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NONOSCILLATORY SOLUTIONS OF $(rx^n)^n \pm f(t, x)x = 0$

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We study the existence and growth rates of positive, monotonic, bounded solutions of the equations

$$(1^{\pm}) \qquad \left(r(t)x^{(n)}\right)^{(n)} \pm f(t, x)x = 0, \qquad f(t, x) > 0.$$

First we prove our results for the linear equation with $f(t, x) = p(t)$, then by a fixed point method we extend these to the nonlinear equation. We also obtain some oscillation results for $(1^{\pm})$.

1. Introduction. Fixed point theorems have proved very effective in solving problems posed for nonlinear equations of the form

$$(1.1) \qquad x' = A(t, x)x.$$

The reason is that if one considers the mapping $\varphi \to T\varphi$, where φ lies in a suitable set in a function space and $T\varphi$ is the solution of the linear equation

$$(1.2) \qquad x' = A(t, \varphi(t))x$$

satisfying a suitable property P, then a fixed point of T will be a solution of (1.1) satisfying P. Thus results for linear equations may be extended to nonlinear equations.

For $A(t, x)$ a matrix-valued function, this method was used by Conti [4] and Opial [14] to solve boundary value problems associated with (1.1). Corduneanu [5] used it to describe the growth behavior of solutions for large t. Kartsatos [10] also studied the growth of solutions and he used fixed point theorems from set-valued mappings — thus eliminating the need that property P describe unique solutions to (1.2). In some recent papers, [2] and [15], the method has been applied to the growth of solutions of nth order nonlinear equations of the type

$$(1.3) \qquad x^{(n)} + a_{n-1}(t, x)x^{(n-1)} + \cdots + a_0(t, x)x = 0.$$

These results become especially interesting when one realizes: (i) any equation $x' = f(t, x)$ with $f(t, 0) \equiv 0$ and f continuously differentiable with respect to x can be put into the form (1.1) (cf. [2], [15]); and (ii) the theorems apply to families of nonlinear equations, not just single equations.

In this paper we apply the fixed point method to oscillation-nonoscillation theory. In §2 we consider the equations

$$(1.4)\qquad (r(t)x^{(n)})^{(n)} \pm f(t,x)x = 0; \qquad f(t,x), r(t) > 0$$

and we obtain necessary conditions and sufficient conditions for the existence of certain monotonic, nonoscillatory solutions (the so-called minimal solutions).

The theorem on minimal solutions relates to work of Hartman and Wintner **[8]**; in §3 we extend their main theorem to nonlinear vector equations.

In §4 we obtain several results about oscillatory solutions. In particular, we derive sufficient conditions for the existence of an oscillatory solution of

$$(1.5)\qquad x^{(4)} = f(t,x)x$$

and we show that if

$$(1.6)\qquad x^{(2n)} = f(t,x)x$$

has a bounded oscillatory solution $x(t)$, with $0 < \overline{\lim}_{t\to\infty} x^{(2n-1)}(t) < \infty$, then every eventually positive solution of (1.4) is either strongly increasing or strongly decreasing. Finally we generalize theorems of Hille and Leighton-Nehari to (1.6).

2. We begin by studying the differential equations

$$(2.1)\qquad (r(t)x^{(n)})^{(n)} + p(t)x = 0$$

and

$$(2.2)\qquad (r(t)x^{(n)})^{(n)} + f(t,x)x = 0$$

where p and r are positive and continuous on $[\tau,\infty)$, $\int_\tau^\infty dt/r(t) = \infty$, and f is positive and continuous on $[\tau,\infty)\times(-\infty,\infty)$.

For now we are interested in nonoscillatory solutions. In (2.2), with $r(t) = 1$, one sees, by means of Rolle's theorem, that if $x(t)$ is a solution satisfying $x(t) > 0$ on $[\rho,\infty)$, then there exists $\sigma \geq \rho$ such that $x^{(k)}(t) \neq 0$ on $[\sigma,\infty)$, $1 \leq k \leq 2n$. If, further, $x(t)$ is bounded on $[\rho,\infty)$, then we may choose σ so that $x^{(k)}(t)x^{(k+1)}(t) < 0$ on $[\sigma,\infty)$, $1 \leq k \leq 2n-1$. (This statement requires the integral condition on r.)

Kusano and Naito **[12]** studied (2.2) with $n = 2$ and Kreith **[11]** studied (2.2) with $r(t) = 1$. Among their results were necessary and sufficient conditions for (2.2) to have a solution satisfying

$$(2.3)\quad \begin{gathered} x(t) > 0, \qquad (-1)^{k+1} x^{(k)}(t) > 0 \quad (1 \le k \le n-1), \\ (-1)^{n+k+1}\left(r(t)x^{(n)}(t)\right)^{(k)} > 0 \qquad (0 \le k \le n-1) \\ \text{all on } [\sigma, \infty) \text{ (for some } \sigma \ge \tau) \quad \text{and} \quad \lim_{t\to\infty} x(t) = c > 0. \end{gathered}$$

We consider this problem and we begin with a similar result for the linear equation:

THEOREM 2.1. *Let $c > 0$ be given. Then* (2.1) *has a solution x_c satisfying* (2.3) *if and only if*

$$(2.4)\qquad \int_\tau^\infty R(\tau, s)p(s)\,ds > -\infty$$

where

$$(2.5)\quad R(t, s) = -\int_t^s \frac{(s-u)^{n-1}(u-t)^{n-1}}{((n-1)!)^2 r(u)}\,du, \qquad \tau \le t \le s < \infty.$$

Further, x_c satisfies

$$(2.6)\quad \begin{cases} \lim_{b\to\infty} b^k x^{(k)}(b) = 0 & (1 \le k \le n-1), \\ \lim_{b\to\infty} \left(r(b)x^{(n)}(b)\right)^{(k)} R^{[n-1-k]}(t, b) = 0 & (0 \le k \le n-1), \end{cases}$$

([ ] *denotes d/db*). *And, for a given c, x_c is unique.*

Proof. (i) A solution of (2.1) satisfies

$$(2.7)\quad \begin{aligned} x(t) - x(b) = {} & \sum_1^{n-1}\left[\frac{(-1)^k (b-t)^k x^{(k)}(b)}{k!}\right] \\ & + \sum_0^{n-1}\left[(-1)^{n+k+1}\left(r(b)x^{(n)}(b)\right)^{(k)} R^{[n-1-k]}(t, b)\right] \\ & + \int_t^b R(t, u)p(u)x(u)\,du. \end{aligned}$$

(If $n = 1$, the first term on the right-hand side is omitted.)

Formula (2.7) follows from Taylor's theorem:

$$\text{(a)}\quad \begin{aligned} x(t) = {} & \sum_0^{n-1} \frac{(-1)^k (b-t)^k x^{(k)}(b)}{k!} \\ & + (-1)^n \int_t^b \frac{(s-t)^{n-1} r(s)x^{(n)}(s)}{(n-1)!\,r(s)}\,ds; \end{aligned}$$

$$\text{(b)}\quad r(s)x^{(n)}(s) = \sum_0^{n-1} \frac{(-1)^k (b-s)^k \left(r(b)x^{(n)}(b)\right)^{(k)}}{k!} + (-1)^{n-1} \int_s^b \frac{(s-u)^{n-1} p(u)x(u)}{(n-1)!}\, du.$$

If we substitute (b) into (a) and reverse the order of integration on the double integral, the result is (2.7).

(ii) Necessity. Suppose (2.1) has a solution satisfying (2.3). In (2.7) hold t fixed and let $b \to \infty$. The limit of the left-hand side exists. On the right-hand side each term is negative, the limit of the whole side exists, and hence the limit of each term exists, i.e.

$$\lim_{b\to\infty} (b-t)^k x^{(k)}(b) = c_k(t),$$

$$\lim_{b\to\infty} \left(r(b)x^{(n)}(b)\right)^{(k)} R^{[n-1-k]}(t,b) = d_k(t),$$

and

$$\int_t^\infty R(t,u)p(u)x(u)\, du > -\infty$$

(hence (2.4) holds). We shall show that the c_k's and d_k's are constant functions. Then (2.7) can be changed to

$$x(t) - c_0 - \int_t^\infty R(t,u)p(u)x(u)\, du = \sum_1^{n-1} \frac{(-1)^k c_k}{k!} + \sum_0^{n-1} (-1)^{n+k+1} d_k$$

and since the left-hand side goes to zero as $t \to \infty$ and each term on the right-hand side has the same sign, each c_k and each d_k equals zero. Hence (2.6) will hold.

$$\text{(a)}\qquad \lim_{b\to\infty} (b-t)^k x^{(k)}(b) = c_k(t)$$

and

$$(b-t)^{k-1} x^{(k)}(b) = \frac{(b-t)^k x^{(k)}(b)}{b-t} \le \frac{(b-\tau)^k x^{(k)}(b)}{b-\tau}, \qquad \tau \le t < b < \infty,$$

implies $\lim_{b\to\infty} (b-t)^{k-1} x^{(k)}(b) = 0$, uniformly in t. Hence we may differentiate within the limit and $dc_k(t)/dt = 0$.

$$\text{(b)}\qquad \lim_{b\to\infty} \left(r(b)x^{(n)}(b)\right)^{(k)} R^{[n-1-k]}(t,b) = d_k(t)$$

and

$$-R^{[n-1-k]}(t, b) < -R^{[n-1-k]}(\tau, b), \qquad \tau \le t < b < \infty.$$

By the Second Mean Value Theorem of Integral Calculus,

$$\int_\tau^b \frac{(b-u)^k (u-\tau)^{n-1}}{r(u)} du = (\xi_b - \tau) \int_\tau^b \frac{(b-u)^k (u-\tau)^{n-2}}{r(u)} du, \qquad \tau < \xi_b < b,$$

and $\xi_b \to \infty$ as $b \to \infty$. So

$$\lim_{b\to\infty} \left(r(b)x^{(n)}(b)\right)^{(k)} \int_t^b \frac{(b-u)^k (u-t)^{n-2}}{k!\,(n-1)!r(u)} du = 0,$$

uniformly in t. Hence $dd_k(t)/dt = 0$.

(iii) Sufficiency. Assume that (2.4) holds and let $\sigma \ge \tau$ be such that $\int_\sigma^\infty R(\sigma, s)p(s)\, ds > -1$. Let X be the space of functions which are bounded and continuous on $[\sigma, \infty)$ and for $x \in X$ let $\|x\| = \sup\{|x(t)| : \sigma \le t < \infty\}$. Define T: $X \to X$ by

$$T[x](t) = c + \int_t^\infty R(t, s)p(s)x(s)\, ds. \tag{2.8}$$

Then T is a strict contraction and its unique fixed point is seen to satisfy (2.1), (2.3).

(iv) Uniqueness. A solution of (2.1), (2.3) was shown, in (ii), to satisfy (2.6). Combining (2.6) and (2.7) we see this solution must also be a fixed point of (2.8) — and hence unique.

THEOREM 2.2. *Let* (2.2) *be given and define* $f_a(t) = \sup\{f(t, x);\ 0 \le x \le a\}$. (i) *If*

$$\int_\tau^\infty R(\tau, s)f_a(s)\, ds > -\infty \quad \textit{for some } a > 0, \tag{2.9}$$

then (2.2), (2.3) (*with* $c = a$), (2.6) *has a solution*. (ii) *If* (2.2), (2.3) *has a solution, this solution satisfies* (2.6). (iii) *If* (2.2), (2.3) *has a solution and if* $x < y$ *implies* $f(\cdot, x) < f(\cdot, y)$, *then this solution is unique and* (2.9) *holds with* $a = c$ (c *given in* (2.3)). (iv) *If* (2.2), (2.3) *has a solution and if* $x < y$ *implies* $f(\cdot, x) > f(\cdot, y)$, *then this solution is unique and* (2.9) *holds for any* $a \in (0, c)$.

Proof. (i) Let X be the Fréchet space of continuous functions on $[\tau, \infty)$ with the compact-open topology (i.e. $\|x_n - x\| \to 0$ means

$\sup|x_n(t) - x(t)| \to 0$ uniformly on each compact $I \subset [\tau, \infty)$). Let $S = \{x \in X\colon 0 \le x(t) \le c \text{ on } [\tau, \infty)\}$; S is closed, convex, and bounded. Define $T\colon S \to S$ by Tu is the solution of the linear equation

$$(r(t)x^{(n)})^{(n)} + f(t, u(t))x = 0, \qquad u \in S, \tag{2.10}$$

which satisfies (2.3), (2.6).

By Theorem 2.1, T is well-defined and, if $x = Tu$,

$$x(t) = c + \int_t^\infty R(t, s)f(s, u(s))x(s)\, ds.$$

Now $u \in S$ and (2.9) holds so from the line above we have $|x(t)|$, $|x'(t)| \le M$ ($M = M(c, I)$) on each compact $I \subset [\tau, \infty)$ and hence, by Ascoli's theorem, TS is relatively compact.

Now T is continuous: Let $\{u_n\}$ be a sequence in S converging to u_0 and let $\{x_n\}$ be the corresponding solutions of (2.10); let $I \subset [\tau, \infty)$ be compact. By compactness, some subsequence of $\{x_n\}$ converges to x_0. On I, a solution of (2.10) is a continuous function of u. Hence the full sequence $\{x_n\}$ converges to x_0.

By Schauder's theorem T has a fixed point which satisfies (2.2), (2.3).

(ii) If u is a solution of (2.2), (2.3), then u is a solution to the linear equation (2.10) and (2.3) and hence, by Theorem 2.1, satisfies (2.6).

(iii) Suppose that x_1 and x_2 are two solutions of (2.2), (2.3) (with the same c) and suppose that $x_2 > x_1$ on $[t, \infty)$. Using (2.7) and (2.6)

$$x_i(t) = c + \int_t^\infty R(t, s)f(s, x_i(s))x_i(s)\, ds.$$

Also $f(\cdot, x_2)x_2 > f(\cdot, x_1)x_1$ and $R(t, s) < 0$ on $t \le s < \infty$. Then

$$\begin{aligned} 0 &< x_2(t) - x_1(t) \\ &= \int_t^\infty R(t, s)\big[f(s, x_2(s))x_2(s) - f(s, x_1(s))x_1(s)\big]\, ds < 0. \end{aligned}$$

Likewise $x_2(a) = x_1(a)$, $x_2(b) = x_1(b)$, $x_2 > x_1$ on (a, b) is impossible:

$$\begin{aligned} 0 &= [x_2(b) - x_1(b)] - [x_2(a) - x_1(a)] \\ &= \int_a^b \text{(same integrand)} < 0. \end{aligned}$$

So $x_1 = x_2$. That (2.9) holds follows from the monotonicity on f.

(iv) The proof is similar to that of (iii).

REMARKS. (1) In (i) we cannot apply the Contraction Mapping Theorem directly to (2.2) without further assumptions on f. In (iii) (and

iv) we are very close to a Lipschitz condition ($x > y \Rightarrow f(\cdot, x)x - f(\cdot, y)y \geq f(t, y)(x - y)$) and this suggests that with monotonicity a contraction-type proof could be constructed.

(2) Theorem 2.2 includes the corresponding results of Kusano and Naito and Kreith and adds the growth condition contained in (2.6) and the uniqueness.

We next consider

$$(2.1) \qquad \left(r(t)x^{(n)}\right)^{(n)} - p(t)x = 0$$

and

$$(2.2)^- \qquad \left(r(t)x^{(n)}\right)^{(n)} - f(t, x)x = 0$$

where r, p and f are as in (2.1), (2.2).

A bounded, positive solution for either of these equations must satisfy

$$(2.11) \qquad \begin{array}{ll} x(t) > 0, \quad (-1)^k x^{(k)}(t) > 0 \quad (1 \leq k \leq n-1), & \\ (-1)^{n+k}\left(r(t)x^{(n)}\right)^{(k)} > 0 \quad (0 \leq k \leq n-1), & \text{all on } [\tau, \infty), \\ \lim_{t\to\infty} x(t) = c \geq 0. & \end{array}$$

For $(2.1)^-$, that such a solution exists is a special case of a theorem of Hartman and Wintner [**8**], its uniqueness was shown by Etgen and Taylor [7]. For $(2.2)^-$ with $n = 2$ and monotonicity conditions on f, necessary and sufficient conditions for the existence of a solution satisfying (2.11) with $c > 0$ were given by Wong [**16**] and existence for arbitrary n, with $r(t) = 1$, follows from a theorem of Chow, Dunninger and Schuur [3]. We now add the following two theorems:

THEOREM 2.3. (i) *Let x_c be a solution of* $(2.1)^-$ *satisfying* (2.11) (*and x_c is known to exist*). *Then x_c satisfies* (2.6). (ii) *For a given $c > 0$, x_c will exist if and only if* (2.4) *holds. This x_c is unique.* (iii) *For $c = 0$, x_c will exist if and only if $\int_\tau^\infty R(t, s)p(s)\, ds = -\infty$. In this case $\int_\tau^\infty R(t, s)p(s)x_c(s)\, ds > -\infty$.* (iv) *If* $(2.1)^-$ *has no oscillatory solutions, then the x_c of* (iii) *is unique up to scalar multiplication.*

Proof. The proof of (i) and (ii) uses (2.7) and is similar to the proof of Theorem 2.1.

To prove (iii) we note that x_c exists, satisfies (2.6) and (2.7), and hence satisfies

$$x_c(t) = c - \int_t^\infty R(t, s)p(s)x_c(s)\, ds.$$

Now x_c is decreasing. If $c = 0$ and if (2.4) holds, then

$$x_c(t) < x_c(t)\left[-\int_t^{\infty} R(t, s)p(s)\,ds\right] < x_c(t) \quad \text{for large } t$$

and this is impossible.

To prove (iv) assume that $x(t)$ and $y(t)$ are two solutions of $(2.1)^-$ satisfying (2.11) with $c = 0$. Let $k = -[x(\tau)/y(\tau)]$ and $z(t) = x(t) - ky(t)$ (so $z(\tau) = 0$). Now $z(t) = -\int_t^{\infty} R(t, s)p(s)z(s)\,ds$, so $z(\sigma) = 0$ for some $\sigma \geq \tau$, $z(t) \neq 0$ on (σ, ∞) is impossible. (Part (iv) was proved by Etgen and Taylor [7].)

3. **Matrix equations.** The Hartman-Wintner result, mentioned at the end of §2, is a special case of a theorem for matrix equations:

THEOREM 3.1 (cf. [8]). *Let $A(t)$ be an n by n matrix of continuous functions satisfying $A(t) \geq 0$, $\tau \leq t < \infty$. Then the equation*

$$x' = -A(t)x \qquad (x \in R^n) \tag{3.1}$$

has a nontrivial solution $x_0(t)$ satisfying

$$x_0(t) \geq 0, \qquad -x_0'(t) \geq 0, \tag{3.2}$$

for $\tau \leq t < \infty$. (For a vector or matrix, $\geq$ means the inequality holds componentwise.)

This suggests two possibilities: applying the fixed point method to the nonlinear matrix equation

$$x' = -F(t, x)x \qquad (x \in R^n), \tag{3.3}$$

or extending Theorems 2.2 and 2.3 to the equations

$$x'' - A(t)x = 0, \qquad A(t) \geq 0. \tag{3.4}$$

$$x'' - F(t, x)x = 0. \tag{3.5}$$

Here F: $[\tau, \infty) \times R^n \to L^n$ (the linear functions from R^n into R^n) is continuous and satisfies

$$x \geq 0 \Rightarrow F(t, x) \geq 0 \quad \text{for each } t \in [\tau, \infty). \tag{3.6}$$

THEOREM 3.2. *Equation* (3.3) *has a nontrivial solution $x_0(t)$ satisfying* (3.2).

Proof. Let X denote the Fréchet space of continuous functions $[\tau, \infty) \to R^n$, topologized by the compact open topology, and let

$$S = \left\{x \in X \colon x(t) \geq 0 \text{ for } t \geq \tau,\ \sum_1^n |x_i(0)| = 1\right\}.$$

For $u \in S$ we have the linear equation

$$x' = -F(t, u(t))x, \qquad F(t, u(t)) \geq 0 \text{ for } t \geq 0. \tag{3.6}$$

Define

$$Tu = \{x \in S \colon x \text{ is a solution of (3.6) and } x(t) \leq x(s) \text{ for } t \geq s\}.$$

We do not, in this case, know that the monotonically decreasing solutions of (3.6) are unique and hence Tu is not necessarily single-valued. In place of the Schauder theorem we must use the corresponding result for set-valued mappings: if S is a closed, convex, nonempty subset of a Fréchet space X and if T satisfies: (i) for each $u \in S$, Tu is a nonempty, compact, convex subset of S; (ii) T is a closed mapping; and (iii) TS is contained in a compact subset of S; then there is a $u \in S$ such that $u \in Tu$.

To verify (i) we note that Tu is convex because (3.6) is a linear equation. For the compactness we let $\{x_j\}$ be a sequence in Tu. The conditions

$$\|x(t)\| = \sum_1^n |x_i(t)| \leq \sum_1^n |x_i(0)| = 1,$$

and

$$\sup_J |x'(t)| \leq \Big(\sup_J |F(t, u(t))|\Big)\|x(t)\|$$

for each compact interval $J \subset [\tau, \infty)$, imply that $\{x_j\}$ contains a subsequence $\{x_k\}$ converging to x_0 in $C^0(J)$. Putting x_k into (3.6) we see that $x_k' \to z$ and that x_0 is a solution of (3.6) with $z = x_0'$. It follows that Tu is compact. That Tu is nonempty follows from Theorem 3.1.

To show (ii) we consider a sequence $\{u_k\} \subset S$ such that $u_k \to u_0$ in $C^0(J)$, and assume that $x_k \in Tu_k$, $x_k \to x_0$. An argument similar to the preceding shows that $x_0 \in Tu_0$, and so T is closed. The proof of (iii) is similar.

The fixed point of T is the solution x_0.

THEOREM 3.3. *Let* $c \in R^n$, $c > 0$, *be given. Then* (3.4) *has a solution* x_c *satisfying*

$$x(t) > 0, \quad x'(t) < 0, \quad \lim_{t\to\infty} x(t) = c > 0 \tag{3.7}$$

if and only if

$$\int_\tau^\infty sA(s)\,ds < \infty. \tag{3.8}$$

Further x_c *satisfies*

$$\lim_{t\to\infty} tx'(t) = 0 \tag{3.9}$$

and x_c *is unique.*

Proof. We observe that a solution $x(t)$ of (3.4) can be written as

$$\begin{aligned} x(t) &= x(b) - x'(b)(b-t) \\ &\quad + \int_t^b (s-t)A(s)x(s)\,ds, \qquad \tau \le t \le b < \infty. \end{aligned}$$

The rest of the proof follows the lines of the proof of Theorem 2.1.

THEOREM 3.4. *Let* (3.5.a) *denote* (3.5) *with the additional condition that* $x \ge y \Rightarrow F(t, x) \ge F(t, y)$ *for all* $t \in [\tau, \infty)$ (*or* $\le$ *for all* t). *Then* (3.5.a), (3.7) *has a solution if and only if*

$$\int_\tau^\infty sF(s, a)\,ds < \infty \quad \textit{for some } a > 0. \tag{3.10}$$

Further, this solution is unique and satisfies (3.9).

Proof. Similar to the proof of Theorem 2.2.

4. Oscillation theorems. A proof of the existence of oscillatory solutions is complicated by the absence of a suitable topological structure in the set of oscillatory solutions. We shall first study the disconnection between oscillatory solutions and a certain type of monotonic solution.

Consider the equations

$$x^{(2n)} - p(t)x = 0 \tag{4.1}$$

and

$$x^{(2n)} - f(t, x)x = 0 \tag{4.2}$$

where $p(t)$ and $f(t, x)$ are as in (2.1), (2.2).

We note that every eventuallly positive solution of (4.1) or (4.2) is of one of the following types:

(i) $x^{(i)}(t) > 0$ for $0 \leq i \leq 2n$ (strongly increasing)

(ii) $(-1)^i x^{(i)}(t) > 0$ for $0 \leq i \leq 2n$ (strongly decreasing); or

(iii) $x^{(i)}(t) > 0$ for $0 \leq i \leq 2k$, $k > 0$, and $(-1)^i x^{(i)}(t) > 0$ for $2k < i \leq 2n$.

Our first theorem collects some known results for these equations.

THEOREM 4.1. (a) *For* $n = 2$ *or* 3: *Equation* (4.1) *has an oscillatory solution if and only if every eventually positive solution is either of type* (i) *or of type* (ii). (b) *Equation* (4.2) *has an oscillatory solution if every eventually positive solution is either of type* (i) *or of type* (ii).

Proof. For $n = 2$, (a) was proved by Ahmad [**1**]. For $n = 3$, (a) follows from results of Edelson and Kreith [**6**] and Jones [**9**]. Part (b) was proved by Edelson and Kreith.

Equations (4.1) and (4.2) always have solutions of types (i) and (ii); Jones [**9**] has shown that for $n > 3$, equation (4.1) may have both oscillatory solutions and solutions of type (iii). We have the following result on oscillatory solutions and solutions of type (iii):

THEOREM 4.1. *If* (4.1) *has a solution of type* (iii), *then no oscillatory solution can satisfy*

$$|x(t)| \leq M, \qquad 0 < \overline{\lim_{t \to \infty}}\, x^{(2n-1)}(t) < \infty. \tag{4.3}$$

Proof. Let $x_1(t)$ be an oscillatory solution, and $x_2(t)$ a solution of type (iii). If $|x_1(t)| \leq M$, then the solution $x(t) = x_1(t) + x_2(t)$ is either strongly increasing or of type (iii). The conditions $\lim_{t \to \infty} x_2^{(2n-1)}(t) = 0$ and $0 < \overline{\lim}_{t \to \infty} x_1^{(2n-1)}(t) < \infty$ imply that $x(t)$ is a solution of type (iii) which satisfies $0 < \overline{\lim}_{t \to \infty} x^{(2n-1)}(t) < \infty$, and this is impossible.

Now we are able to give quantitative criteria for oscillation of (4.2), in the case $n = 2$.

THEOREM 4.2. *If* $f(t, x)$ *is non-decreasing in* x, *and if*

$$\int^{\infty} t^q f(t, x)\, dt = \infty \tag{4.4}$$

for every $c > 0$, *and for some* $q < 3$, *then* (4.2), *with* $n = 2$ *is oscillatory and every nonoscillatory solution is either strongly increasing or strongly decreasing.*

Proof. We will show that if (4.4) holds, then (4.2) has no solutions of type (iii). If, on the contrary, $x_0(t)$ is a solution of type (iii), then the corresponding linear equation

$$x^{(4)} = f(t, x_0(t))x \tag{4.5}$$

has a solution of type (iii) and is therefore nonoscillatory. By Theorem 4.59 of [**13**], we must have $\int^\infty t^q f(t, x_0(t))\, dt < \infty$ for any $q < 3$, but since $x_0(t)$ is positive and increasing, and $f(t, x)$ is nondecreasing in x, this contradicts (4.4).

We note that the equation $x^{(4)} = \frac{9}{16} t^{-4} x$ is nonoscillatory, and therefore the conclusion of Theorem 4.2 fails when $q = 3$.

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