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THE CALCULATION OF AN INVARIANT FOR TOR

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Let λ be a limit ordinal such that λ is not cofinal with ω and let $G = \mathrm{Tor}(A, B)$ where A and B are reduced p-groups. It is shown that the invariant defined to be the dimension of the Z/pZ-vector space $p^\lambda \mathrm{Ext}(Z(p^\infty), G/p^\lambda G)/p^{\lambda+1}\mathrm{Ext}(Z(p^\infty), G/p^\lambda G)$ is zero. If A, B and $\mathrm{Tor}(A, B)$ are three totally projective p-groups then either A or B must be the direct sum of countable p-groups.

1. Introduction. Warfield introduced in [6] the class of S-groups and showed that these groups can be distinguished by a collection of invariants. These invariants for the group G consisted of the classical Ulm invariants and the invariant $k(p^\lambda, G)$, defined to be the dimension of the Z/pZ-vector space $p^\lambda \mathrm{Ext}(Z(p^\infty), G/p^\lambda G)/p^{\lambda+1}\mathrm{Ext}(Z(p^\infty), G/p^\lambda G)$ where λ is a limit ordinal which is not cofinal with ω. In [7], it was shown that the S-groups are the p-groups projective relative to a class of short exact sequences. Since the class of S-groups has a projective characterization and contains the totally projective p-groups, and since each totally projective p-group is p^α-projective for some ordinal α, it was conjectured that an S-group would also be p^α-projective for some ordinal α. However, it will be shown in this paper that an S-group is p^α-projective only if it is totally projective; in fact, it is a summand of a group of the form $\mathrm{Tor}(A, B)$ where A and B are reduced p-groups only if it is totally projective, [Corollary 3.6]. These results will follow once it is shown that the invariant $k(p^\lambda, \mathrm{Tor}(A, B))$ is zero for all reduced p-groups A and B, [Corollary 3.4]. Finally, it is shown that if A, B and $\mathrm{Tor}(A, B)$ are three totally projective p-groups then either A or B is the direct sum of countable p-groups, [Corollary 3.8].

2. Notation. If G is a group then let $c(G)$ denote the cotorsion hull of G, i.e., $c(G) = \mathrm{Ext}(Z(p^\infty), G)$ where $Z(p^\infty)$ is the divisible torsion p-group of Q/Z.

If G is a reduced p-group then $l(G)$ will denote the length of G, i.e., $l(G)$ is the least ordinal for which $p^{l(G)}G = 0$. Let Ω denote the first uncountable ordinal.

3. Results. Dr. R. Nunke has communicated orally to me the following result and proof which will be used in the proof of Lemma 3.2.

THEOREM 3.1. *Let $\{A_\alpha,\ \alpha \in \Gamma\}$ be a collection of p-groups, and λ a limit ordinal such that λ is not cofinal with ω. If for each $\alpha \in \Gamma$, $p^\lambda c(A_\alpha) = 0$, then $p^\lambda c(\bigoplus_{\alpha\in\Gamma} A_\alpha) = 0$.*

Proof. Let $A = \bigoplus_{\alpha\in\Gamma} A_\alpha$ and $e \in p^\lambda c(A)$ represent the exact sequence $e\colon 0 \to A \xrightarrow{\nu} M \to Z(p^\infty) \to 0$. For each $\alpha \in \Gamma$, the pushout sequence by the projection map $\pi_\alpha\colon A \to A_\alpha$ is p^λ-pure; consequently, there exists a map $\phi_\alpha\colon M \to A_\alpha$ such that the following diagram commutes.

$$\begin{array}{ccccccccc} e: 0 & \to & A & \to & M & \to & Z(p^\infty) & \to & 0 \\ & & \pi_\alpha \downarrow & \swarrow \phi_\alpha & & & & & \\ & & A_\alpha & & & & & & \end{array}$$

To show that the sequence e splits, it must be shown that $\phi = \bigoplus_{\alpha\in\Gamma} \phi_\alpha\colon M \to \bigoplus_{\alpha\in\Gamma} A_\alpha = A$ is a homomorphism, i.e., that for each x in M the set $\{\alpha \mid \phi_\alpha(x) \neq 0,\ \alpha \in \Gamma\}$ is finite. Suppose there exists an x in M and a sequence $\{\alpha_i \in \Gamma\}$ such that $\phi_{\alpha_i}(x) \neq 0$. Let β be any ordinal which satisfies the inequalities $\lambda > \beta$ and $\beta >$ height of $\phi_{\alpha_i}(x)$ for each i. The ordinal β exists since $\lambda > \alpha_i$ for each i and λ is a limit ordinal which is not cofinal with ω. The sequence e being p^β-pure and the group $Z(p^\infty)$ being divisible imply there exists an element a in the subgroup $\nu(A) = \nu(\bigoplus_{\alpha\in\Gamma} A_\alpha)$ of M, and an element b in $p^\beta M$ for which $x = a + b$, [**3**, 87]. For some α_i, $\phi_{\alpha_i}(a) = 0$; hence, $\phi_{\alpha_i}(x) = \phi_{\alpha_i}(b)$. This is a contradiction since the height of $\phi_{\alpha_i}(x)$ is less than β whereas the height of $\phi_{\alpha_i}(b)$ is greater than or equal to β.

The following lemma will be used in the proof of Theorem 3.3.

LEMMA 3.2. *Let λ be a limit ordinal which is not cofinal with ω and let G be a p-group such that $p^\lambda G = 0$. Then $p^\lambda c(\mathrm{Tor}(G, X)) = 0$ for any reduced group X.*

Proof. There exists a reduced p^λ-injective group I and a p^λ-pure sequence $0 \to G \to I \to U \to 0$, [**3**, 84]. It follows that $0 \to \mathrm{Tor}(G, X) \to \mathrm{Tor}(I, X) \to \mathrm{Tor}(U, X) \to 0$ is a pure sequence of reduced groups, and the sequence

$$0 \to c(\mathrm{Tor}(G, X)) \to c(\mathrm{Tor}(I, X)) \to c(\mathrm{Tor}(U, X)) \to 0$$

is exact. Once it is shown that $p^\lambda c(\mathrm{Tor}(I, X)) = 0$, then the lemma is proved. To show this let D be the injective hull of X and $0 \to X \to D \to D'$ $\to 0$ be the resulting exact sequence. Consequently, $0 \to \mathrm{Tor}(I, X) \to \mathrm{Tor}(I, D) \to \mathrm{Tor}(I, D')$ is an exact sequence of reduced groups. The group $\mathrm{Tor}(I, D)$ is isomorphic to the direct sum of γ copies of $t(I)$, the torsion subgroup of I, where γ is the dimension of the Z/pZ-vector space $D[p]$, and $p^\lambda c(t(I)) \subseteq p^\lambda c(I) = 0$. Hence, it follows from Theorem 3.1 that $p^\lambda c(\mathrm{Tor}(I, X)) \subseteq p^\lambda c(\mathrm{Tor}(I, D)) = 0$.

THEOREM 3.3. *If λ is a limit ordinal such that λ is not cofinal with ω, then $p^\lambda c(\mathrm{Tor}(A, B)) = c(p^\lambda \mathrm{Tor}(A, B))$ whenever A and B are reduced groups.*

Proof. It need only be shown that $p^\lambda c(\mathrm{Tor}(A, B))$ is a subset of $c(p^\lambda \mathrm{Tor}(A, B))$, since there is an exact sequence

$$0 \to c\big(p^\lambda \mathrm{Tor}(A, B)\big) \to p^\lambda c(\mathrm{Tor}(A, B))$$
$$\to p^\lambda c\big(\mathrm{Tor}(A, B)/p^\lambda \mathrm{Tor}(A, B)\big) \to 0,$$

[**1**, 56.1].

The sequence

$$0 \to \mathrm{Tor}(p^\lambda A, B) \to \mathrm{Tor}(A, B) \xrightarrow{\pi} \mathrm{Tor}(A/p^\lambda A, B) \xrightarrow{\delta} (p^\lambda A) \otimes B$$

is exact. If X is the image of π and Y the image of δ, then X and Y are reduced subgroups of $\mathrm{Tor}(A/p^\lambda A, B)$ and $(p^\lambda A) \otimes B$, respectively. Therefore it follows that the sequences

$$e\colon 0 \to c\big(\mathrm{Tor}(p^\lambda A, B)\big) \to c(\mathrm{Tor}(A, B)) \to c(X) \to 0$$

and

$$f\colon 0 \to c(X) \to c\big(\mathrm{Tor}(A/p^\lambda A, B)\big) \to c(Y) \to 0$$

are exact sequences of reduced groups.

$$p^\lambda c(\mathrm{Tor}(A, B)) \subseteq c\big(\mathrm{Tor}(p^\lambda A, B)\big)$$

since

$$p^\lambda c(X) \subseteq p^\lambda c\big(\mathrm{Tor}(A/p^\lambda A, B)\big) = 0,$$

[Lemma 3.2]. Similarly,

$$p^\lambda c(\mathrm{Tor}(A, B)) \subseteq c\big(\mathrm{Tor}(A, p^\lambda B)\big).$$

The conclusion follows from the identity

$$c\big(\mathrm{Tor}(A, p^\lambda B)\big) \cap c\big(\mathrm{Tor}(p^\lambda A, B)\big) = c(p^\lambda \mathrm{Tor}(A, B)),$$

[**1**, 64.2].

COROLLARY 3.4. *If G is a summand of* $\mathrm{Tor}(A, B)$ *where A and B are reduced p-groups, then $p^{\lambda}c(G) = c(p^{\lambda}G)$ (equivalently, $p^{\lambda}c(G/p^{\lambda}G) = 0 = k(p^{\lambda}, G)$) for every limit ordinal λ such that λ is not cofinal with ω. Consequently, if G is p^{α}-projective for some ordinal α then $p^{\lambda}c(G) = c(p^{\lambda}G)$ and $p^{\alpha}G = 0$.*

Proof. Since all the functors commute with direct sums, $p^{\lambda}c(G) = c(p^{\lambda}G)$.

If G is p^{α}-projective for some ordinal α then there is a reduced group H such that $p^{\alpha}H = 0$ and G is a summand of $\mathrm{Tor}(G, H)$. Hence, $p^{\lambda}c(G) = c(p^{\lambda}G)$. Also, $p^{\alpha}G = 0$ since $p^{\alpha}\mathrm{Tor}(G, H) = 0$.

The equivalence of $p^{\lambda}c(G) = c(p^{\lambda}G)$ and $p^{\lambda}c(G/p^{\lambda}G) = 0$ follows from the exact sequence $0 \to c(p^{\lambda}G) \to p^{\lambda}c(G) \to p^{\lambda}c(G/p^{\lambda}G) \to 0$, [**1**, 56.1].

COROLLARY 3.5. *If $0 \to Z \to M \to H_{\lambda} \to 0$ is a sequence which represents p^{λ} where λ is a limit ordinal which is not cofinal with ω, then the torsion subgroup of M is not p^{α}-projective for any ordinal α.*

Proof. Let G be the torsion subgroup of M. G is a λ-elementary S-group and in [**6**] it is shown that $k(p^{\lambda}, G) \neq 0$, [Corollary 3.4].

COROLLARY 3.6. *If G is an S-group, then G is p^{α}-projective if and only if G is a totally projective p-group and $p^{\alpha}G = 0$. Also, G is not a summand of a group* $\mathrm{Tor}(A, B)$ *where A and B are reduced groups, unless G is totally projective.*

Proof. This result follows from Corollary 3.4 and the fact that an S-group is totally projective if and only if $k(p^{\lambda}, G) = 0$ for every limit ordinal λ which is not cofinal with ω, [**6**].

THEOREM 3.7. *If A and B are two totally projective p-groups such that $l(A) \geq l(B) = \alpha > \Omega$, then* $\mathrm{Tor}(A, B)$ *is not totally projective.*

Proof. The proof will be by transfinite induction on the ordinal α.

Case 1. $\alpha = \lambda + n + 1$ where λ is a limit ordinal and n is a finite ordinal. Let T be a $p^{\lambda+n}$-high subgroup of the group A and e: $0 \to T \to A \to D \to 0$ be the resulting exact sequence. Since the sequence e is p^{α}-pure, [**3**, 92], and the group Tor(D, B) is p^α-projective, [**3**, 82], the sequence f: $0 \to \mathrm{Tor}(T, B) \to \mathrm{Tor}(A, B) \to \mathrm{Tor}(D, B) \to 0$ is p^α-pure, [**5**, 2], and splits. Hence, $\mathrm{Tor}(A, B) \simeq \mathrm{Tor}(T, B) \oplus \mathrm{Tor}(D, B)$. The group T is an S-group because $p^\lambda T$ and $T/p^\lambda T$ are both S-groups, [**6**, 5.3]. Three subcases will now be considered. In each of the subcases Tor(A, B) will be shown to be not totally projective by showing that it has a summand which is not totally projective.

Case 1.1. $\alpha = \Omega + 1$. The sequence e being p^α-pure implies that the sequence $0 \to \mathrm{Hom}(Z(p^\infty), D) \to p^{\Omega+1}c(T) \to p^{\Omega+1}c(A) = 0$ is exact, [**3**, 89]. Since D is a non-trivial divisible p-groups, $p^\Omega c(T) \neq c(p^\Omega T) = 0$ and the group T is not p^Ω-projective, [Corollary 3.6]. Since $l(T) < l(B)$, Tor(T, B) is not totally projective, [**4**, 3.4].

Case 1.2. $\alpha > \Omega + 1$ and the group T is totally projective. Since $\Omega < l(T) = \lambda + n < l(B)$, induction is used to show that Tor(T, B) is not totally projective.

Case 1.3. $\alpha > \Omega + 1$ and the group T is not totally projective. By Corollary 3.6, the group T is not $p^{\lambda+n}$-projective. Consequently, Tor(T, B) is not totally projective, [**4**, 3.4].

Case 2. α is a limit ordinal greater than Ω. There exists a summand W of B such that the group W is totally projective and $\Omega < l(W) < \alpha$, [**2**, 83.1(e)]. By induction, Tor(A, B) has a summand which is not a totally projective p-group.

COROLLARY 3.8. *If A, B and* Tor(A, B) *are three totally projective p-groups then either A or B is the direct sum of countable p-groups.*

Proof. This corollary follows from Theorem 3.7 and noting that any totally projective p-group with length at most Ω is the direct sum of countable p-groups.

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