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COMPACT CONNECTED LIE GROUPS ACTING ON SIMPLY CONNECTED 4-MANIFOLDS

HAE SOO OH

Suppose a compact connected Lie group G acts effectively on a simply connected 4-manifold M. Then we show that G is one of the groups SO(5), SU(3)/$Z(G)$, SO(3) $\times$ SO(3), SO(4), SO(3) $\times$ T^1, (SU(2) $\times$ T^1)/D, SU(2), SO(3), T^2, T^1, and that the representatives of the conjugacy classes of the principal isotropy groups for these groups on M are, respectively, SO(4), $U(2)$, T^2, SO(3), S^1, S^1, $\widetilde{\mathrm{SO}}(2)$ or e, SO(2) or D_{2n}, e, and e. We also show that in each of these cases M is a connected sum of copies of S^4, $S^2 \times S^2$, CP^2, and $-CP^2$ (except when G is T^1, see Theorem 2.6).

1. Introduction. All manifolds in this paper are assumed to be closed, connected and orientable. Also all actions are assumed to be effective and locally smooth. Orlik-Raymond **[O-R]** showed that if a simply connected 4-manifold admits an action of the two-dimensional torus group T^2, then M is a connected sum of copies of S^4, $S^2 \times S^2$, CP^2, and $-CP^2$. Fintushel **[$\mathbf{F_2}$]** proved that if M admits a circle action and the orbit space M^* is not a counterexample of Poincaré's conjecture, then M is also a connected sum of copies of these manifolds.

In this paper we determine all Lie groups which can act on a simply connected 4-manifold M, and dually we classify all simply connected 4-manifolds which admit an action of a given compact connected Lie group G.

An isotropy group H is a *principal* if H is conjugate to a subgroup of each isotropy group (that is, G/H is a maximum orbit type for G on M). One denotes by $G(x)$ the orbit of G through x, and by G_x the isotropy group at x. A *maximal torus* T is a compact connected abelian Lie subgroup which is not properly contained in any larger such subgroup. We denote the normalizer of G by $N(G)$, and the centralizer of G by $Z(G)$. Let $\chi(M)$ denote the euler characteristic of a space M. Then it is well known that $\chi(G/T)$ is the order of $N(T)/T$.

2. The rank of a Lie group G which can act on a simply connected 4-manifold M. Suppose K is a subgroup of G which acts on a topological space X. Then the action of G on X may not be effective even if the action restricted to K is effective. But the maximal torus theorem gives rise to the following.

LEMMA 2.1. *A compact connected Lie group G acts effectively on a topological space X if and only if the action restricted to a maximal torus T of G is effective.*

Proof. Suppose G does not act effectively. Then there exists at least one element $g \neq e$ in G such that $gx = x$, for all $x \in X$. It follows from the maximal torus theorem that there exists an element $h \in G$ such that $g \in hTh^{-1}$. Hence $h^{-1}gh \in T$. Thus we have $(h^{-1}gh)x = h^{-1}g(hx) = h^{-1}hx = x$, for all $x \in X$, which says that the action restricted to T is not effective. $\square$

By *the rank of a Lie group G*, we mean the dimension of a maximal torus of G.

LEMMA 2.2. *If a compact connected Lie group G acts on a simply connected 4-manifold M, then the rank of G is less than 3.*

Proof. Suppose the rank of G is ≥ 3. Then M admits an effective T^3-action. By [**P**], M is homeomorphic to either T^4 or $L(p, q) \times T^1$, which contradicts the simple connectivity of M. $\square$

It is known that every compact connected Lie group of dimension ≤ 6 can be represented as a factor group G/F, where $G = G_1 \times G_2 \times \cdots \times G_n$ is a product; each factor G_i is either SO(2) or SU(2) ($= S^3$), and F is a finite subgroup of the center of G.

From now on G is a compact connected Lie group acting on a simply connected 4-manifold M, and H is a principal isotropy group for G on M. (Note: any two principal isotropy groups are conjugate to each other. Actually H denotes a *representative group of the conjugacy class* of principal isotropy groups.)

LEMMA 2.3. *Suppose the rank of G is 2 and the rank of H is 0. Then G is the two-dimensional torus group T^2.*

Proof. From [**B**, p. 195], we have the following inequality:

$$(*) \quad \dim M - \dim G/H - (\operatorname{rank} G - \operatorname{rank} H) \leq \dim M - 2 \operatorname{rank} G.$$

Since we assumed rank $H = 0$, then $\dim G/H \leq 4$. Hence the inequality gives rise to $4 \geq \dim G/H \geq \operatorname{rank} G = 2$. Since $\dim G - \operatorname{rank} G$ should be an even integer, $\dim G$ ($= \dim G/H$) must be either 4 or 2.

If $\dim G$ is 4, then G acts transtively on M (that is, $M = G/H$). Since a compact connected Lie group of dimension 4 and of rank 2 is either $\mathrm{SU}(2) \times \mathrm{SO}(2)$ or a factor group of this by a finite subgroup, G/H cannot be simply connected. We thus have $\dim G = 2 = \operatorname{rank} G$. Hence G is T^2. □

LEMMA 2.4. *If a compact connected Lie group G acts on a simply connected 4-manifold M, then we have the following*:

(i) *if* $\operatorname{rank} G = 2$ *and* $\operatorname{rank} H = 2$, *then* $\dim G$ *is* 10, 8, *or* 6;

(ii) *if* $\operatorname{rank} G = 2$ *and* $\operatorname{rank} H = 1$, *then* $\dim G/H = 3$ *and* $\dim G$ *is either* 6 *or* 4;

(iii) *if* $\operatorname{rank} G = 2$ *and* $\operatorname{rank} H = 0$, *then* $G = T^2$;

(iv) *the orbit space M^* is a simply connected manifold with boundary.*

Proof. From [**B**, p. 195], we have an inequality,

$$(**) \qquad 4 \geq \dim G/H \geq \operatorname{rank} G + \operatorname{rank} H.$$

It is known [**M-Z**] that if the maximal dimension of any orbit is k, then $\dim G \leq k(k+1)/2$. Thus $\dim G \leq 10$. Since $\dim G - \operatorname{rank} G$ is an even integer, $\dim G$ is 10, 8, 6, 4, or 2, provided $\operatorname{rank} G$ is 2.

(i) If $\operatorname{rank} G = 2$ and $\operatorname{rank} H = 2$, then it follows from inequality $(**)$ that $\dim G/H = 4$. Hence $\dim G \geq 6$.

(ii) If $\operatorname{rank} G = 2$ and $\operatorname{rank} H = 1$, then by $(**)$, $\dim G/H$ is either 3 or 4. Suppose $\dim G/H = 4$. Then $\dim H$ $(= \dim G - \dim G/H)$ is 6, 4, or 2. On the other hand, $\operatorname{rank} H = 1$ implies that the identity component of H is SO(2), SO(3), or SU(2). Hence $\dim G/H$ should be 3. By [**M-Z**], $\dim G \leq \frac{1}{2}(\dim G/H)(\dim G/H + 1) = 6$.

(iii) was shown in Lemma 2.3.

(iv) If $\operatorname{rank} G = 2$ and $\operatorname{rank} H \geq 1$, then $(**)$ implies that $\dim G/H$ is either 4 or 3. If $\operatorname{rank} G = 2$, and $\operatorname{rank} H = 0$, then by (iii), we have $G = T^2$.

Thus if $\operatorname{rank} G = 2$, the orbit space M^* is D^0, D^1, or D^2 (cf. Lemma 5.1 [**O-R**]). If $\operatorname{rank} G = 1$, then G is SO(2), SO(3), or SU(2). Since any proper subgroups of SO(3) and SU(2) are of dimension ≤ 1, if G is either SO(3) or SU(2), then $\dim G/H$ should be ≥ 2. Hence M^* is D^1, D^2, or S^2. If $G = \mathrm{SO}(2)$, then by Lemma 3.1 [$\mathbf{F_1}$], M^* is a simply connected 3-manifold with boundary. □

If an abelian group G acts effectively on a manifold M, then the principal isotropy group H is trivial. We have shown that if $\operatorname{rank} G = 2$

and rank $H = 0$, then G is T^2, hence H is trivial. In this case, the manifolds are determined by the following theorem.

THEOREM 2.5. [**O-R**] *If M is a simply connected 4-manifold supporting an effective T^2-action, then M has k (≥ 2)-fixed points, and*

$$M \approx \begin{cases} S^4, & \text{if } k = 2; \\ CP^2 \text{ or } -CP^2, & \text{if } k = 3; \\ S^2 \times S^2,\ CP^2\#CP^2,\ CP^2\#-CP^2, \text{ or } -CP^2\#-CP^2, & \text{if } k = 4; \\ \text{a connected sum of copies of these spaces}, & \text{if } k > 4. \end{cases}$$

THEOREM 2.6. [$\mathbf{F_2}$] *Let* SO(2) *act locally smoothly and effectively on the simply connected 4-manifold M, and suppose the orbit space M^* is not a counterexample to the 3-dimensional Poincaré conjecture. Then M is a connected sum of copies of S^4, CP^2, $-CP^2$, and $S^2 \times S^2$.*

Suppose a compact Lie group G acts on a compact connected manifold M so that the orbit space M/G is a closed interval [0, 1], and let $G(x)$ and $G(y)$ be the orbits corresponding to 0 and 1 respectively. Then $G(x)$ and $G(y)$ are singular orbits and all other orbits are principal orbits of type G/H. Moreover, we may assume $H \subset G_x$ and $H \subset G_y$. The following lemma was proved by Mostert [**Mo**].

LEMMA 2.7. [**Mo**] *If a Lie group G acts locally smoothly and effectively on a manifold M so that M/G is a closed interval, then G_x/H and G_y/H are spheres.*

3. The case of rank $G = 2$.

3A. Suppose rank $G = 2$ and rank $H = 2$. Then by Lemma 2.4, $\dim G$ is 10, 8, or 6. Inequality ($**$) implies $\dim G/H = 4$ and hence M is a homogeneous space.

(i) It follows from [**E**, p. 239] that if $\dim G = 10$, then M is S^4 or RP^4. Since M is simply connected, M is S^4. Hence [**Wo**, p. 282] gives rise to $G = \mathrm{SO}(5)$ and $H = \mathrm{SO}(4)$.

(ii) It is known [**Wa**] that if $n(n-1)/2 + 1 < \dim G < n(n+1)/2$, $n = \dim M$, then $n = 4$. Mann [**Ma**] proved that the effective action of $\mathrm{SU}(3)/Z(\mathrm{SU}(3))$ of dimension 8 on the complex projective plane $CP^2 = \mathrm{SU}(3)/U(2)$ is the only exceptional possibility for $n = 4$.

(iii) If $\dim G = 6$, then $\dim H$ should be 2. Since G is assumed to be connected and $G/H = M$ is assumed to be simply connected, the homotopy exact sequence of the fibre bundle implies that H is also connected, hence H is T^2. The Lie group G of dimension 6 and of rank 2 is either $\mathrm{SU}(2) \times \mathrm{SU}(2)$, or a factor group of this by a finite subgroup. Since $Z(\mathrm{SU}(2) \times \mathrm{SU}(2)) = \{(1,1), (-1,1), (1,-1), (-1,-1)\}$ is contained in a maximal torus (and hence in H), $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is not admissible. For similar reasons, $\mathrm{SO}(3) \times \mathrm{SU}(2)$, $\mathrm{SU}(2) \times \mathrm{SO}(3)$, and $\mathrm{SO}(4)$ are not admissible. Hence $(\mathrm{SU}(2) \times \mathrm{SU}(2))/\text{the center} = \mathrm{SO}(3) \times \mathrm{SO}(3)$ is the only admissible group. Hence M is $S^2 \times S^2$. $\square$

We recall some properties of $\mathrm{SO}(3)$ (see [**R**]).

(1) Every subgroup of $\mathrm{SO}(3)$ is conjugate to one of the following: $\mathrm{SO}(2)$, $N(\mathrm{SO}(2))$, the cyclic group Z_k of order k, the dihedral group D_n of order $2n$, the groups T, O, I of all rotational symmetries of the tetrahedron, octahedron, and icosahedron, respectively.

(2) If V is a finite subgroup of $\mathrm{SO}(3)$, then $\mathrm{SO}(3)/V$ is an orientable 3-manifold with $H_2(\mathrm{SO}(3)/V) = 0$. Using the double covering $\pi\colon \mathrm{SU}(2) \to \mathrm{SO}(3)$ we can calculate the first homology group of $\mathrm{SO}(3)/V$:

$$\begin{aligned} &H_1(\mathrm{SO}(3)/Z_k) = Z_{2k}, && H_1(\mathrm{SO}(3)/D_{2n}) = Z_2 + Z_2, \\ &H_1(\mathrm{SO}(3)/D_{2n+1}) = Z_4, && H_1(\mathrm{SO}(3)/T) = Z_3, \\ &H_1(\mathrm{SO}(3)/O) = Z_2, && H_1(\mathrm{SO}(3)/I) = 0. \end{aligned}$$

In the following $\tilde{K}$ denotes the preimage of $K \subset \mathrm{SO}(3)$ under the covering map.

3B. Suppose $\operatorname{rank} G = 2$ and $\operatorname{rank} H = 1$. Then by Lemma 2.4, $\dim G$ is either 6 or 4 and $\dim G/H$ is 3.

(I) If $\dim G = 4$, then G is $\mathrm{SU}(2) \times T^1$ or a factor group of this by a finite subgroup. Since $\dim G/H$ is 3, $\dim H$ is 1. Since any 1-dimensional subgroup of $\mathrm{SU}(2) \times T^1$ contains a non-trivial element of $Z(\mathrm{SU}(2) \times T^1) = \{1, -1\} \times T^1$, it is not admissible. The remaining possibilities are $\mathrm{SO}(3) \times T^1$ and $(\mathrm{SU}(2) \times T^1)/D$, where $D = \{(1,1), (-1,-1)\}$.

(Ia) Suppose G is $(\mathrm{SU}(2) \times T^1)/D$. Then the identity component H_0 of H cannot be included in $(\mathrm{SU}(2) \times 1)/D$ since $(\widetilde{\mathrm{SO}}(2) \times 1)/D$ contains $(-1, 1)/D$ $(\in Z(G))$. Nor can H_0 be $(1 \times T^1)/D$ since $(1 \times T^1)/D$ is a subgroup of $Z(G)$. Hence by using an argument similar to that of 8.1 of [**R**], we can show that H is included in a maximal torus of G.

Since $\dim G/H$ is 3, the orbit space M^* is a closed interval $[0, 1]$. That is, the orbit space M^* is as shown below.

$$G_x \overset{H}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet} G_y$$

By Lemma 2.7, G_x/H and G_y/H are spheres. But $((\widetilde{N\mathrm{SO}(2)} \times T^1)/D)/H$ is not a sphere. Hence G_x (and also G_y) must be $(\widetilde{\mathrm{SO}}(2) \times T^1)/D$ or G.

(i) If G_x and G_y are maximal tori, then the number of fixed points of the action restricted to G_x is either 2 or 4 since the order of $N(G_x)/G_x$ is $\chi(G/G_x) = 2$. Now it follows from Theorem 2.5 that M is S^4 or an S^2-bundle over S^2 according as the number of fixed points is 2 or 4. Let $A = p^{-1}([0, \frac{1}{2}])$ and $B = p^{-1}([0, \frac{1}{2}])$, where $p: M \to M^* = [0, 1]$ is the orbit map. From the Mayer-Vietoris sequence for (M, A, B), we have

$$0 \to H_3(M) \to H_2(G/H) \to Z \oplus Z \to H_2(M) \to H_1(G/H) \to 0.$$

Now we have

$$\begin{aligned}(G/H)/\{[(\widetilde{\mathrm{SO}}(2) \times T^1)/D]/H\} &\\ \approx [(\mathrm{SU}(2) \times T^1)/D]/[(\widetilde{\mathrm{SO}}(2) \times T^1)/D] &\\ \approx (\mathrm{SU}(2) \times T^1)/(\widetilde{\mathrm{SO}}(2) \times T^1) \approx S^2 &\end{aligned}$$

(see [**B**, p. 87]). Since $[(\widetilde{\mathrm{SO}}(2) \times T^1)/D]/H$ is a topological group, the fundamental group of this is abelian. From a homotopy exact sequence of the fibre bundle $[(\widetilde{\mathrm{SO}}(2) \times T^1)/D]/H \to G/H \to S^2$, we can see that $\pi_1(G/H)$ is abelian, hence $H_1(G/H) = \pi_1(G/H)$.

From the homotopy sequence of the fibre bundle $H \to G \to G/H$, we have

$$0 \to \pi_2(G/H) \to Z \to Z \to \pi_1(G/H) \to \pi_0(H) \to 0.$$

If M is S^4, then from the homology sequence we have $\pi_2(G/H) = Z \oplus Z$ which contradicts the homotopy sequence. Hence the number of fixed points must be 4. Therefore we have $G_x = G_y$ which implies $\pi_1(G/H) = 1$, and hence H is connected. Thus H is S^1 and M is either $S^2 \times S^2$ or $CP^2\#-CP^2$.

(ii) If G_x and G_y are G (i.e. x and y are fixed points), then the homotopy exact sequence of a fibre bundle $H \to G \to G/H = G_x/H \approx S^3$ yields $H \approx S^1$. Furthermore, the number of fixed points of the action restricted to $(\widetilde{\mathrm{SO}}(2) \times T^1)/D$ is two. Hence, by Theorem 2.5, we have $M = S^4$ (alternatively,

$$M \approx p^{-1}([0, \tfrac{1}{2}]) \cup p^{-1}([\tfrac{1}{2}, 1]) \approx D^4 \cup D^4 \approx S^4).$$

(iii) If G_x is G and G_y is a maximal torus, then by an argument similar to that used in (ii), H is connected and hence H is S^1. The number of fixed points of the action restricted to G_y is 3 and hence it follows from Theorem 2.5 that M is CP^2.

(Ib) Suppose G is $\mathrm{SO}(3) \times T^1$. Then by 8.1 of [**R**], H is contained in a maximal torus or conjugate to either $\mathrm{SO}(2) \times 1$ or $N(\mathrm{SO}(2)) \times 1$. But

$(\mathrm{SO}(3) \times 1)/(N(\mathrm{SO}(2)) \times 1) = RP^2 \times S^1$ is not orientable and hence by [**B**, p. 188], H cannot be $N(\mathrm{SO}(2)) \times 1$. (1) If H is contained in a maximal torus, then neither G_x nor G_y can be G since $(\mathrm{SO}(3) \times T^1)/H$ is not a sphere. Hence by an argument similar to that of (Ia), H is S^1 and M is $S^2 \times S^2$ or $CP^2\#{-CP^2}$. (2) If H is $\mathrm{SO}(2) \times 1$, then by Lemma 2.7, there are three possibilities:

(i) $G_x \approx \mathrm{SO}(2) \times T^1 \approx G_y$, which implies $M = S^2 \times S^2$.

(ii) $G_x \approx \mathrm{SO}(3)$ and $G_y \approx \mathrm{SO}(2) \times T^1$, which implies $M = [(S^2 \times D^2) \cup (D^3 \times S^1)] = S^4$.

(iii) $G_x \approx \mathrm{SO}(3) \approx G_y$, which implies $M = S^3 \times S^1$, not admissible.

(II) If $\dim G = 6$, then $\dim H$ should be 3. Since the rank of G is 2, G is $\mathrm{SU}(2) \times \mathrm{SU}(2)$, $\mathrm{SO}(3) \times \mathrm{SU}(2)$, $\mathrm{SU}(2) \times \mathrm{SO}(3)$, or $(\mathrm{SU}(2) \times \mathrm{SU}(2))/D$, where $D = \{(1,1), (-1,-1)\}$.

Assertion. Suppose H_0 is the identity component of a 3-dimensional subgroup H of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and let p_i be the projection onto the ith factor, for $i = 1, 2$. Then $p_i | H_0$, the restriction of p_i to H_0, is either a trivial map or an isomorphism.

To prove this Assertion, first of all we have to show that $p_i | H_0$ is either trivial or surjective. Suppose $p_i | H_0$ is neither surjective nor trivial. Then $p_i(H_0)$ should be either $\mathrm{SO}(2)$ or $N(\mathrm{SO}(2))$, and hence the kernel of $p_i | H_0$ is a two-dimensional normal subgroup of H_0. This is impossible. Hence $p_1 | H_0$ or $p_2 | H_0$ must be surjective. Suppose $p_1 | H_0$ is surjective and let K be the kernel of $p_1 | H_0$. Then $H_0/K \approx \mathrm{SU}(2)$. Since $\mathrm{SO}(3)$ is simple, H_0 cannot be $\mathrm{SO}(3)$. If H_0 is $\mathrm{SU}(2)$, then $K = \pi_1(H_0/K) = \pi_1(\mathrm{SU}(2)) = 1$. Thus $p_1 | H_0$ is an isomorphism and $H_0 \approx \mathrm{SU}(2)$. □

(IIa) If either $p_1 | H_0$ or $p_2 | H_0$ is trivial, then $H \approx \mathrm{SU}(2) \times V$, for a finite subgroup V, which contains a normal subgroup of $\mathrm{SU}(2) \times \mathrm{SU}(2)$. Since H cannot contain a normal subgroup of $\mathrm{SU}(2) \times \mathrm{SU}(2)$, $p_1 | H_0$ and $p_2 | H_0$ must be isomorphisms. Therefore, H contains the two elements central subgroup D. Thus $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is not admissible.

(IIb) If G is $\mathrm{SO}(4)$ $(\approx (\mathrm{SU}(2) \times \mathrm{SU}(2))/D)$, then a principal isotropy group is H/D, where H is a three-dimensional subgroup of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ such that $p_1(H) = \mathrm{SU}(2) = p_2(H)$.

If x^* and y^* are the endpoints of a closed interval $M/\mathrm{SO}(4)$, then x and y should be fixed points so that G_x (and also G_y) could contain H as a conjecture subgroup. In fact, suppose K is a subgroup of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ such that $H \subset K$ and $\dim K \geq 4$. Then $\dim K$ is either 4 or 6 since rank G is 2. If $\dim K$ is 4, then the kernel of P_1 is an 1-dimensional subgroup of K. So K contains $1 \times \widetilde{\mathrm{SO}}(2)$. For any $g \in \mathrm{SU}(2)$, there exists $h \in \mathrm{SU}(2)$

such that $(h, g) \in K$. Moreover, $(h, g)^{-1}(1 \times \widetilde{SO}(2))(h, g) = 1 \times g^{-1}\widetilde{SO}(2)g \subset K$. By the maximal torus theorem, we have $1 \times SU(2) \subset K$. Similarly, $SU(2) \times 1 \subset K$. Hence $K = SU(2) \times SU(2)$. Since $G/(H/D)$ must be S^3 (by Theorem 2.7), by a homotopy exact sequence of $H/D \to G \to S^3$, H/D is connected. Since $H_0/D \approx SU(2)/D \approx SO(3)$, H/D is SO(3) and hence M is S^4.

(IIc) If G is $SO(3) \times SO(3)$, then by an argument similar to the Assertion, we can show that x and y should be fixed points so that G_x (and G_y) can contain a non-normal 3-dimensional subgroup H as a conjugate subgroup. But $(SO(3) \times SO(3))/H$ cannot be a sphere. Hence $SO(3) \times SO(3)$ is not admissible.

(IId) If G is $SU(2) \times SO(3)$, then by an argument similar to that used in the proof of the Assertion, $P_1 | H_0$ is either a trivial map or an isomorphism. If $P_1 | H_0$ is trivial, then H is $V \times SO(3)$ for a finite subgroup V of SU(2), which contains a normal subgroup $1 \times SO(3)$. If $P_1 | H_0$ is an isomorphism, then H contains $\{(-1, 1), (1, 1)\}$ $(\subset Z(G))$. Hence $SU(2) \times SO(3)$ is not admissible. $\square$

As a summary we have the table:

TABLE I

dim G	rank H	G	H	M
10	2	SO(5)	SO(4)	S^4
8	2	$SU(3)/Z(G)$	$U(2)/Z(G)$	CP^2
6	2	$SO(3) \times SO(3)$	T^2	$S^2 \times S^2$
6	1	SO(4)	SO(3)	S^4
4	1	$SO(3) \times T^1$	S^1	$S^2 \times S^2$, S^4, $CP^2\#-CP^2$
4	1	$SU(2) \times T^1/D$	S^1	S^4, CP^2, $S^2 \times S^2$, $CP^2\#-CP^2$
2	0	T^2	e	Theorem 2.5

Here S^1 is a circle subgroup and D is the two element central subgroup $\{(1, 1), (-1, -1)\}$.

4. The case of rank $G = 1$. If a compact connected Lie group G is of rank 1, then G is T^1, SO(3), or SU(2), and the rank of H must be either 1 or 0.

4A. Suppose rank $H = 1$. Then G is either SO(3) or SU(2).

(i) If $G = SO(3)$, then H is either SO(2) or $N(SO(2))$. Since $SO(3)/N(SO(2)) = RP^2$ is not orientable, H should be SO(2). Since $SO(3)/SO(2)$ is S^2, the orbit space M^* is either S^2 or D^2. If M^* is S^2, then M is an S^2-bundle over S^2. If M^* is D^2, then ∂D^2 corresponds to the fixed points and int D^2 corresponds to the principal orbits. Hence M is S^4.

(ii) If $G = \mathrm{SU}(2)$, then by an argument similar to that used in (i), H is $\widetilde{\mathrm{SO}}(2)$, and M is either S^4 or an S^2-bundle over S^2.

4B. Suppose rank $H = 0$. Then G is T^1, SO(3), or SU(2).

(i) If G is T^1, then H must be trivial and M was described in Theorem 2.6.

(ii) If $G = \mathrm{SO}(3)$ and x^* and y^* are the endpoints of M^*, then G_x and G_y are conjugate to SO(2), $N(\mathrm{SO}(2))$, or SO(3). By Lemma 2.7, none of x and y are fixed points and G_x should be conjugate to G_y.

(iia) If G_x and G_y are conjugate to $N(\mathrm{SO}(2))$, then H is a dihedral group D_{2n} (since G_x/H and G_y/H must be spheres). Richardson [**R**] showed that S^4 admits an action of SO(3) such that $S^4/\mathrm{SO}(3) = [x^*, y^*]$, a closed interval, $H = D_{2n}$, $(\mathrm{SO}(3))(x) = RP^2 = (\mathrm{SO}(3))(y)$. Since the orbit maps $M \to M/G$ and $S^4 \to S^4/\mathrm{SO}(3)$ have cross-sections ([**Mo**], [**R**]), M is equivariantly homeomorphic to S^4.

(iib) If G_x and G_y are conjugate to SO(2), then H should be a cyclic group Z_k and M is the space $[0, 1] \times \mathrm{SO}(3)/Z_k$ with $0 \times \mathrm{SO}(3)/Z_k$ collapsed to $\mathrm{SO}(3)/G_x$ ($\approx S^2$) and $1 \times \mathrm{SO}(3)/Z_k$ collapsed to $\mathrm{SO}(3)/G_y$ ($\approx S^2$). Let p be the orbit map. Let $A = p^{-1}([0, \frac{1}{2}])$ and $B = p^{-1}([\frac{1}{2}, 1])$. From the Mayer-Vietoris sequence for (M, A, B), we have $H_2(M; Q) = Q \oplus Q$ and hence $\chi(M) = 4$. Now we consider the action restricted to G_x ($\approx T^1$). The set of fixed points under the restricted action is contained in $G(x) \cup G(y)$. Since $N(G_x)/G_x$ is Z_2, there are only two fixed points for G_x on $G(x)$, and hence there are at most four fixed points under the restricted action. Let $F(G_x, M)$ denote the set of fixed points. Then it is well known that $\chi(F(G_x, M)) = \chi(M) = 4$. Therefore there are four fixed points for G_x on M, which implies $G_x = G_y$. Since $H^3(M; Z) = H_1(M; Z) = 0$, $H_2(M; Z)$ is torsion free and hence $H_2(M; Z)$ is $Z \oplus Z$. The Mayer-Vietoris sequence gives rise to

$$0 \to Z \oplus Z \xrightarrow{i_* \oplus j_*} Z \oplus Z \to H_1(\mathrm{SO}(3)/Z_k; Z) \to 0.$$

Here i_* and j_* are induced by inclusions $i\colon A \to M$ and $j\colon B \to M$ respectively. Since $G_x = G_y$, i and j are virtually the same maps. Hence $(Z \oplus Z)/\mathrm{im}(i_* \oplus j_*) = Z_n \oplus Z_n$ for an integer n, which contradicts $H_1(\mathrm{SO}(3)/Z_k; Z) = Z_{2k}$.

(iii) If G is SU(2) and π is the double covering map, then the only subgroups of SU(2) which do not contain the kernel of π are cyclic subgroups of odd order. Hence every subgroup K of SU(2) contains $Z(\mathrm{SU}(2))$ unless K is a cyclic group of odd order. Since the action was assumed to be effective, H is either Z_{2k+1} or e. By an argument similar to that used in (ii) of 4B, we can show that H cannot be Z_{2k+1}. If H is the identity, then by Lemma 2.7, there are three possibilities:

(a) x and y are fixed points, which implies $M \approx S^4$.

(b) G_x is conjugate to $\widetilde{\mathrm{SO}}(2)$ and y is a fixed point, which implies $M \approx CP^2 \# S^4$ (Recall: $\mathrm{SU}(2) \to \mathrm{SU}(2)/\widetilde{\mathrm{SO}}(2)$ is the Hopf bundle).

(c) G_x and G_y are conjugate to $\widetilde{\mathrm{SO}}(2)$, which implies $M \approx CP^2 \# -CP^2$. $\square$

We summarize these in the following table:

TABLE II

dim G	rank H	G	H	M^*	M
3	1	SO(3)	SO(2)	D^2 S^2	S^4 $S^2 \times S^2$, $CP^2 \# -CP^2$
3	1	SU(2)	$\widetilde{\mathrm{SO}}(2)$	D^2 S^2	S^4 $S^2 \times S^2$, $CP^2 \# -CP^2$
3	0	SO(3)	D_{2n}	D^1	S^4
3	0	SU(2)	e	D^1	S^4, CP^2, $CP^2 \# -CP^2$
1	0	T^1	e		Theorem 2.6

5. Conclusion.

Suppose a compact connect Lie group G acts on a simply connected 4-manifold M. Then it was shown in §2 that the rank of G is either 1 or 2. Let H denote a representative subgroup of the conjugacy class of principal isotropy groups. Then G, M, and H are completely determined in §§3 and 4 in the cases of rank $G = 2$ and rank $G = 1$, respectively. Thus we have proved the following.

THEOREM 5.1. *If a Lie group G, a subgroup H, and a manifold M are those denoted above, then G, H, and M must be one of the cases represented in Table* I (§3) *and Table* II (§4).

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MICHIGAN STATE UNIVERSITY
EAST LANSING, MI 48824