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SEWN-UP r-LINK EXTERIORS

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Suppose two solid handlebodies, each of genus r, are disjointly embedded in S^3. If the interiors of the handlebodies are removed and the boundary components of the remaining space are identified via an orientation reversing homeomorphism, then a closed connected orientable 3-manifold results. Such a manifold is called a *sewn-up r-link exterior*. The main result of this paper is that a closed connected orientable 3-manifold M can be realized as a sewn-up r-link exterior if and only if the first homology of M is infinite.

The extend to which this theorem can be used to demonstrate Property R for knots is discussed.

Introduction. A sewn-up r-link exterior is similar in construction to a manifold of the following type. Instead of embedding two solid genus r handlebodies in the same copy of S^3, start with two copies of S^3 and embed one handlebody in each. Then remove the interiors of the two handlebodies and identify the boundary components of the resulting spaces. This is known as "sewing together two r-knot exteriors." Since the complements of the handlebodies may or may not be handlebodies themselves, this construction generalizes the well-known Heegaard construction.

In the first part of this paper it is shown how to obtain a framed link description for a manifold constructed in either of these ways. In the case of sewing together two r-knot exteriors this generalizes Lickorish's proof that any manifold obtained via the Heegaard construction can also be obtained by Dehn surgery on a link in S^3 [**8**]. The proof given here is apparently different from Lickorish's since our viewpoint is largely 4-dimensional whereas his is strictly 3-dimensional. However, our proof still rests squarely on Lickorish's analysis of the selfhomeomorphisms of a closed orientable 2-manifold [**9**]. The main result is then proven by showing that a closed connected orientable 3-manifold with infinite first homology has a framed link description like one obtained from sewing up an r-link exterior.

Suppose that K is a knot in S^3. Denote by K_0 the manifold obtained from S^3 by zero surgery on K. Since $H_1(K_0) \simeq \mathbf{Z}$, K_0 may be viewed as a sewn-up r-link exterior. This can sometimes be used to show that $K_0 \not\approx S^1 \times S^2$ and hence that K satisfies Property R. Sewn-up r-link exteriors were probably first investigated by Brakes in [**1**] and [**2**], who used them in this way to show that many superslice knots satisfy Property R. The extent to which the theory of sewn-up r-link exteriors can be applied to demonstrate Property R for knots in general is explored in §6. Other applications to knot theory, such as the calculation of the Alexander invariant of a knot, are also discussed.

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1. Definitions. Throughout this paper all maps and manifolds are assumed to be piecewise linear. All homology groups are assumed to have integer coefficients.

Let $H^{(r)}$ be the solid genus r handlebody shown in Figure 1.1. Let $K^{(r)}$ be the oriented subcomplex contained in the interior of $H^{(r)}$ that consists of the 2-disk D, with one side labeled "$+$" and the other "$-$", together with the oriented cores c_i of the 1-handles as indicated. The oriented curves l_i and m_i lie on $\partial H^{(r)}$ and are called the *longitudes* and *meridians* of $K^{(r)}$. Notice that l_i is isotopic to c_i in $H^{(r)}$, is oriented the same as c_i and $\mathrm{lk}(l_i, c_i) = 0$. Furthermore, all the longitudes lie on the "$+$" side of D. The meridian m_i is oriented so that $\mathrm{lk}(m_i, c_i) = 1$.

By an *r-knot* $K^{(r)}$ in S^3 we shall mean an embedding $f\colon H^{(r)} \to S^3$ such that, for each i, $f(l_i)$ is a preferred longitude of the knot $f(c_i)$ in the usual sense (i.e. $\mathrm{lk}(f(l_i), f(c_i)) = 0$). When drawing a picture of an r-knot in S^3 we would like, for the sake of convenience, to only have to draw the embedded subcomplex $f(K^{(r)})$. We can do this if we adopt the following convention. Given an embedding of $K^{(r)}$ we will recover an embedding of $H^{(r)}$ as follows. First thicken $f(D)$ to obtain an embedding of $D \times [-1, 1]$. Then attach 1-handles, $B^1 \times B^2$'s, to $f(D) \times [-1, 1]$ with the curves $f(c_i)$ serving as the cores of the handles. Furthermore, attach these 1-handles so that a product fibre $B^1 \times p \in B^1 \times B^2$ with $p \in \partial B^2$ together with $f(c_i \cap D) \times \{1\}$ forms a preferred longitude of $f(c_i)$. We will refer to this embedding as the r-knot $K^{(r)}$. If $r = 1$ the disk D is omitted and $f(c_1)$ is an oriented knot in the usual sense with $f(l_1)$ and $f(m_1)$ a preferred longitude and meridian. By $N(K^{(r)})$ we shall mean $f(H^{(r)})$, which in the case $r = 1$ is just a tubular neighborhood of the knot K. The *exterior* $E(K^{(r)})$ is $\overline{S^3 - N(K^{(r)})}$. An *r-link* $L^{(r)} = \{K_1^{(r)}, \dots, K_n^{(r)}\}$ of n components in S^3 is a collection of n disjoint r-knots in S^3. The *exterior* $E(L^{(r)})$ is $\overline{S^3 - \bigcup_i N(K_i^{(r)})}$. The exteriors of r-knots and r-links will always be oriented by means of the standard right-handed orientation on S^3.

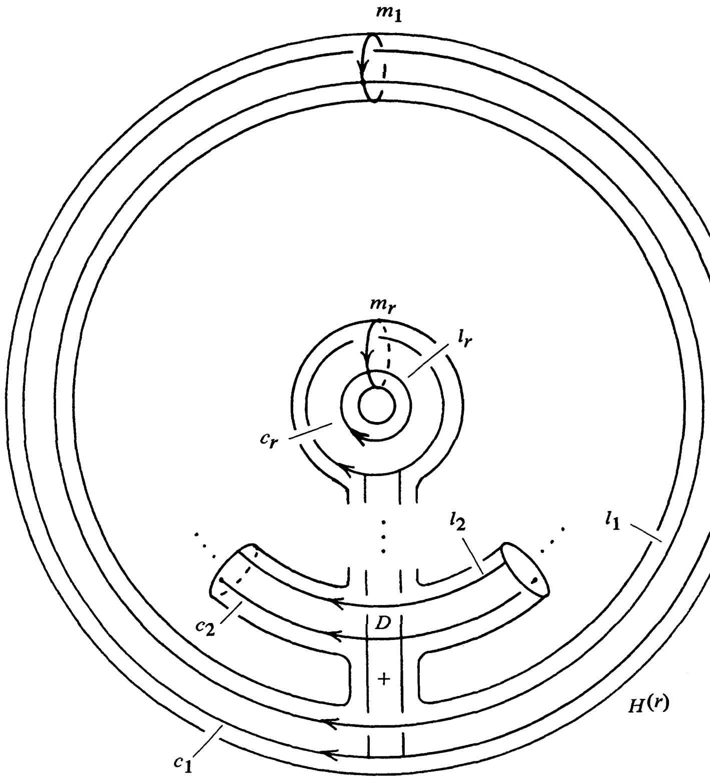


FIGURE 1.1

Of course if $r = 1$ these concepts are the usual ones for knots and links and r will be omitted from the notation.

Let $K_1^{(r)}$ and $K_2^{(r)}$ be two r-knots in separate copies of S^3. Then $\partial E(K_1^{(r)})$ and $\partial E(K_2^{(r)})$ are two surfaces of the same genus and hence homeomorphic. If g: $\partial E(K_1^{(r)}) \to \partial E(K_2^{(r)})$ is any orientation reversing homeomorphism then we may form $M(K_1^{(r)}, K_2^{(r)}; g)$ by attaching $E(K_1^{(r)})$ to $E(K_2^{(r)})$ along $\partial E(K_i^{(r)})$ via g. Thus $M(K_1^{(r)}, K_2^{(r)}; g)$ is a closed orientable 3-manifold.

Consider those homeomorphisms g from $\partial H^{(r)}$ to $\partial H^{(r)}$ obtained by Dehn twists about the longitude and meridian of each handle. If l_i is sent to the class of $a_i l_i + c_i m_i$, and m_i is sent to the class of $b_i l_i + d_i m_i$, then

$$A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$$

completely classifies (up to isotopy) the action of the homeomorphism on the ith handle. Hence we may write $g = (A_1, \ldots, A_r)$ for some r 2×2 integral matrices A_i. Since g is orientation preserving, det $A_i = 1$ for all i.

Since we are interested in orientation reversing homeomorphisms we shall compose a homeomorphism of this type with the reflection ρ through the plane of symmetry orthogonal to that of D.

The action of ρ on each handle is described by the matrix $\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$ since meridians are preserved while longitudes are reversed. Thus the orientation reversing homeomorphism

$$(A_1,\ldots,A_r)\circ\rho=\left(A\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\ldots,A_r\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right)$$

is classified by r 2×2 integral matrices, each having determinant -1.

Throughout the rest of the paper let $J=\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$. Let j be the orientation reversing homeomorphism

$$j=(J,J,\ldots,J)=\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\ldots,\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)\circ\rho.$$

If $f=(A_1,\ldots,A_r)$ and $g=(B_1,\ldots,B_r)$ then $f\circ g=(A_1B_1,\ldots,A_rB_r)$ regardless of whether or not f and g are orientation preserving.

If $g=(A_1,\ldots,A_r)$ then we shall rewrite $M(K_1^{(r)},K_2^{(r)};\ g)$ as $M(K_1^{(r)},K_2^{(r)};A_1,\ldots,A_r)$. We will often denote this manifold as

$$K_1^{(r)}\overset{A_1,\ldots,A_r}{\nrightarrow}K_2^{(r)}.$$

In the case of specific r-knots we can represent the manifold as in Figure 1.2. The slashes on the arrows are used to indicate that $K_1^{(r)}$ and $K_2^{(r)}$ lie in separate copies of S^3.

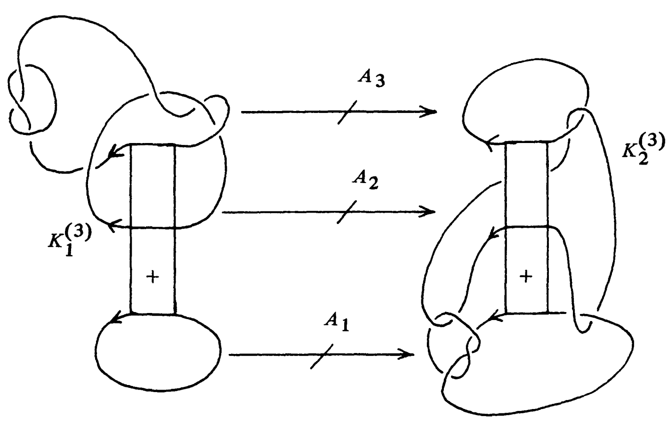


FIGURE 1.2

Notice that if $r = 1$ then every orientation reversing homeomorphism g corresponds to a matrix. In this case $M(K_1, K_2; A)$ is obtained by sewing together two "honest" knot exteriors. (See Gordon [3] for a thorough study of this construction.)

If, in addition to $K_1^{(r)}$ and $K_2^{(r)}$, we consider n 2-component r-links $L_i^{(r)} = \{K_{i,1}^{(r)}, K_{i,2}^{(r)}\}$ we may form a *chain of sewn-up r-knot and r-link exteriors*

$$K_1^{(r)} \overset{g_0}{\nrightarrow} K_{1,1}^{(r)}, K_{1,2}^{(r)} \overset{g_1}{\nrightarrow} \cdots \overset{g_{n-1}}{\nrightarrow} K_{n,1}^{(r)}, K_{n,2}^{(r)} \overset{g_n}{\nrightarrow} K_2^{(r)}.$$

If the g_i's are orientation reversing homeomorphisms simply attach $E(K_1^{(r)})$ to $E(L_1^{(r)})$ along $\partial N(K_{1,1}^{(r)})$ via g_0, $E(L_i^{(r)})$ to $E(L_{i+1}^{(r)})$ along $\partial N(K_{i,2}^{(r)})$ and $\partial N(K_{i+1,1}^{(r)})$ via g_i and $E(L_n^{(r)})$ to $E(K_2^{(r)})$ along $\partial N(K_{n,2}^{(r)})$ via g_n. The integer n is called the *length* of the chain.

Now suppose that $L^{(r)} = \{K_1^{(r)}, K_2^{(r)}\}$ is a 2-component r-link in S^3. Given any orientation reversing homeomorphism g: $\partial N(K_1^{(r)}) \to \partial N(K_2^{(r)})$ we may form the *sewn-up r-link exterior* $\mathcal{S}(K_1^{(r)}, K_2^{(r)}; g)$, or $\mathcal{S}(L^{(r)}; g)$, by starting with $E(L^{(r)})$ and identifying $\partial N(K_1^{(r)})$ with $\partial N(K_2^{(r)})$ via g. Since g is orientation reversing $\mathcal{S}(K_1^{(r)}, K_2^{(r)}; g)$ is a closed orientable manifold. If $g = (A_1, \ldots, A_r)$ then we shall write $\mathcal{S}(K_1^{(r)}, K_2^{(r)}; A_1, \ldots, A_r)$. (See Brakes [1] and [2] for more information about this construction.)

We will often denote this manifold as $K_1^{(r)} \xrightarrow{A_1, \ldots, A_r} K_2^{(r)}$. In the case of specific r-knots we can represent the manifold as in Figure 1.3. The absence of slashes on the arrows indicates that $K_1^{(r)}$ and $K_2^{(r)}$ lie in the same copy of S^3 and would be the only way of distinguishing $\mathcal{S}(K_1^{(r)}, K_2^{(r)}; g)$ from $M(K_1^{(r)}, K_2^{(r)}; g)$ if $K_1^{(r)}$ and $K_2^{(r)}$ were split. Recall that an r-link $L^{(r)} = \{K_1^{(r)}, K_2^{(r)}\}$ is called *split* if there exists some

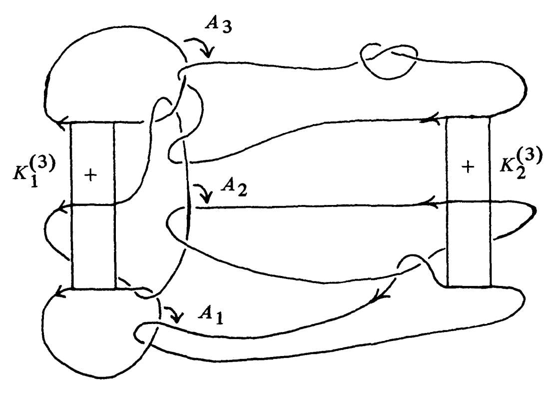


FIGURE 1.3

embedded 2-sphere in S^3 that separates $K_1^{(r)}$ from $K_2^{(r)}$. Otherwise $L^{(r)}$ is called *nonsplit*. $L^{(r)}$ is called *strongly nonsplit* if, in addition to being nonsplit, each boundary component of $E(L^{(r)})$ is incompressible in $E(L^{(r)})$. Notice that strongly nonsplit and nonsplit are equivalent when $r = 1$. This is not true for $r > 1$.

If $K_1^{(r)}$ and $K_2^{(r)}$ are split then it is not hard to see that

$$\mathbb{S}\left(K_1^{(r)}, K_2^{(r)}; g\right) \simeq S^1 \times S^2 \# M\left(K_1^{(r)}, K_2^{(r)}; g\right).$$

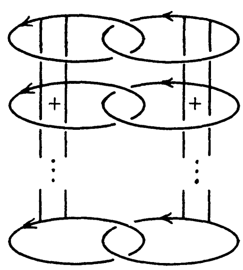
Starting with n 2-component r-links $L_i^{(r)} = \{K_{i,1}^{(r)}, K_{i,2}^{(r)}\}$ we may also form a *circular chain of sewn-up r-link exteriors* in the obvious way. This is denoted

$$\underbrace{K_{1,1}^{(r)}, K_{1,2}^{(r)} \overset{g_1}{\nrightarrow} K_{2,1}^{(r)}, K_{2,2}^{(r)} \overset{g_2}{\nrightarrow} \cdots \overset{g_{n-1}}{\nrightarrow} K_{n,1}^{(r)}, K_{n,2}^{(r)}}_{g_n}$$

Finally, for the sake of completeness, we could define a sewn-up 0-link exterior by considering as a 0-link an embedding of two 3-balls. Then every 0-link exterior is homeomorphic to $S^2 \times I$ and the only sewn-up 0-link exterior is $S^1 \times S^2$.

2. Constructing a framed link description of $\mathbb{S}(L^{(r)}; g)$.

LEMMA 2.1. *The following two chains are homeomorphic*;

$K_1^{(r)} \overset{f}{\nrightarrow}$  $\overset{g}{\nrightarrow} K_2^{(r)}$

and

$$K_1^{(r)} \overset{g \circ j \circ f}{\nrightarrow} K_2^{(r)}.$$

Proof. Notice that the exterior of the r-link $L^{(r)}$ pictured above is homeomorphic to a closed surface of genus r cross the unit interval. This product structure induces the homeomorphism j between the components of $\partial E(L^{(r)})$. □

The r-link $L^{(r)}$ pictured above is called the *Hopf r-link*.

Consider $M(K_1^{(r)}, K_2^{(r)}; A_1,\ldots,A_r)$ in the case where each

$$A_i = \begin{pmatrix} -1 & 0 \\ -n_i & 1 \end{pmatrix}.$$

Since meridians are taken to meridians the homeomorphism from $\partial N(K_1^{(r)})$ to $\partial N(K_2^{(r)})$ can be naturally extended to a homeomorphism from all of $N(K_1^{(r)})$ to $N(K_2^{(r)})$. If $K_1^{(r)}$ and $K_2^{(r)}$ are thought of as lying in the boundaries of two disjoint 4-balls then a 4-manifold may be formed by glueing together the two 4-balls along the handlebodies using the homeomorphism given by $(A_1,\ldots,A_r)$. The boundary of this 4-manifold is $M(K_1^{(r)}, K_2^{(r)};\ A_1,\ldots,A_r)$. The following lemma, in combination with Lemma 2.1, shows that a chain with sewing homeomorphisms given by matrices can always be lengthened if necessary so that all the matrices are of this type.

LEMMA 2.2. *If $(A_1,\ldots,A_r)$ are r 2×2 integral matrices with* $\det A_i = -1$ *for all i then there exists an integer m such that $A_i = B_{im}JB_{im-1}\cdots JB_{i1}$ for all i and each*

$$B_{ij} = \begin{pmatrix} -1 & 0 \\ -n_{ij} & 1 \end{pmatrix}.$$

Proof. According to Gordon [3, Lemma 4, p. 159] each A_i has an expansion of the form $A_i = B_rJB_{r-1}\cdots JB_1$ where each

$$B_i = \begin{pmatrix} -1 & 0 \\ -n_i & 1 \end{pmatrix}.$$

However, these expansions may not all be the same length. But the following two identities show that any expansion can be increased in length by multiples of either three or four.

$$\begin{pmatrix} -1 & 0 \\ -s & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -s-1 & 1 \end{pmatrix} J \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} J \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} J \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 \\ -s & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -s & 1 \end{pmatrix} J \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} J \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} J \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} J \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using these two identities it is easy to show that each expansion can be lengthened if necessary until they all agree in length. □

Now suppose that $K_1^{(r)}$ and $K_2^{(r)}$ are two r-knots lying in the boundaries of two separate copies of B^4. Let $K_i^{(r)} = D_i \cup c_{i1} \cup \cdots \cup c_{ir}$ where the

c_{ij}'s are the cores of the handles. Suppose that

$$A_i = \begin{pmatrix} -1 & 0 \\ -n_i & 1 \end{pmatrix}$$

for each i. In order to form $M(K_1^{(r)}, K_2^{(r)}; A_1,\ldots,A_r)$ we must identify $N(K_1^{(r)})$ with $N(K_2^{(r)})$ via $(A_1,\ldots,A_r)$. To do this imagine sewing one end of $H^{(r)} \times I$ to one copy of B^4 along $N(K_1^{(r)})$ and the other end to the other copy of B^4 along $N(K_2^{(r)})$. Suppose $H^{(r)} \times \{0\}$ is attached to $N(K_1^{(r)})$ via the identity while $H^{(r)} \times \{1\}$ is attached to $N(K_2^{(r)})$ via $(A_1,\ldots,A_r)$. Now $H^{(r)}$ may be thought of as a 0-handle with r 1-handles attached. The 0-handle is a 3-ball B^3 and each 1-handle is a 3-ball viewed as $B^1 \times B^2$ attached to B^3 along $\partial B^1 \times B^2 \simeq S^0 \times B^2$. Hence $H^{(r)} \times I$ may be viewed as $B^3 \times I$ union r pieces, each a $B^1 \times B^2 \times I$ attached to $B^3 \times I$ along $S^0 \times B^2 \times I$. Now attaching $H^{(r)} \times I$ to the two 4-balls can be viewed as two steps. First the $B^3 \times I$ is attached and then the remaining r pieces are attached. Attaching the $B^3 \times I$ amounts to adding a 1-handle between the two 4-balls. Hence a 4-ball results. Next a $B^1 \times B^2 \times I$ is attached along $S^0 \times B^2 \times I \cup B^1 \times B^2 \times S^0 \simeq \partial B^2 \times B^2$. Thus attaching each of the remaining r pieces amounts to adding r 2-handles. Notice that each 2-handle runs geometrically twice and algebraically zero times over the 1-handle. Furthermore each 2-handle is clearly attached along the connected sum $c_{1i}\#c_{2i}$. Since

$$\begin{pmatrix} 1 & 0 \\ -n_i & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ n_i & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

the surgery coefficient on $c_{1i}\#c_{2i}$ must be n_i.

We may draw this framed link by placing $K_1^{(r)}$ and $K_2^{(r)}$ in separate hemispheres of S^3 and then forming the connected sums of the cores of the handles as illustrated in the following example.

Suppose we begin with the 3-link exterior and homeomorphism shown in Figure 2.1. After attaching the 1 and 2-handles the 4-manifold appears as in Figure 2.2. However the feet of the 1-handle lie on separate copies of B^4 and hence the 4-manifold may be viewed as a single B^4 with three 2-handles attached. To draw it in this manner we proceed in the following way. Label the two 2-spheres in Figure 2.2 S_1 and S_2. In each copy of $S^3 = \partial B^4$, S_i bounds two 3-balls, say B_{i1} and B_{i2}, with B_{11} and B_{21} the feet of the 1-handle. Deleting B_{11} from S^3 we may turn S_1 inside out and draw B_{12} in place of B_{21} with the correct identification between S_1 and S_2. A more convenient way of drawing the resulting link however, is to turn both S_1 and S_2 inside out and to identify B_{12} with B_{22} by means of drawing in an $S^2 \times I$, with product fibres forming the connected sums $c_{1j}\#c_{2j}$. This is shown in Figure 2.3. In practice we may dispense with drawing the spheres S_1 and S_2 and simply place $K_1^{(r)}$ and $K_2^{(r)}$ in separate hemispheres of S^3, delete D_1 and D_2 from each and then band connect together the cores of the handles. It is important to note that the bands are drawn as product fibres of the $S^2 \times I$ as described above. This can most easily be done by always first positioning $K_1^{(r)}$ and $K_2^{(r)}$ with the "+" sides of D_1 and D_2 up and then drawing the bands above $K_1^{(r)}$ and $K_2^{(r)}$ in the obvious way.

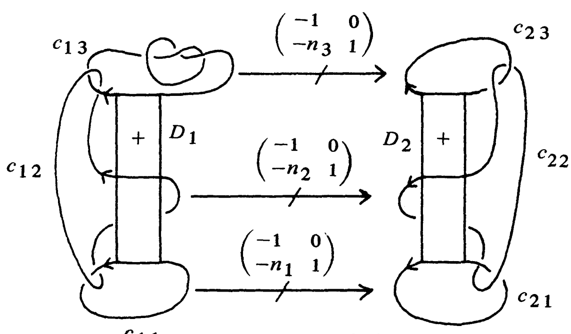


FIGURE 2.1

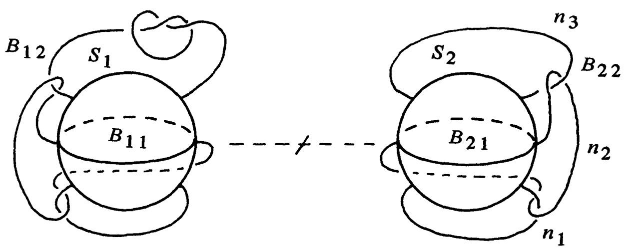


FIGURE 2.2

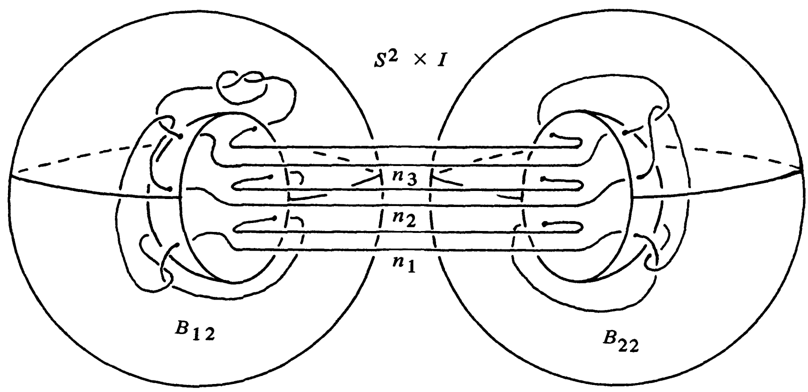
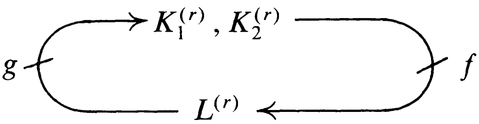


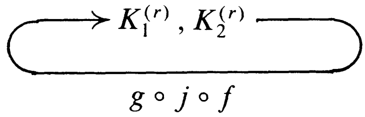
FIGURE 2.3

Notice that we may now pass from any chain whose homeomorphisms correspond to matrices to a framed link. In the case $r = 1$ this includes all chains.

LEMMA 2.3. *Let $L^{(r)}$ be the Hopf r-link. The following two circular chains are homeomorphic*



and



Proof. The proof is similar to that given for Lemma 2.1. □

Now in order to find a framed link description of

$$\mathcal{S}\left(K_1^{(r)}, K_2^{(r)}; A_1, \ldots, A_r\right)$$

it only remains to do so in the case where each A_i is of the form $\left(\begin{smallmatrix}-1 & 0\\ -n_i & 1\end{smallmatrix}\right)$. But in this case the homeomorphism represented by $(A_1, \ldots, A_r)$ can naturally be extended to a homeomorphism from all of $N(K_1^{(r)})$ to $N(K_2^{(r)})$. Hence we may think of $K_1^{(r)}$ and $K_2^{(r)}$ as lying in the boundary of B^4 and then identify $N(K_1^{(r)})$ with $N(K_2^{(r)})$ via $(A_1, \ldots, A_r)$. The boundary of this 4-manifold will be $\mathcal{S}(K_1^{(r)}, K_2^{(r)}; A_1, \ldots, A_r)$.

We may proceed just as we did in the case of $M(K_1^{(r)}, K_2^{(r)}; A_1, \ldots, A_r)$. Namely, we think of attaching $H^{(r)} \times I$ to the 4-ball with $H^{(r)} \times \{0\}$ attached to $N(K_1^{(r)})$ via the identity and $H^{(r)} \times \{1\}$ attached to $N(K_2^{(r)})$ via $(A_1, \ldots, A_r)$. Again think of this as being done in two stages. First a 1-handle is attached and then r 2-handles are attached. This time however, the addition of the 1-handle does not yield a 4-ball but rather $S^1 \times B^3$. The 2-handles are then added with each 2-handle being attached along $c_{1i} \#_b c_{2i}$, the band connected sum of c_{1i} and c_{2i}, with the band running geometrically once over the 1-handle. Since we are only interested in the boundary of this 4-manifold we may trade the 1-handle for a 2-handle to obtain a framed link for $\mathcal{S}(K_1^{(r)}, K_2^{(r)}; A_1, \ldots, A_r)$. To see what the surgery coefficient on each $c_{1i} \#_b c_{2i}$ must be, let l_i be the (not necessarily preferred) longitude of $c_{1i} \#_b c_{2i}$ obtained by band connect summing l_{1i} and

l_{2i}, the longitudes of c_{1i} and c_{2i} respectively. Now *with respect to* l_i the 2-handle is attached to $c_{1i}\#_b c_{2i}$ with a surgery coefficient of n_i. But since $\mathrm{lk}(l_i, c_{1i}\#_b c_{2i}) = 2\,\mathrm{lk}(c_{1i}, c_{2i})$ the true surgery coefficient (i.e. with respect to a *preferred* longitude of $c_{1i}\#_b c_{2i}$) must be $n_i + 2\mathrm{lk}(c_{1i}, c_{2i})$. The following example illustrates this process.

Beginning with the 2-link exterior and homeomorphism shown in Figure 2.4, we first obtain a handlebody description that involves only one 0-handle, one 1-handle and two 2-handles. The 1-handle is then traded for a 2-handle yielding the framed link description of $\mathcal{S}(L^{(r)}; A_1, A_2)$ that is illustrated in Figure 2.5. This is done by introducing the zero framed unknot and by band connecting the cores of the handles along a path that passes through this unknot exactly once. It is well known that the choice of this path is immaterial as can be seen from the fact that any crossings in the path can be changed by sliding over the zero framed unknot. It is important to notice however, that the bands connect either the " + " sides of D_1 and D_2 or the " − " sides, just as in the case of sewing together two r-knot exteriors.

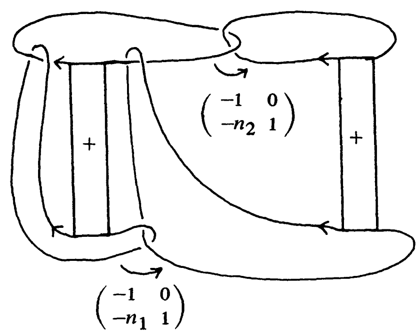


FIGURE 2.4

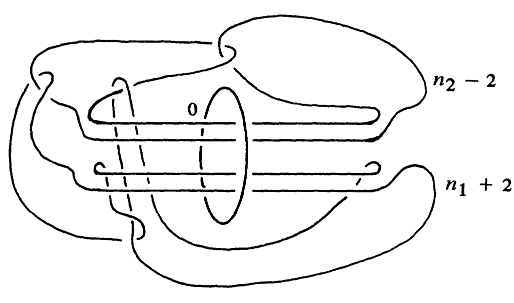


FIGURE 2.5

Notice that if $K_1^{(r)}$ and $K_2^{(r)}$ are split then the bands can be chosen so that $c_{1_i} \#_b c_{2_i}$ is the ordinary connected sum $c_{1_i} \# c_{2_i}$. Furthermore each surgery coefficient is n_i since $\mathrm{lk}(c_{1_i}, c_{2_i}) = 0$ for all i. Finally all the surgery curves can be "pulled through" (i.e. isotoped away from) the zero surgered unknot. This is in keeping with the fact that in this case

$$\mathcal{S}(K_1^{(r)}, K_2^{(r)}; A_1, \ldots, A_r) \simeq S^1 \times S^2 \# M(K_1^{(r)}, K_2^{(r)}; A_1, \ldots, A_r).$$

We now see how to pass from any circular chain of sewn-up r-link exteriors whose homeomorphisms correspond to matrices to a framed link. Of course in the case $r = 1$ this includes all circular chains.

Now suppose that g is an arbitrary orientation reversing homeomorphism. We would like to find a framed link description for $M(K_1^{(r)}, K_2^{(r)}; g)$ or $\mathcal{S}(L^{(r)}; g)$. According to Lickorish [9] g may be written as the reflection ρ followed by Dehn twists about the meridians m_i, longitudes l_i and the additional curves $\gamma_{i,i+1}$ shown in Figure 2.6. Hence g might be written as

$$\begin{aligned} g &= \begin{pmatrix}\text{twists about}\\ \gamma\text{'s}\end{pmatrix} \circ \begin{pmatrix}\text{twists about}\\ l\text{'s and } m\text{'s}\end{pmatrix} \circ \cdots \circ \begin{pmatrix}\text{twists about}\\ \gamma\text{'s}\end{pmatrix} \circ \rho \\ &= \left(\begin{pmatrix}\text{twists about}\\ \gamma\text{'s}\end{pmatrix} \circ j\right) \circ j \circ \left(\begin{pmatrix}\text{twists about}\\ l\text{'s and } m\text{'s}\end{pmatrix} \circ j\right) \circ j \circ \cdots \\ &\qquad \cdots \circ \left(\begin{pmatrix}\text{twists about}\\ \gamma\text{'s}\end{pmatrix} \circ \rho\right). \end{aligned}$$

Therefore, by Lemma 2.1, g may be factored through Hopf r-link exteriors with each new sewing homeomorphism one of the four types listed below.

(i) (twists about γ's) $\circ\, j$
(ii) (twists about γ's) $\circ\, \rho$
(iii) (twists about l's and m's) $\circ\, j$
(iv) (twists about l's and m's) $\circ\, \rho$.

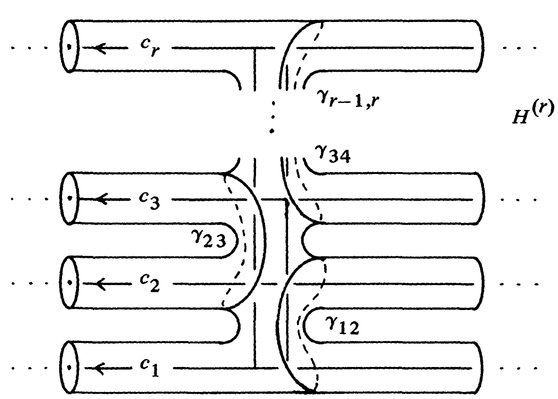


FIGURE 2.6

But j itself is a homeomorphism of type (iv) and hence every homeomorphism of type (iii) reduces to one of type (iv). We have already seen how to obtain a framed link in this case. Thus it only remains to show how to pass to a framed link description of $M(K_1^{(r)}, K_2^{(r)}; g)$ or $\mathbb{S}(L^{(r)}; g)$ when g is a homeomorphism of types (i) or (ii). By further introductions of the homeomorphism j and factoring through Hopf r-link exteriors we may further reduce the situation to the case where g is a homeomorphism of one of the following two types.

(i′) (a single twist about $\gamma_{i,i+1}$) $\circ\, j$

(ii′) (a single twist about $\gamma_{i,i+1}$) $\circ\, \rho$.

Now suppose $g = \gamma_{i,i+1}^{\pm 1} \circ j$ where $\gamma_{i,i+1}^{\pm 1}$ denotes a right- or left-handed Dehn twist, respectively, about $\gamma_{i,i+1}$. Then $g = (\gamma_{i,i+1}^{\pm 1} \circ \rho) \circ j \circ (j \circ \rho \circ j)$. But $j \circ \rho \circ j$ can be expressed as a homeomorphism of type (iv) since

$$\begin{aligned} j \circ \rho \circ j &= \left(J\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \ldots, J\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right) \circ \rho \circ \rho \circ (J, \ldots, J) \\ &= \left(J\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} J, \ldots, J\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} J \right) \\ &= \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \\ &= \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \ldots, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right) \circ \rho. \end{aligned}$$

Hence by an additional factoring through a Hopf r-link exterior we see that it only remains to consider the case where g is a homeomorphism of type (ii′).

Let $L^{(r)} = \{K_1^{(r)}, K_2^{(r)}\}$ be an r-link exterior and suppose $g\colon \partial N(K_1^{(r)}) \to \partial N(K_2^{(r)})$ is given by $g = \gamma_{i,i+1}^{\pm 1} \circ \rho$. This homeomorphism extends to the solid handlebody $N(K_1^{(r)})$, so again we may think of sewing up the r-link exterior by attaching a copy of $H^{(r)} \times I$ to B^4 along $N(K_1^{(r)})$ and $N(K_2^{(r)})$. Suppose $H^{(r)} \times \{0\}$ is attached to $N(K_1^{(r)})$ via the identity and $H^{(r)} \times \{1\}$ is attached to $N(K_2^{(r)})$ via g.

Instead of thinking of $H^{(r)}$ as a single 0-handle union r 1-handles as we did before, think of $H^{(r)}$ as decomposed as in Figure 2.7. What was formerly the 0-handle has been broken into three 0-handles (or two if $\gamma_{i,i+1} = \gamma_{12}$ or $\gamma_{r-1,r}$) and two 1-handles (or one). Hence we may think of attaching $H^{(r)} \times I$ to the 4-ball in two steps. First the 1-handles are attached and then the 2-handles. Since $g = \gamma_{i,i+1}^{\pm 1} \circ \rho$, two of the 1-handles are attached in the ordinary way and the 2-handles that pass over them do so just as before. One of the 1-handles however has its feet attached not

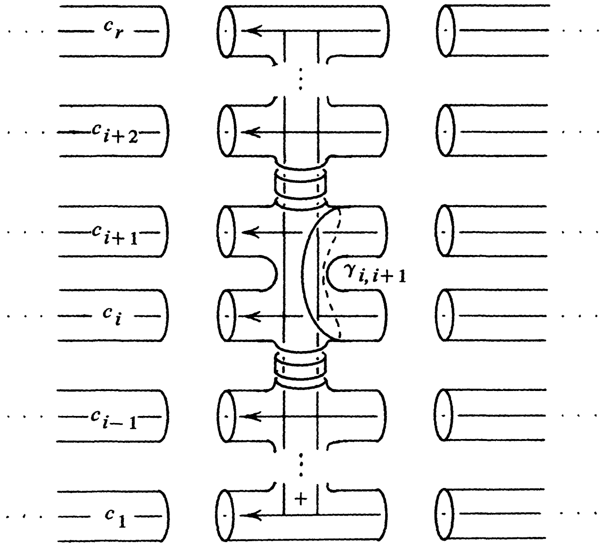


FIGURE 2.7

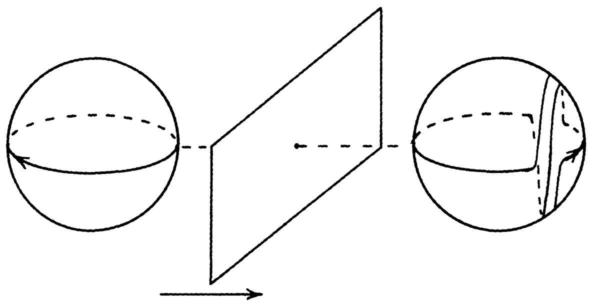


FIGURE 2.8

simply by reflection but also by the Dehn twist about $\gamma_{i,i+1}$. Considering only this 1-handle we may represent it as in Figure 2.8. The associated 3-manifold is obtained by removing the interiors of the feet of the 1-handle and identifying the two S^2 boundary components by reflection through an equidistant S^2 composed with the Dehn twist. The action of the Dehn twist can be assumed to lie entirely in a narrow annulus A so that off of A the spheres are identified by reflection. If we perform the identification off of A leaving only the identification on A to yet take place we arrive at the situation illustrated in Figure 2.9. The sphere on the left and the inner sphere on the right are identified simply by reflection, just as an ordinary 1-handle would be attached. Furthermore the identification along A that still remains to be accomplished amounts to simply doing ± 1 surgery along a curve just beneath the centerline of A. ($+1$ surgery if $g = \gamma_{i,i+1}^{1} \circ \rho$ and -1 surgery if $g = \gamma_{i,i+1}^{-1} \circ \rho$.) Hence we may obtain a framed link description of $\mathcal{S}(L^{(r)};\, g)$ as illustrated by the following example.

Consider the 4-link exterior and homeomorphism shown in Figure 2.10. Adding the 1 and 2-handles gives the handlebody description illustrated in Figure 2.11. We may trade the 1-handles for 2-handles to obtain the framed link shown in Figure 2.12. But this framed link is equivalent to the framed link in Figure 2.13.

Finding a framed link description of $M(K_1^{(r)}, K_2^{(r)};\, g)$ is similar. The difference is that the zero framed unknot does not appear as below and the surgery coefficients do not involve $\operatorname{lk}(c_{1i}, c_{2i})$.

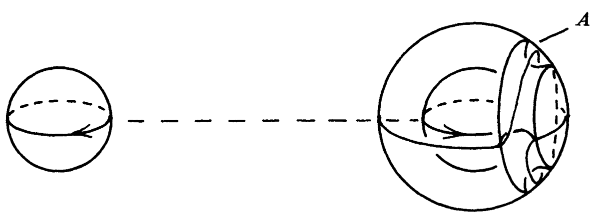


FIGURE 2.9

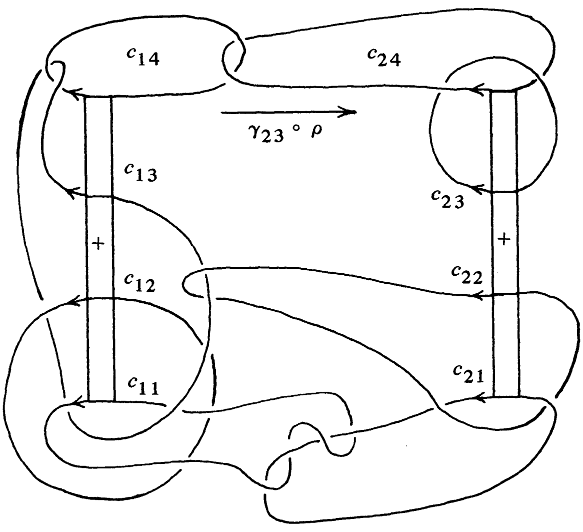


FIGURE 2.10

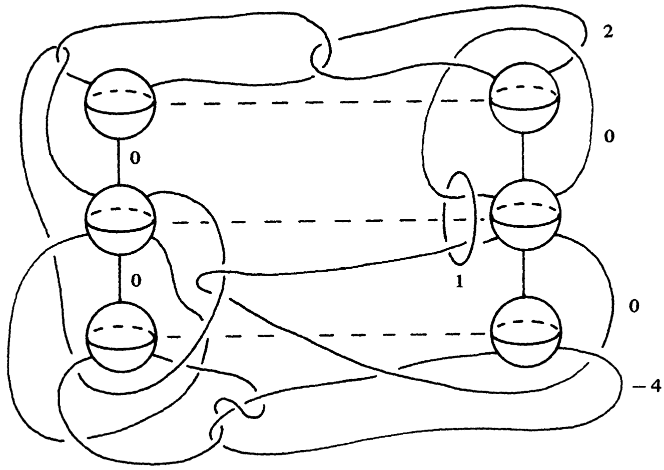


FIGURE 2.11

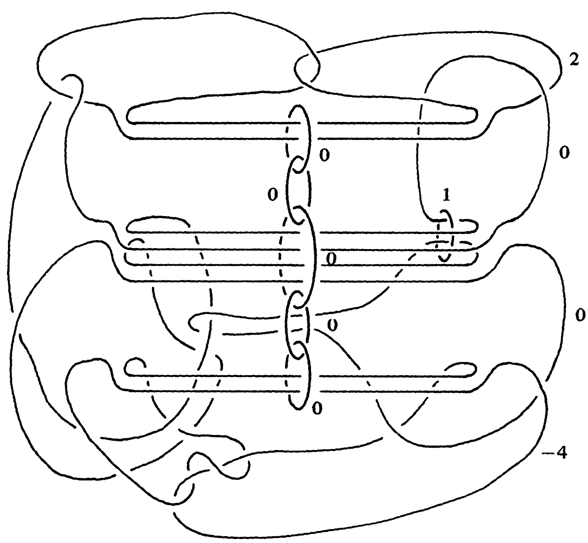


FIGURE 2.12

Notice that the ± 1 framed unknot that appears in the final framed link as a result of the Dehn twist about $\gamma_{i,i+1}$ can be blown down. This yields a framed link that represents a sewn-up r-link exterior whose sewing homeomorphism is of type (iv). This suggests that homeomorphisms of type (iv) alone might suffice to build all possible sewn-up r-link exteriors. This is indeed the case and is proven in the following Proposition.

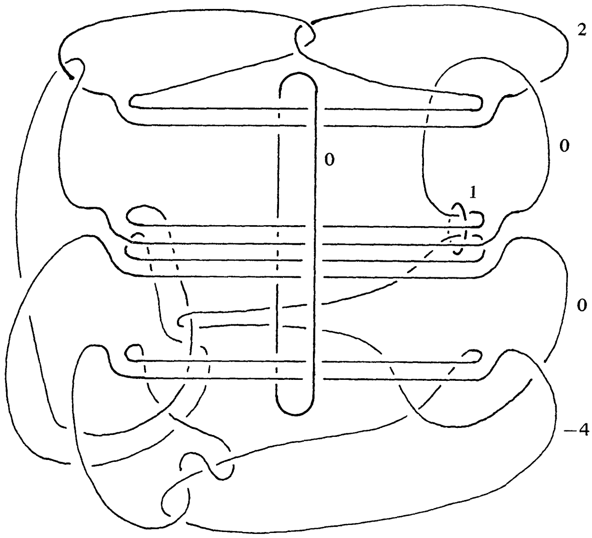


FIGURE 2.13

PROPOSITION 2.4. *Suppose M is an arbitrary circular chain of sewn-up r-link exteriors. Then there exists some s-link $L^{(s)}$ and homeomorphism $(A_1, A_2, \ldots, A_s)$ such that $M \cong \mathbb{S}(L^{(s)}; A_1, \ldots, A_s)$. Similarly, suppose N is an arbitrary chain of sewn-up r-link exteriors. Then there exist some t-knots $K_1^{(t)}$ and $K_2^{(t)}$ and a homeomorphism $(B_1, \ldots, B_t)$ such that $N \cong M(K_1^{(t)}, K_2^{(t)}; B_1, \ldots, B_t)$.*

Proof. Suppose M is an arbitrary circular chain of sewn-up r-link exteriors. We have seen how to pass to a framed link L representing M. One component of L, say C, is unknotted and has a framing of zero. Every other component of the link passes through C either geometrically zero times or algebraically zero times and geometrically twice. By means of an isotopy that leaves C fixed we may change L so that every component other than C passes through C algebraically zero times and geometrically twice. Hence L represents a sewn-up s-link exterior for some s-link $L^{(s)}$ and homeomorphism $(A_1, A_2, \ldots, A_s)$.

The proof in the case of an arbitrary chain N is similar. □

3. The interface surface. Suppose that $\mathbb{S}(K_1^{(r)}, K_2^{(r)}; g)$ is a sewn-up r-link exterior. Then $\partial N(K_1^{(r)})$ and $\partial N(K_2^{(r)})$ are identified to form what we shall call the *interface surface* S in $\mathbb{S}(K_1^{(r)}, K_2^{(r)}; g)$. We have seen how to construct a framed link representing the sewn-up r-link exterior and it is not hard to visualize S within this framed link. Let's assume that $g = (A_1, \ldots, A_r)$ and that each

$$A_i = \begin{pmatrix} -1 & 0 \\ -n_i & 1 \end{pmatrix}.$$

Again, think of sewing up the r-link exterior by first adding one 1-handle and then r 2-handles to the 4-ball. After the addition of the 1-handle think of $\partial N(K_i^{(r)})$ as two pieces, the boundary of one foot of the 1-handle minus $2r$ meridional disks of $N(K_i^{(r)})$, which we shall call Y_i, and the rest of $\partial N(K_i^{(r)})$, which we shall call X_i. So X_i consists of r tubes, $S^1 \times I$, one on each handle while Y_i is a 2-sphere with $2r$ holes. After the r 2-handles are added, $\partial N(K_i^{(r)})$ still appears the same. Finally when the 1-handle is traded for a 2-handle, Y_i is traded for the core of that 2-handle union a punctured spanning disk of the unknot along which that 2-handle is attached. The following example illustrates this process.

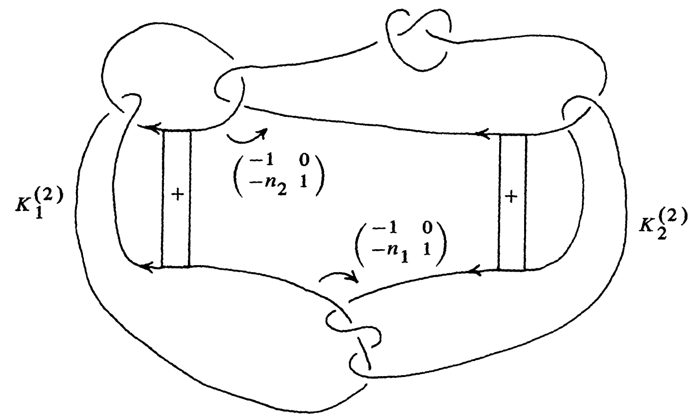


FIGURE 3.1

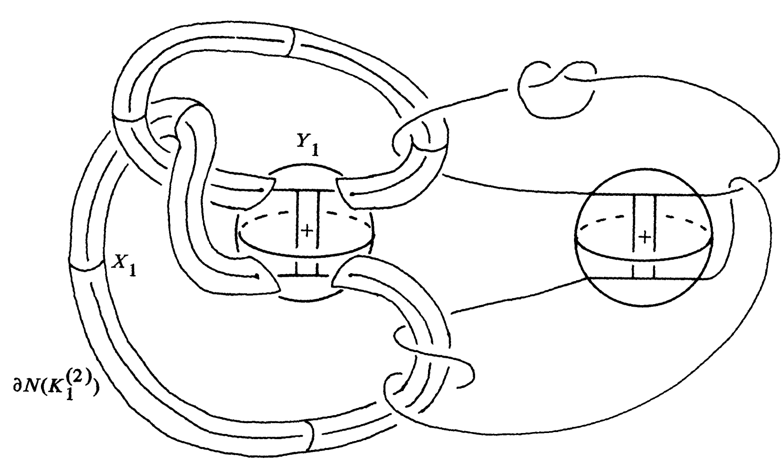


FIGURE 3.2

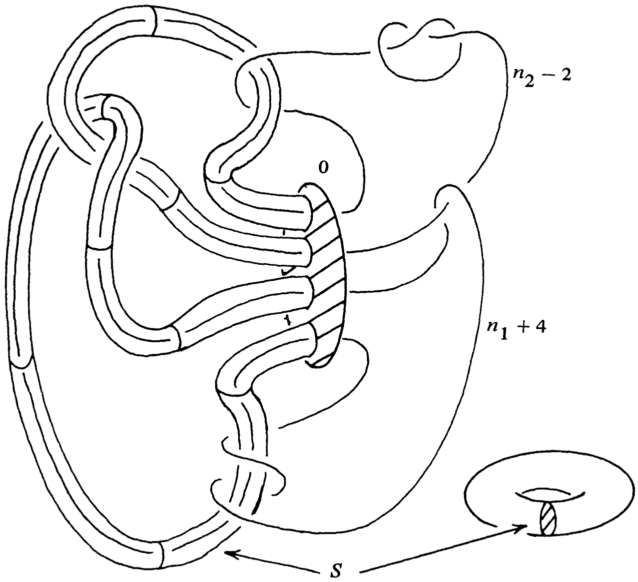


FIGURE 3.3

Consider the 2-link exterior and homeomorphism in Figure 3.1. After the addition of the 1-handle, $\partial N(K_1^{(2)})$ appears as in Figure 3.2. When the 1-handle is traded for a 2-handle, $\partial N(K_1^{(2)})$ becomes the surface pictured in Figure 3.3 union a meridional disk of the solid torus that is sewn in to replace the tubular neighborhood of the unknot.

In the above example, if we had begun by considering $\partial N(K_2^{(2)})$ we would have obtained an apparently different surface for S. There is no ambiguity however, because the two surfaces are isotopic in $\mathbb{S}(K_1^{(2)}, K_2^{(2)}; A_1, A_2)$. Actually X_1 and X_2 are isotopic rel $\partial X_1 = \partial X_2$ as can be seen by sliding over the 2-handles.

Notice that the manifold produced by doing all but the zero surgery may be a homology sphere. In this case S consists of a Seifert surface for the knot on which zero surgery is performed, together with a meridional disk of the solid torus used to do the surgery.

4. A description of K_0 as a sewn-up r-link exterior. If K is a knot in S^3 then we shall denote by K_0 the 3-manifold obtained by performing zero surgery on K.

PROPOSITION 4.1. *If K is any knot in S^3 then $K_0 \simeq \mathbb{S}(L^{(r)}; A_1, \ldots, A_r)$ for some r-link $L^{(r)}$ and homeomorphism $(A_1, \ldots, A_r)$. Furthermore the interface surface S is a Seifert surface of K union a meridional disk of the solid torus which is attached to $E(K)$ so as to form K_0.*

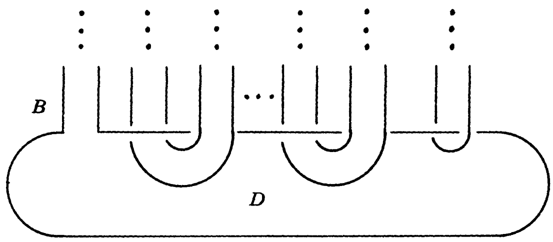


FIGURE 4.1

Proof. Suppose that D is a 2-disk and that B is a band, i.e. 2-disk, such that $\partial D \cap \partial B$ is a small arc and $\partial D \cap \operatorname{int} B$ is a single point. Then $\partial D \cup \partial B - (\partial D \cap \partial B)$ is a knot of unknotting number one and may be pictured as in Figure 4.1.

Now it is shown in [6] that any knot K can be obtained by attaching n disjoint bands to a disk D as described above where n is the unknotting number of K. With the addition of more bands the ribbon type intersections between the bands and D can be eliminated. To see this consider a band B having ribbon intersections with D. Starting at the "foot" of B, i.e. where $\partial B \cap \partial D \neq \varnothing$, proceed along B to the first ribbon intersection with D. Introduce two more bands to eliminate this intersection as shown in Figure 4.2. Now slide the feet of these bands along B back to the foot of B and then onto ∂D. These two new bands have no ribbon intersections with D. The next ribbon intersection of B with D, if any exists, can now be eliminated by the introduction of yet another pair of bands. Hence we may assume that K is obtained from D by attaching many disjoint bands each of which has no ribbon intersection with D.

Now K can be untied by introducing surgery curves that each link K zero times and have surgery coefficients of ± 1. Figure 4.3 shows how these must appear, depending on how many half twists are in each band. Straightening out K yields the surgery description shown in Figure 4.4.

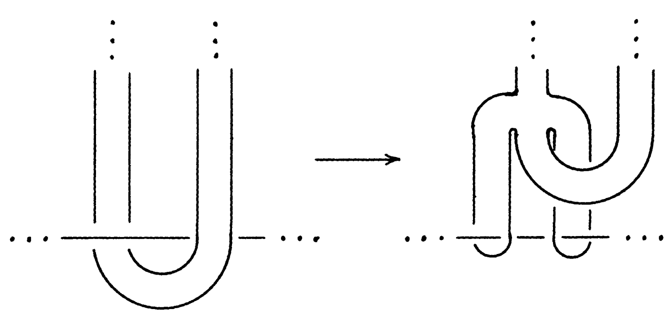


FIGURE 4.2

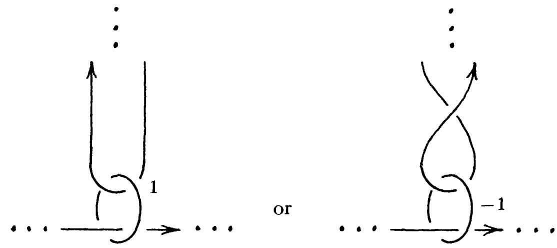


FIGURE 4.3

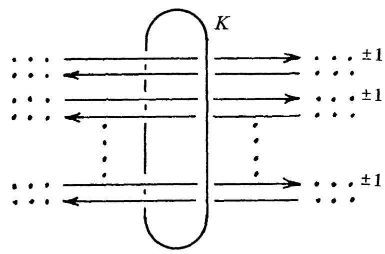


FIGURE 4.4

Finally zero surgery on K can be realized by adding a zero surgery coefficient to K in Figure 4.4 since all the other surgery curves link K zero times. This framed link is now like one obtained from a sewn-up r-link exterior. The interface surface is clearly a Seifert surface of K union a meridional disk of the solid torus which has been attached to $E(K)$. $\square$

5. A description of M^3 as a sewn-up r-link exterior. It is not hard to see that the rank of the first homology of any sewn-up r-link exterior must be one or more. (A simple Mayer-Vietoris sequence argument can be used to show that the first homology can be mapped onto $\mathbf{Z}$.) The following theorem states that the converse of this fact is true.

THEOREM 5.1. *A closed connected orientable* 3-*manifold* M^3 *is homeomorphic to a sewn-up r-link exterior if and only if* $\operatorname{rank}(H_1(M)) \geq 1$.

COROLLARY 5.2. *If K is a knot in a homology sphere H^3 then* $K_0 \simeq \mathfrak{S}(L^{(r)}; g)$ *for some r-link $L^{(r)}$ and homeomorphism g.*

Notice that if M is a sewn-up r-link exterior then by Proposition 2.4 we may assume that the sewing homeomorphism is of the form $(A_1, \ldots, A_r)$. This will also follow from the proof of the theorem.

The corollary follows easily from the theorem since $H_1(K_0) \simeq \mathbf{Z}$.

Before proceeding with the proof of the theorem some preliminary results are necessary.

Suppose L is an oriented framed link in S^3. Let $B = (b_{ij})$ be the linking matrix of L. So b_{ij} is the linking number of the ith component of L with the jth component of L $(i \neq j)$ and b_{ii} is the surgery coefficient of the ith component.

LEMMA 5.3. *Suppose L is an oriented framed link with linking matrix B. If the kth row and kth column of B consist entirely of zeros for some k then L represents a sewn-up r-link exterior* $\mathbb{S}(L^{(r)}; A_1,\ldots,A_r)$.

Proof. Let $L = \{K_1, K_2,\ldots,K_n\}$. By relabeling if necessary we may assume that $b_{j1} = b_{1j} = 0$ for all j. Untie K_1 as described in the proof of Proposition 4.1. This will leave K_1 as an unknot with zero surgery coefficient, many ± 1 surgered curves that each pass through K_1 exactly twice, once in each direction, plus all the other original components of L. These components still each link K_1 zero times and hence must each pass through K_1 an even number of times, half in each direction. The framed link must appear as it does in Figure 5.1.

Suppose $2m$ strands of K_2 pass through K_1 with $m > 1$. Two of the strands must be connected by an arc α of K_2 such that α does not pass through K_1 and furthermore the two strands pass through K_1 in opposite directions.

Now α itself may be knotted and linked with other strands of K_2 or components of the link. Consider all the undercrossings of α and in particular those undercrossings where α does not cross under itself. These crossings can be changed to overcrossings by the introduction of $+1$

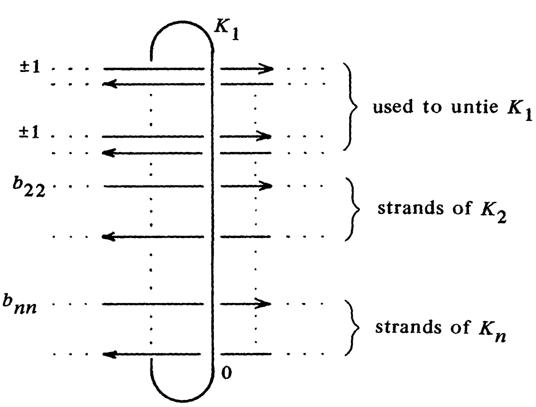


FIGURE 5.1

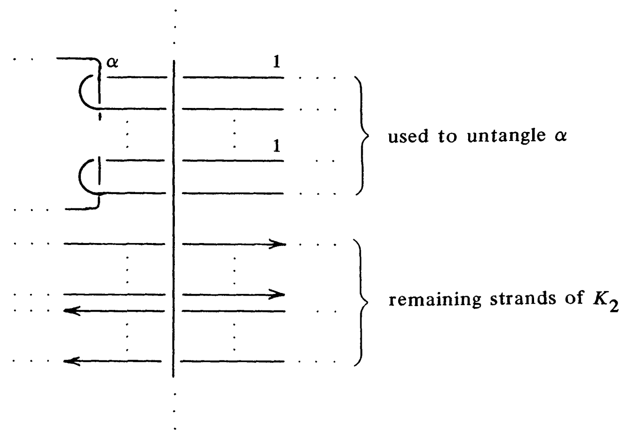


FIGURE 5.2

surgered curves. This will not change the zero surgery coefficient on K_1. Now α can be pulled through K_1, dragging the $+1$ surgered curves along with it and so yielding the framed link in Figure 5.2. Clearly this process can be continued until finally every component of the framed link passes through K_1 exactly twice, once in each direction. The framed link is now in the form of a sewn-up r-link exterior $\mathbb{S}(L^{(r)}; A_1,\dots,A_r)$. $\square$

We must now understand the effect of sliding handles of L on the linking matrix B. Let E_{ij}, $i \neq j$, be the square matrix with a 1 in the ith row and jth column and zeroes elsewhere. If the jth handle of L is slid over the ith handle then the new linking matrix is $(I \pm E_{ji})B(I \pm E_{ij})$. Notice that the symmetry, determinant and rank of B are all preserved.

LEMMA 5.4. *Suppose A is an $n \times n$ symmetric matrix over $\mathbf{Z}$ with* $\det A = 0$. *Let* $r = \operatorname{rank} A$. *Then A can be transformed by handle slides to a matrix of the form* $\left(\begin{smallmatrix} B & 0 \\ 0 & 0 \end{smallmatrix}\right)$ *where B is an $r \times r$ matrix and* $\det B \neq 0$.

Proof. Since $\det A = 0$ there exists a nonzero primitive vector $x \in \mathbf{Z}^n$ such that $Ax = 0$. Now there exists $Q = (I \pm E_{ij}) \cdots (I \pm E_{k1})$ such that $Q(0,0,\dots,0,\pm 1)^t = x$. To see this suppose $x = (a_1, a_2,\dots,a_n)^t$, not all $a_i = 0$. Hence some a_i is nonzero and smallest in absolute value. By multiplying x on the left by matrices of the form $(I \pm E_{ji})$, multiples of a_i may be added or subtracted from all the other nonzero entries until all of them are less than a_i in absolute value. If a_i is the only nonzero entry remaining then $a_i = \pm 1$ since x is primitive. If not then some other a_j is now the smallest nonzero entry and the argument may be repeated. Eventually this process must yield a vector with only one nonzero entry of

± 1. Finally, multiplication on the left by $(I - E_{jn})(I + E_{nj})$ will move the ± 1 from the jth position to the nth position.

Now $Q^t A Q(0, 0, \ldots, 0, \pm 1)^t = Q^t A x = Q^t \cdot 0 = 0$. Hence

$$Q^t A Q = \left(\begin{array}{c|c} & 0 \\ B & \vdots \\ & 0 \\ \hline 0 \cdots 0 & 0 \end{array} \right).$$

If $\det B = 0$ then the process may be continued in the obvious way. □

Proof of Theorem 5.1. Suppose $\operatorname{rank}(H_1(M)) \geq 1$. M has some framed link representation L with linking matrix B. Now $\det B = 0$ since the rank of $H_1(M)$ is the nullity of B. Hence L can be transformed by handle slides until B satisfies the hypothesis of Lemma 5.3. □

6. Applications to property R. A knot K in S^3 is said to have *Property R* if K_0 is not homeomorphic to $S^1 \times S^2$. Related to this is the concept of homotopy Property R and the Poenaru Property. K is said to have *homotopy Property R* if K_0 is not homotopy equivalent to $S^1 \times S^2$ and is said to have the *Poenaru Property* if there is no properly embedded planar surface in $E(K)$ whose boundary circles are longitudes of K having algebraic sum one. It is known that if K has the Poenaru Property then it has homotopy Property R, and clearly if K has homotopy Property R then it has Property R. However it is not known whether the reverse implications are true. (See [**4**, problem 1.17] and [**7**, lemma 1].)

The unknot fails to have Property R and is conjectured to be unique in this respect. Kirby and Melvin [**5**] have shown that only slice knots can possibly fail to have Property R. In an addendum to [**5**] credited to Taylor and Freedman it is further shown that if a knot fails to have Property R it must be superslice (that is, of the form $S^3 \cap S$ where S is an unknotted 2-sphere in S^4 invariant under the reflection of S^4 through its equator S^3). Brakes [**2**] has since shown that many superslice knots have the Poenaru Property.

The method employed by Brakes was the following. Starting with a specific knot K it was shown that K_0 was a sewn-up strongly nonsplit r-link exterior. Brakes then appealed to the following lemma.

LEMMA 6.1. *If $L^{(r)}$ is a strongly nonsplit r-link then $\pi_2(\mathbb{S}(L^{(r)}; g)) = 0$ for all orientation reversing homeomorphisms g.*

Proof. The proof is straightforward and is given in [**2**]. □

Now it $\pi_2(K_0) = 0$ it is not hard to show that K has the Poenaru Property. (See [2].)

This technique is nicely illustrated in the following proof that all doubled knots have the Poenaru Property. (This fact is originally due to L. Moser [10].)

Consider zero surgery on the doubled knot shown in Figure 6.1. It is easy to see that this link is strongly nonsplit if and only if the original knot is nontrivial. Hence all nontrivial doubled knots have the Poenaru Property.

As a further example consider zero surgery on the superslice knot shown in Figure 6.2. Now drawing only 1-dimensional cores of the handlebodies we see that the 2-link exterior is isotopic to the 2-link exterior in Figure 6.3. This 2-link can be shown to be strongly nonsplit and hence the original knot K has the Poenaru Property.

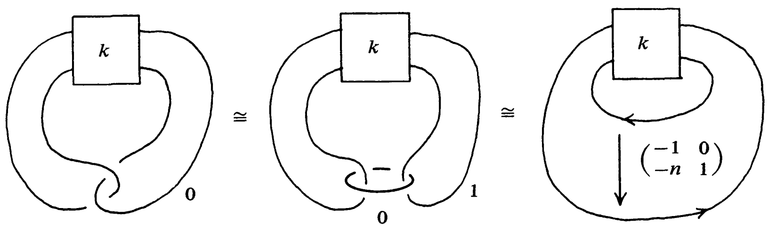


FIGURE 6.1

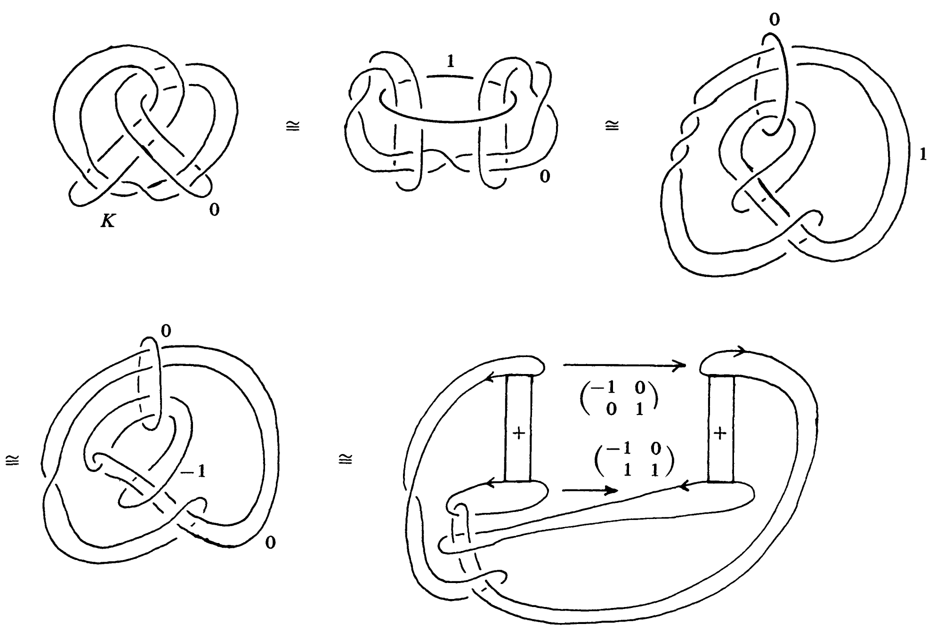


FIGURE 6.2

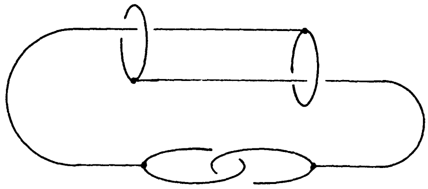


FIGURE 6.3

This knot K was shown to have the Poenaru Property by Brakes. However, instead of showing that K_0 was a sewn-up 2-link exterior Brakes showed that a 2-fold cover M_2 of K_0 was a sewn-up strongly nonsplit 2-link exterior. Hence $\pi_2(K_0) \simeq \pi_2(M_2) \simeq 0$ and so K has the Poenaru Property.

Notice that in each of the previous examples the interface surface of the sewn-up r-link exterior consists of a Seifert surface of K union a meridional disk of the solid torus sewn to $E(K)$ to form K_0.

Suppose K is a knot in S^3 with Seifert surface F. If we attach a solid torus to $E(K)$ to form K_0 then F union a meridional disk of that torus is a closed 2-sided surface $\bar{F}$ embedded in K_0. $\bar{F}$ does not separate K_0 so that we may cut K_0 open along $\bar{F}$ to obtain a connected manifold X with two homeomorphic boundary components. When is X an r-link exterior? Or, in other words when can X be embedded in S^3 so that its complement is two solid handlebodies? If X is an r-link exterior then clearly K_0 is a sewn-up r-link exterior.

For example if K is a fibered knot then there exists some Seifert surface F such that X is a product and hence homeomorphic to a Hopf r-link exterior. The Hopf r-link is clearly strongly nonsplit and so all fibered knots have the Poenaru Property.

We may reinterpret Proposition 4.1 as saying that given any knot K there exists some Seifert surface F such that we may cut K_0 open along $\bar{F}$ to obtain an r-link exterior. If the r-link so obtained is split then $K_0 \simeq S^1 \times S^3 \# H^3$ for some homology sphere H^3. Furthermore it is relatively easy to see that K fails to have the Poenaru Property. For suppose that upon cutting K_0 open along $\bar{F}$ the split r-link $L^{(r)} = \{K_1^{(r)}, K_2^{(r)}\}$ results. Now $K_0 \simeq E(K) \cup S^1 \times D^2$ where the solid torus is sewn to $E(K)$ according to the zero framing. Thus we may think of $E(L^{(r)})$ as the union of $E(K)$ cut open along F, and the solid torus cut open along a meridional disk. These two pieces are $\pi^{-1}(E(K))$ and $\pi^{-1}(S^1 \times D^2)$, respectively, where π: $E(L^{(r)}) \to K_0$ is the obvious quotient map. Let α be the core of $S^1 \times D^2$. Then $\beta = \pi^{-1}(\alpha)$ is a properly embedded arc whose endpoints lie in separate components of $\partial E(L^{(r)})$

and which are identified by π. Hence $\pi^{-1}(S^1 \times D^2)$ is a tubular neighborhood of β. Since $L^{(r)}$ is split there exists some sphere $\bar{S}$ that separates $K_1^{(r)}$ from $K_2^{(r)}$. We may assume that β meets $\bar{S}$ transversely in an odd number of points. Hence $S \simeq \pi(\bar{S} \cap \pi^{-1}(E(K)))$ is a planar surface properly embedded in $E(K)$ such that ∂S is a collection of longitudes on K. Furthermore these longitudes have algebraic sum ± 1 since β has algebraic intersection ± 1 with $\bar{S}$. Hence K fails to have the Poenaru Property. We have proven the following proposition.

PROPOSITION 6.2. *Let K be a knot in S^3 and F a Seifert surface for K. If K_0 can be cut open along $\bar{F}$ to obtain a split r-link exterior then $K_0 \simeq S^1 \times S^2 \# H^3$ for some homology sphere H^3 and K fails to have the Poenaru Property.* □

The proof of Proposition 4.1 gives an algorithm by which we may represent K_0 as a sewn-up r-link exterior for any knot K. When is this r-link strongly nonsplit? As a partial answer to this question we begin with the following lemma.

LEMMA 6.3. *Let K be a knot in S^3 and express K_0 as a sewn-up r-link exterior according to the algorithm given in §4. If the r-link so obtained is split then K is the unknot.*

COROLLARY 6.4. *Given any knot K in S^3 (including the unknot), $K_0 \simeq \mathbb{S}(L^{(r)}; A_1, \ldots, A_r)$ for some nonsplit r-link $L^{(r)}$ and some homeomorphism $(A_1, \ldots, A_r)$.*

Proof of Lemma 6.3. Proceeding as in the proof of Proposition 4.1, K has a surgery description like that shown in Figure 6.4. Each surgery curve has surgery coefficient ± 1 and links K zero times. Above the dashed line labeled "a" all the bands are disjoint. If there is only one surgery curve then K is a doubled knot and we have already seen that the lemma is true. So we may assume that there are two or more surgery curves.

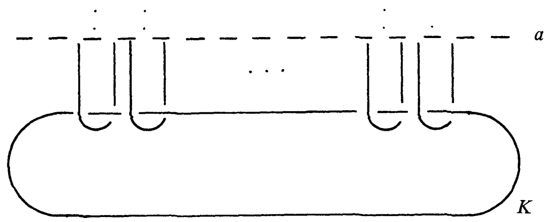


FIGURE 6.4

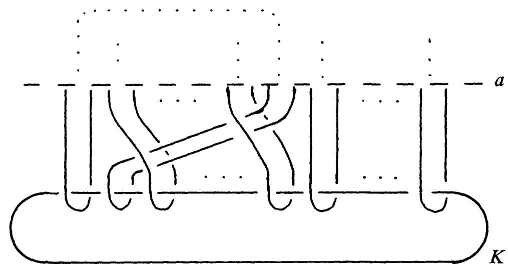


FIGURE 6.5

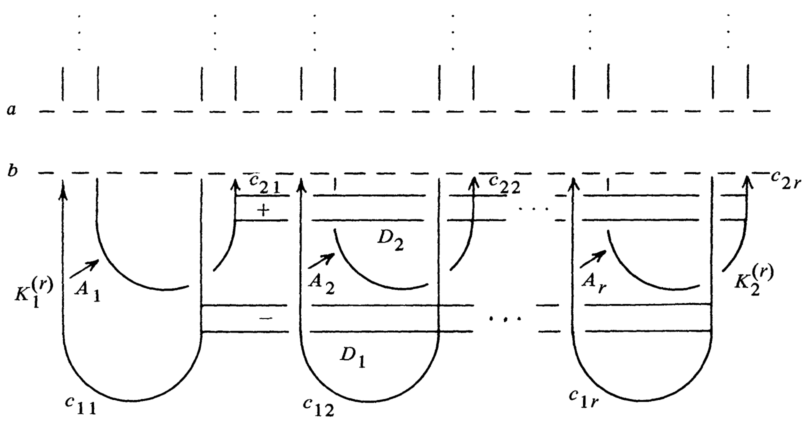


FIGURE 6.6

In order to pass to a sewn-up r-link exterior for K_0 we must first bring the "ends" of each band together on K. Starting with the leftmost band bring its other end over to it as shown in Figure 6.5. Continuing in this manner we may bring the ends of all the bands together.

K_0 is then equivalent to the sewn-up r-link exterior shown in Figure 6.6. Above the dashed line a the bands are disjoint and appear exactly as in Figure 6.4. Between lines a and b the bands have ribbon intersections as described earlier.

Now c_{11} and c_{21} cobound an annulus A as illustrated in Figure 6.7. As indicated in Figure 6.7 we may label one side of A positive and the other negative. We may think of A as being properly embedded in the r-link exterior.

Since we are assuming that the r-link is split it follows that the link $\{c_{11}, c_{21}\}$ is split. Hence both c_{11} and c_{21} are unknotted and A is embedded in an unknotted untwisted way. We will show that $c_{11} \#_b c_{21}$ actually slips off of K as it appears in Figure 6.4 and can be blown down without affecting the rest of the link.

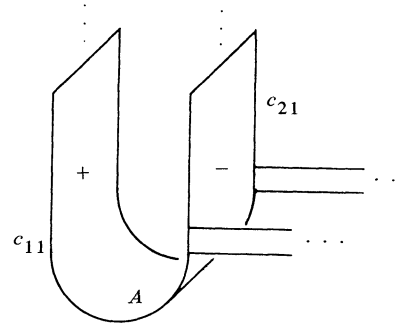


FIGURE 6.7

Now there exists a 2-sphere S that separates the components of the r-link. Hence $S \cap A \neq \varnothing$. We may assume that S meets A transversely in a collection of circles. Consider an innermost circle of intersection on S. This circle bounds a disk d on S such that $\operatorname{int}(d)$ lies in the r-link exterior and does not meet A. Suppose that ∂d is trivial on A. Then ∂d bounds a disk d' on A. Now $d \cap d' = \partial d = \partial d'$ so that $S' = d \cup d'$ is an embedded 2-sphere in the r-link exterior. S' bounds a ball in the r-link exterior and d may be isotoped across this ball to d' and then through A. This isotopy will not move the r-link although it may move other portions of S. Proceeding in this manner we must arrive at an innermost circle of intersection on S which is a nontrivial curve on A. Thus there exists a disk d embedded in the r-link exterior with ∂d a nontrivial curve on A and $\operatorname{int}(d)$ disjoint from A. We may assume that ∂d is the centerline of A and that in a product neighborhood $A \times [-1, 1]$ of A, d is just $\partial d \times [0, -1]$ or $\partial d \times [0, 1]$.

Suppose first that $d \cap (A \times [-1, 1]) = \partial d \times [0, -1]$ or that in Figure 6.7 d meets A on the negatively labeled side. Consider the rectangular 3-ball B pictured in Figure 6.8. Think of B as $D^2 \times I$ where $D^2 \times \{0\} = D_1$ and $D^2 \times \{1\} = D_2$. Now $\partial d \cap B = \partial d \cap B \cap A$ consists entirely of the arc pictured in Figure 6.8. Furthermore $\operatorname{int}(d)$ misses $(B \cap A) \cup D_1 \cup D_2 \subset \partial B$. Hence we may isotope d, keeping the r-link fixed, so that $d \cap B$ appears as it does in Figure 6.9. Now push d slightly off itself in each direction to obtain two disks d_1 and d_2 each parallel to d. Finally let $d_i' = \overline{d_i - (d_i \cap B)}$. Figure 6.10 shows d_1' and d_2' as they appear in the surgery description of K. We may now band together d_1' and d_2' to obtain a disk $\tilde{d}$ that spans $c_{11} \#_b c_{21}$ as shown in Figure 6.11. Since the disk $\tilde{d}$ lies completely in the complement of the framed link we can clearly blow down $c_{11} \#_b c_{21}$ without affecting the rest of the link.

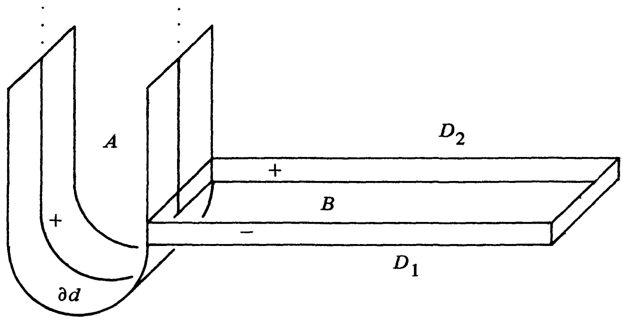


FIGURE 6.8

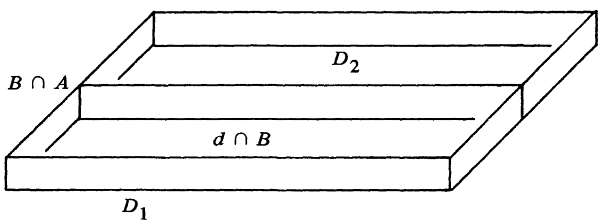


FIGURE 6.9

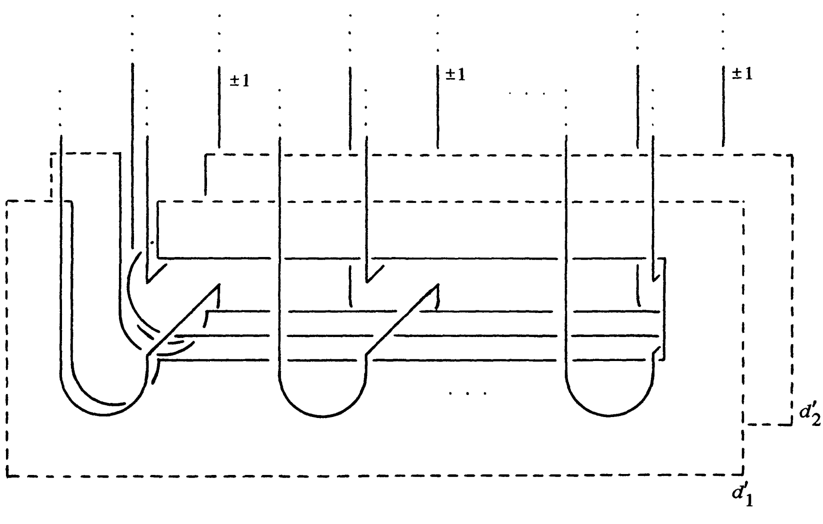


FIGURE 6.10

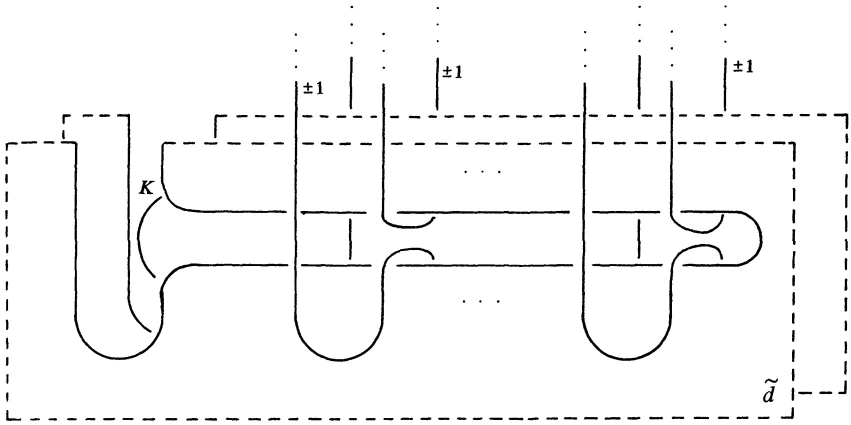


FIGURE 6.11

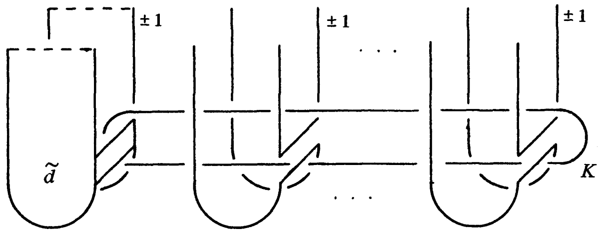


FIGURE 6.12

If $d \cap (A \times [-1, 1]) = \partial d \times [0, 1]$ then we may isotope d so that $d \cap \operatorname{int}(B) = \varnothing$. Let d_1 and d_2 be two parallel copies of d such that d_1 spans c_{11} and d_2 spans c_{21}. Then we may return to a surgery description of K and band connect d_1 to d_2 to obtain the disk $\tilde{d}$ which spans $c_{11} \#_b c_{21}$ as shown in Figure 6.12. Again, since $\tilde{d}$ lies completely in the complement of the framed link we may blow down $c_{11} \#_b c_{21}$ without affecting the rest of the link.

In both cases we have been able to eliminate $c_{11} \#_b c_{21}$ and hence obtain a surgery description of K similar to the original one but with one less surgery curve. This new surgery description yields the sewn-up $(r-1)$-link exterior gotten from Figure 6.6 by simply deleting c_{11} and c_{21}. Hence this $(r-1)$-link is still split and so the process may be repeated until no surgery curves remain and K is obviously the unknot. □

Proof of Corollary 6.4. To prove the corollary it only remains to show that $S^1 \times S^2$ can be realized as a sewn-up r-link exterior employing a nonsplit r-link. Consider the knot $K(k)$ pictured in Figure 6.13. The

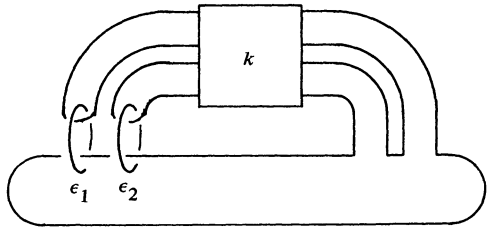


FIGURE 6.13

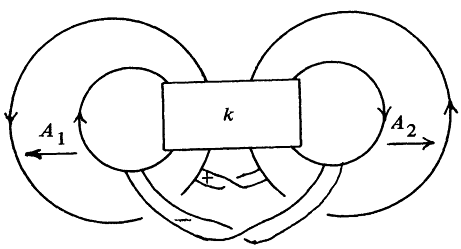


FIGURE 6.14

surgery coefficients ε_i are either ± 1. If $\varepsilon_1\varepsilon_2 = -1$ then $K(k)$ is the unknot regardless of what k is. Now K_0 is the sewn-up 2-link exterior shown in Figure 6.14. We may choose k so that this 2-link is nonsplit and further choose ε_1 and ε_2 so that the sewn-up 2-link exterior is $S^1 \times S^2$. □

So we have seen that the algorithm of §4 always yields a nonsplit r-link if we start with a nontrivial knot. The last example, however, shows that this is insufficient to conclude that the knot has Property R. The following proposition identifies at least one situation when we can be sure the r-link is strongly nonsplit.

PROPOSITION 6.5. *Let K be a knot in S^3 and suppose that K can be obtained from a disk D by attaching r disjoint bands as described in* §4 *with the further restriction that each band has no ribbon type intersections with D. Furthermore suppose that the genus of K is r. Then $\pi_2(K_0) = 0$ and K has the Poenaru Property.*

Proof. Express K_0 as a sewn-up r-link exterior according to the algorithm of §4 and the beginning of the proof of Lemma 6.3 so that the r-link $L^{(r)} = \{K_1^{(r)}, K_2^{(r)}\}$ appears as in Figure 6.6. We have already proven that $L^{(r)}$ is nonsplit. We shall prove that in fact it is strongly nonsplit.

We would like to show that $\partial E(L^{(r)})$ is incompressible in $E(L^{(r)})$. So suppose that d is a properly embedded disk in $E(L^{(r)})$ with $\partial d \subset N(K_1^{(r)})$ say. Let A be the annulus defined in the proof of Lemma 6.3. Put d in regular position with respect to A. Now $d \cap A$ consists of circles and properly embedded arcs having both endpoints on c_{11}. Consider an innermost circle of intersection on d. If this curve is trivial in A we may eliminate it from $d \cap A$ just as we did in the proof of Lemma 6.3. If it is nontrivial we could eliminate $c_{11} \#_b c_{21}$ from the surgery description of K also as we did before. But this is impossible since r is the genus of K. Hence $d \cap A$ consists entirely of arcs. Now consider an innermost arc γ on d. So $\partial d - \partial\gamma$ consists of two arcs δ_1 and δ_2 such that $\gamma \cup \delta_1$ bounds a disk d' in d with $\operatorname{int}(d') \cap A = \varnothing$. Now since $\partial\gamma$ lies in c_{11} there exists an arc μ of c_{11} such that $\gamma \cup \mu$ bounds a disk d'' on A. Now $d' \cup d''$ is a disk with boundary $\mu \cup \delta_1$ on $\partial N(K_1^{(r)})$. Furthermore, by pushing d'' slightly off of A we may assume that $d' \cup d''$ misses A. Now, as in the proof of Lemma 6.3, $d' \cup d''$ can be isotoped out of the ball B of Figure 6.8. Hence we may return to Figure 6.4 with $d' \cup d''$ a disk in $E(K)$ with $\partial(d' \cup d'')$ contained in a genus r Seifert surface F of K. Since the genus of K is assumed to be r, F is incompressible in $E(K)$. Hence $\partial(d' \cup d'')$ bounds a disk in F. Therefore, returning to Figure 6.6, $\partial(d' \cup d'')$ must bound a disk in $\partial N(K_1^{(r)})$. Hence we may isotope d' across to d'' and then through A thus eliminating at least one arc from $d \cap A$. Thus we may assume that $d \cap A = \varnothing$. Now by repeating the above argument we have that ∂d bounds a disk in $\partial N(K_1^{(r)})$ and so $\partial N(K_1^{(r)})$ is incompressible in $E(L^{(r)})$. Therefore $L^{(r)}$ is strongly nonsplit. Thus $\pi_2(K_0) = 0$ and so K has the Poenaru Property. $\square$

Notice that in general every knot K has a description like that given in Proposition 6.5 except that $r \geq$ genus K. The proposition shows that knots with "minimal" descriptions of this type have the Poenaru Property. This is essentially the best general result we can expect from these methods. For suppose K is a knot of unknotting number n and is represented by a disk D with n bands attached as described in the beginning of the proof of Proposition 4.1. Suppose at least one band has a ribbon intersection with D. Then it is not hard to show that the r-link yielded by the algorithm of §4 will never be strongly nonsplit. The way in which the ribbon intersection is removed by the introduction of two more bands always causes the $\partial E(L^{(r)})$ to fail to be incompressible.

7. The Alexander invariant. Suppose K is a knot in S^3 and $K_0 \simeq \mathfrak{S}(K_1^{(r)}, K_2^{(r)}; A_1,\ldots,A_r)$ for some r-link $L^{(r)} = \{K_1^{(r)}, K_2^{(r)}\}$ and homeomorphism $(A_1,\ldots,A_r)$. Suppose further that the interface surface of the sewn-up r-link exterior is $\bar{F}$, where F is a Seifert surface for K. We have seen that given any knot K this is always possible. In this section we show how to compute the Alexander invariant of K from the r-link $L^{(r)}$ and homeomorphism $(A_1, \ldots, A_r)$.

In order to construct the infinite cyclic cover $\tilde{X}$ of $E(K)$ we must cut $E(K)$ open along F and glue infinitely many copies of this space together end to end. But as we have already seen in the previous section, $E(K)$ cut open along F is embedded in K_0 cut open along $\bar{F}$. If π: $E(L^{(r)}) \to K_0 \simeq E(K) \cup S^1 \times D^2$ is the quotient map then

$$Y = \pi^{-1}(E(K)) = \overline{E(L^{(r)}) - \pi^{-1}(S^1 \times D^2)}$$

is $E(K)$ cut open along F. Hence to form $\tilde{X}$ we may begin by glueing end to end infinitely many copies of $E(L^{(r)})$ and then removing $\pi^{-1}(S^1 \times D^2)$ from each. Each of the sewing homeomorphisms used in this infinitely long chain will be given by $(A_1, \ldots, A_r)$.

Now Y consists of the complement of the two solid handlebodies $K_1^{(r)}$ and $K_2^{(r)}$ minus a tube that runs between them. Hence Y is the complement of a genus $2r$ handlebody. The first homology of Y is generated by the $2r$ meridians of this handlebody which are just the meridians of $K_1^{(r)}$ and $K_2^{(r)}$. If c_j is the core of the jth handle of $K_1^{(r)}$ let m_j be its meridian and l_j its longitude. Similarly, let m_{r+j} and l_{r+j} be the meridian and longitude of c_{r+j}, the core of the jth handle of $K_2^{(r)}$.

We want to compute $H_1(\tilde{X})$ as a module over $\Lambda(t)$ the ring of Laurent polynomials. To do this let $\{Y_i\}$, $i = \ldots, -1, 0, 1, 2, \ldots$ be infinitely many copies of Y. The first homology of Y_i is freely generated by $\{t^i m_j\}$, $j = 1, \ldots, 2r$. Now Y_0 is sewn to Y_1 via $(A_1, \ldots, A_r)$ thus introducing the relations

$$\begin{pmatrix} l_i \\ m_i \end{pmatrix} = A_i^T \begin{pmatrix} tl_{r+i} \\ tm_{r+i} \end{pmatrix}, \qquad i = 1, \ldots, r.$$

The longitudes however can be written in terms of the meridians. To do this let $B = (b_{ij})$ be the matrix given by

$$b_{ij} = \begin{cases} 0, & i = j, \\ \operatorname{lk}(c_i, c_j), & i \neq j. \end{cases}$$

Then $l_i = \sum_{j=1}^{2r} b_{ij} m_j$. Hence a presentation for $H_1(\tilde{X})$ is

$$H_1(\tilde{X}) \simeq \left\langle m_1, \ldots, m_{2r}; \begin{pmatrix} \sum_{j=1}^{2r} b_{ij} m_j \\ m_i \end{pmatrix} = A_i^T \begin{pmatrix} t \sum_{j=1}^{2r} b_{r+ij} m_j \\ tm_{r+i} \end{pmatrix} \right\rangle.$$

Notice that if t is set equal to one in the above relations then the presentation represents the group $H_1(K_0)/\mathbf{Z} \simeq 0$. This coincides with the fact that $\Delta_K(1) = 1$, where $\Delta_K(t)$ is the Alexander polynomial of K.

Since there are $2r$ generators and $2r$ relations and each term of each relation is at most linear in t, we have that $\deg \Delta_K(t) \leq 2r$.

If $B = 0$ then it is not hard to see that the Alexander polynomial is trivial.

Also if each

$$A_i = \begin{pmatrix} -1 & 0 \\ -n_i & 1 \end{pmatrix}$$

and $L^{(r)}$ is split, then the Alexander polynomial is trivial.

As an example consider the knot K in Figure 6.2. We have that

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So

$$\begin{aligned} H_1(\tilde{X}) &\simeq \langle m_1, m_2, m_3, m_4;\, 0 = -tm_2,\, m_1 = tm_3,\, m_3 = tm_4,\, m_2 = tm_4 \rangle \\ &\simeq \langle m_1;\, m_1 = 0 \rangle \simeq 0. \end{aligned}$$

So $\Delta_K(t) = 1$, just as we would expect since K is a superslice knot.

8. Questions. (1) Does there exist a knot K such that $K_0 \simeq \mathbb{S}(L^{(r)}; f)$ with $r <$ genus (K)? If not then all nontrivial knots have Property R since $S^1 \times S^2$ is a sewn-up 0-link exterior.

(2) If $K_0 \simeq \mathbb{S}(L^{(r)}; f)$ then does the interface surface of the sewn-up r-link exterior contain a Seifert surface for K? In other words is $E(L^{(r)})$ obtained by cutting K_0 open along $\overline{F}$, where F is a Seifert surface for K? If the answer to this is yes then the answer to (1) is no and hence all nontrivial knots have Property R.

(3) Suppose that F is an incompressible Seifert surface for K. If K_0 can be cut open along $\overline{F}$ to obtain an r-link exterior $E(L^{(r)})$ does this imply that K has Property R? Or is $L^{(r)}$ strongly nonsplit? Of course if F is not incompressible then $L^{(r)}$ is not strongly nonsplit.

(4) Is there a nontrivial knot K such that K_0 cannot be expressed as a sewn-up r-link exterior using a strongly nonsplit r-link?

(5) Is there a set of "moves" between sewn-up r-link exteriors whereby two sewn-up r-link exteriors will be homeomorphic if and only if one can be reached from the other by a finite sequence of these moves?

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