0 such that for all $t \in N_2$ and for all $h \in H$, $|h(t) - h(t_0)| < \varepsilon\|h\|$. By the equicontinuity of $\{f^t : t \in T\}$ we can shrink the neighborhood N_1 if necessary so that $|f^t(s) - f^t(s_0)| < \varepsilon$ for all $s \in N_1$ and all $t \in T$. Then $\|f_s - f_{s_0}\| < \varepsilon$. Let $N = N_1 \times N_2$. If $(s, t) \in N$ then

$$\begin{aligned}
&|(\Gamma u)(s, t) - (\Gamma u)(s_0, t_0)| \\
&\quad \leqq |(\Gamma u)(s, t) - (\Gamma u)(s_0, t)| + |(\Gamma u)(s_0, t) - (\Gamma u)(s_0, t_0)| \\
&\quad = |A(f_s - u_s)(t) - A(f_{s_0} - u_{s_0})(t)| + |A(f_{s_0} - u_{s_0})(t) - A(f_{s_0} - u_{s_0})(t_0)| \\
&\quad \leqq \|A(f_s - u_s) - A(f_{s_0} - u_{s_0})\| + \|A(f_{s_0} - u_{s_0})\|\varepsilon \\
&\quad \leqq \lambda\|(f_s - u_s) - (f_{s_0} - u_{s_0})\| + 2\|f_{s_0} - u_{s_0}\|\varepsilon \\
&\quad \leqq \lambda\{\|f_s - f_{s_0}\| + \|u_s - u_{s_0}\|\} + 2\{\|f\| + \|u\|\}\varepsilon \\
&\quad \leqq \lambda\{\varepsilon + \|\textstyle\sum [g_i(s) - g_i(s_0)]x_i\|\} + 2\|f\|\varepsilon + 2k\varepsilon \\
&\quad \leqq \lambda\{\varepsilon + n\varepsilon k\} + 2\|f\|\varepsilon + 2k\varepsilon . \qquad \square
\end{aligned}$$

In a separate paper, we have given examples of subspaces having Lipschitzian proximity maps in a space $C(T)$. These can be of any finite dimension or infinite dimensional. The situation is rather complicated, however, and the topological structure of T must be taken into account.

In several of the following theorems we require the equation

$$X^* \otimes_\lambda Y^* \subset \mathscr{L}(X, Y^*) = (X \otimes_\gamma Y)^* .$$

The identifications made here are as follows. With an element $\sum \varphi_i \otimes \psi_i$ in $X^* \otimes Y^*$ (uncompleted tensor product) we associate an operator $A \in \mathscr{L}(X, Y^*)$ whose defining equation is $Ax = \sum \langle \varphi_i, x \rangle \psi_i$. With an arbitrary operator B in $\mathscr{L}(X, Y^*)$ we associate a functional Φ in $(X \otimes_\gamma Y)^*$ by putting $\Phi(\sum x_i \otimes y_i) = \sum \langle Bx_i, y_i \rangle$.

The weak*-topology in $\mathscr{L}(X, Y^*)$ is the weak topology induced by the duality of $X \otimes_\gamma Y$ with $(X \otimes_\gamma Y)^*$. Convergence of a net A_α to 0 in this topology means $\langle A_\alpha x, y \rangle \to 0$ for all $x \in X$ and $y \in Y$. This topology is also called the weak*-operator topology.

THEOREM 17. *Let P be a projection on a Banach space X. Let Y be any other Banach space. Then the subspace*

$$M = \{A \circ P \colon A \in \mathscr{L}(X, Y^*)\}$$

is complemented, weak-closed, and hence proximinal in $\mathscr{L}(X, Y^*)$.*

Proof. For $A \in \mathscr{L}(X, Y^*)$, define $p(A) = A \circ P$. Then p is a bounded linear mapping of $\mathscr{L}(X, Y^*)$ into M. Since $p(A \circ P) = A \circ P \circ P = A \circ P$, it follows that p acts like the identity on M. Therefore p is a projection and M is complemented.

In order to show that M is weak*-closed, we note first that M is the null-space of $i - p$, where i denotes the identity map on $\mathscr{L}(X, Y^*)$. Next we observe that p (and hence $i - p$) is weak*-continuous. Indeed, if A_α is a net in $\mathscr{L}(X, Y^*)$ which converges in the weak*- topology to 0, then $\langle A_\alpha x, y\rangle \to 0$ for all $(x, y) \in X \times Y$. Hence $\langle p(A_\alpha)x, y\rangle = \langle A_\alpha Px, y\rangle \to 0$ for all (x, y), and $p(A_\alpha)$ converges to 0 in the weak*-topology. □

A completely analogous proof establishes the next result.

THEOREM 18. *Let Q be a projection on a Banach space Y. Let X be any other Banach space. Then the subspace*

$$N = \{Q^* \circ A \colon A \in \mathscr{L}(X, Y^*)\}$$

is complemented, weak-closed and hence proximinal in $\mathscr{L}(X, Y^*)$.*

THEOREM 19. *Let P and Q be projections on Banach spaces X and Y respectively. Then*

$$\{A \circ P + Q^* \circ B \colon A, B \in \mathscr{L}(X, Y^*)\}$$

is complemented, weak-closed, and proximinal in $\mathscr{L}(X, Y^*)$.*

Proof. It is sufficient to verify that the projections p and q defined by $p(A) = A \circ P$ and $q(A) = Q^* \circ A$ commute with each other. But this is obviously true:

$$p(q(A)) = (Q^* \circ A) \circ P = Q^* \circ (A \circ P) = q(p(A)) \,. \qquad \square$$

REMARKS. Theorem 19 was suggested to us by an anonymous referee for the Mathematical Proceedings of the Cambridge Philosophical Society. We had, prior to his suggestion, established only the following theorem by a different argument.

THEOREM 20. *Let G and H be finite-dimensional subspaces in* *conjugate Banach spaces* X^* *and* Y^* *respectively. Then* $G \otimes Y^* + X^* \otimes H$ *is complemented, weak*-closed, and proximinal in* $\mathscr{L}(X, Y^*)$. *It is therefore complemented and proximinal in* $X^* \otimes_\lambda Y^*$.

Proof. We prove first that $G \otimes Y^* = \{A \circ P: A \in \mathscr{L}(X, Y^*)\}$ for an appropriate projection $P: X \to X$. Indeed, select a basis $\{g_1, \cdots, g_n\}$ for G and then select $x_1, \cdots, x_n$ in X so that $\langle g_i, x_j \rangle = \delta_{ij}$. Put $Px = \sum \langle x, g_i \rangle x_i$. If $\sum g_i \otimes \psi_i$ is any element of $G \otimes Y^*$, let A be an element of $\mathscr{L}(X, Y^*)$ such that $Ax_i = \psi_i$. Then $A \circ P = \sum g_i \otimes \psi_i$. Conversely, if $A \in \mathscr{L}(X, Y^*)$, then $A \circ P = \sum g_i \otimes Ax_i \in G \otimes Y^*$.

A similar argument applies to $X^* \otimes H$, and then Theorem 19 establishes the desired conclusion. □

In approximation problems, it is a fortunate circumstance when a subspace of functions being used as approximants has a *linear* proximity map. Of course, this is the rule in Hilbert space, but the exception in other spaces, although proximinal hyperplanes always have linear proximity maps in any normed space. In spaces $C(T)$, a finite-dimensional subspace can have a linear proximity map, but if this happens, T must possess isolated points.

If a proximity map P from a normed space X onto a subspace V is *linear*, then P is a projection (i.e., a bounded, linear, idempotent, surjective map.) It is elementary to prove that for a projection P the properties of being a proximity map and satisfying the equation $\|I - P\| = 1$ are equivalent.

Another elementary result is that if P and Q are projections on a normed space X, and if $QP = PQP$, then $P + Q - PQ$ is a projection onto the vector sum of the ranges of P and Q. This vector sum must then be complemented and closed. We can now prove:

THEOREM 21. *If* P *and* Q *are linear proximity maps, then the same is true of the Boolean sum* $P + Q - PQ$, *provided that* $PQP = QP$.

Proof. It is only necessary to verify that $\|I - P - Q + PQ\| = 1$. This follows from writing the operator in question in the form $(I - P)(I - Q)$. □

LEMMA. *If* $A \in \mathscr{L}(X, X)$ *and* $B \in \mathscr{L}(Y, Y)$ *then the operator* $A \otimes B$ *defined on* $X \otimes Y$ *by the equation*

$$(A \otimes B) \sum x_i \otimes y_i = \sum Ax_i \otimes By_i$$

has a unique extension $A \otimes_\alpha B$ *in* $\mathscr{L}(X \otimes_\alpha Y, X \otimes_\alpha Y)$, *for any uniform cross norm* α.

THEOREM 22. *If G and H are subspaces having linear proximity maps in Banach spaces X and Y respectively, then $G \otimes_\alpha Y + X \otimes_\alpha H$ is proximinal in $X \otimes_\alpha Y$, for any uniform cross-norm α.*

Proof. Suppose that $P: X \twoheadrightarrow G$ and $Q: Y \twoheadrightarrow H$ are linear proximity maps. Then $P \otimes_\alpha I_Y$ and $I_X \otimes_\alpha Q$ are linear proximity maps from $X \otimes_\alpha Y$ onto $G \otimes_\alpha Y$ and $X \otimes_\alpha H$, respectively. They commute, by the lemma which follows. Hence by the preceding theorem, their Boolean sum is a linear proximity map. Its range is the sum of the ranges of the constituent maps, i.e., $G \otimes_\alpha Y + X \otimes_\alpha H$. $\square$

LEMMA. *Let A_1 and A_2 be commuting elements of $\mathscr{L}(X, X)$. Let B_1 and B_2 be commuting elements of $\mathscr{L}(Y, Y)$. Then $A_1 \otimes_\alpha B_1$ commutes with $A_2 \otimes_\alpha B_2$ for any uniform cross-norm α.*

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