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ON NON-CONTRACTIBLE VALUED MULTIFUNCTIONS

J. F. McCLENDON

The purpose of this paper is to prove various selection and fixed point theorems for *r*-open graph multifunctions.

A set valued function $m: X \to Y$ with $m(x)$ non-empty for all x is a multifunction. The graph $G(m)$ is $\{(x, y) \in X \times Y | y \in m(x)\}$. m is said to be an open graph multifunction if $G(m)$ is open in $X \times Y$. In the present paper we consider a larger class of multifunctions — the *r-open graph* multifunctions. m is r-open graph if its graph is an X-retract of an open subset of $X \times Y$. Actually, many of the results are valid in a somewhat more general setting and these generalizations are discussed in §4. The values of the multifunctions are sometimes assumed contractible and sometimes not. The results of the paper are new even for open graph multifunctions. Note that open graph multifunctions seem to occur in a natural way in mathematical economics [**4**, **15**]. Some of the results of the present paper will be applied in a separate paper to the study of minimax theorems.

Two representative fixed point theorems are the following. The first assumes infinitely connected values and generalizes [**11**, Cor. 3.3].

3.2. THEOREM. *Suppose m: $X \to X$ is an r-open graph multifunction with infinitely connected values. Suppose $m(X) \subset Y \subset X$ with Y an acyclic compact finite dimensional ANR. Then m has a fixed point.*

By way of contrast the next theorem definitely allows noncontractible values.

3.1. THEOREM. *Suppose m: $X \to X$ is an r-open graph multifunction with $m(X) \subset Y \subset X$, Y a compact finite dimensional ANR. Suppose also that $m(x) \subset Y$ is a weak homotopy equivalence for all x in X. Then m has a fixed point.*

Other results are proved which weaken somewhat the requirement on $m(x) \subset Y$ but have other hypotheses.

Section 1 gives definitions and some needed results on fibrations. Selection theorems are given in §2 and fixed point theorems in §3. Section 4 discusses generalizations.

1. Terminology, background facts. Suppose X and Y are topological spaces. Let $\pi = \pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ be the projections. $\mathrm{id} = \mathrm{id}_X \colon X \to X$ is the identity map. If $m \colon X \to Y$ is a multifunction and $S \subset Y$ then $m^{-1}(S) = \{x \in X | m(x) \cap S \neq \varnothing\}$. The graph of m is $G(m) = \{(x, y) | y \in m(x)\}$. Denote the composite $G(m) \subset X \times Y \to X$ by p.

1.1. Definition. A multifunction $m \colon X \to Y$ is *r-open graph* if there is an open nbhd U of $G(m)$ in $X \times Y$ and a map $r \colon U \to G(m)$ with $r(x, y) = (x, y)$ all $(x, y) \in G(m)$ (i.e., a retraction) and satisfying $pr = \pi$. (Call r an *X-retraction*.)

If $G(m)$ is open then $r = \mathrm{id}$ shows that every open graph multifunction is r-open graph. Recall that $m \colon X \to Y$ is lower semi-continuous if m^{-1}(open) is open. It is easily checked that r-open graph multifunctions are lower semi-continuous, so open graph $\subset$ r-open graph $\subset$ lsc. The results of the present paper are all stated for r-open graph multifunctions but are valid, with the same proofs, for somewhat more general situations (see §4).

ANR will mean ANR(metric). Many theorems will have the hypothesis that X is a "compact finite dimensional ANR". The reader may find it helpful to remember that for subsets X of Euclidean space this can be replaced by "compact locally contractible" (see [Borsuk, **1**, p. 122]). One fact about ANR's that will be used is the following.

1.2. Proposition. (*See* [*Borsuk*, **1**, *p.* 122].) *Finite dimensional compact ANR's are exactly the retracts of polyhedra.*

A space X is called *N-connected* if every map $\partial I^{j+1} \to X$ can be extended to a map $I^{j+1} \to X$, $0 \leq j \leq N$. X is *infinitely connected* if it is N-connected for all N. A map $f \colon X \to Y$ is a *weak homotopy equivalence* if $f \colon \pi_k(X, x) \to \pi_k(Y, f(x))$ is isomorphic for all $x \in X$, all k. A map $p \colon T \to B$ has the *covering homotopy property* for a space X if for given maps $H \colon X \times I \to B$ and $u \colon X \to T$ with $pu(x) = H(x, 0)$ there is always a map $G \colon X \times I \to B$ with $pG = H$, $G(x, 0) = u(x)$. Note that if $p \colon T \to B$ has the covering homotopy property for X and $f \colon X \to B$, and $g' \colon X \to T$ are maps such that pg' is homotopic to f then there is a map $g \colon X \to T$, homotopic to g', with $pg = f$.

A map $p\colon T \to B$ is a *Serre fibration* (or weak fibration) if it has the covering homotopy property for all cubes (or all simplexes) (see [**16**, p. 375]). It is shown in [**16**, p. 416, Th. 9] that a Serre fibration has the covering homotopy property for all CW complexes.

If $p\colon T \to B$, let $T(b)$ denote $p^{-1}(b)$. For $E \subset T$, $E \to B$ is the composite $E \subset T \to B$. We will use the following propositions.

1.3. PROPOSITION. (*See Cor.* 3.1 *of* [**12**].) *Suppose $T \to B$ is a Serre fibration, E a B-retract of an open subset of T, and $E(b) \to T(b)$ a weak homotopy equivalence for all b in B. Then $E \to B$ is a Serre fibration.*

Consider $j\colon\ E \subset B \times Z$, $\pi_2\colon\ B \times Z \to Z$, $i = i_b\colon\ E(b) \subset E$. Let $M_k(e) = \text{Image}\,(\pi_2 j)_*\colon \pi_k(E, e) \to \pi_k(Z, \pi_2 e)$.

1.4. PROPOSITION. (*See Cor.* 3.2 *of* [**12**].) *Suppose E is a B-retract of an open subset of $B \times Z$ and $(\pi_2 ji)_*\colon\ \pi_k(E(b), e) \to \pi_k(Z, \pi_2 e)$ monic with image $M_k(e)$ for all b, all e in $E(b)$, all k. Then $E \to B$ is a Serre fibration.*

1.5. PROPOSITION. (*Th.* 3.3 *of* [**12**].) *Suppose E is a B-retract of an open subset of T, where $T \to B$ is a Serre fibration, and each $E(b)$ is infinitely connected. Then $E \to B$ is a Serre fibration.*

Maps are continuous functions. All spaces are topological and assumed to be non-empty.

2. Selection Theorems. Recall that a map $f\colon X \to Y$ is a *selection* for a multifunction $m\colon X \to Y$ if $f(x) \in m(x)$ for all $x \in X$. In this section four selection theorems for r-open graph multifunctions are proved. These results should be compared to those of Michael [**13, 14**]. The main differences are (1) the multifunction hypotheses ("r-open graph") used here is more restrictive than that of [**13, 14**] ("lower semi-continuous"); (2) results here involve finite dimensional assumptions whereas some of those of [**13, 14**] do not; but (3) the restrictions on values are different here — e.g., values $m(x)$ in [**13, 14**] are assumed to be complete whereas here that is not the case — and also non-contractible values are sometimes allowed here. The proofs here are different from those of [**13, 14**].

2.1. THEOREM. *Suppose that $m\colon X \to Y$ is an r-open graph multifunction with $m(x) \subset Y$ a weak homotopy equivalence for all x in X and suppose X a finite dimensional compact ANR. Let α be any homotopy class of maps from X to Y. Then m has a selection f with $f \in \alpha$.*

Proof. In Prop. 1.3 take $T = X \times Y$, $B = X$, $G(m) = E$. The fiber $E(x)$ is then $x \times m(x)$ and $T(x)$ is $x \times Y$. It follows from Prop. 1.3 that $G(m) \to X$ is a Serre fibration. Let j: $G(m) \to X \times Y$ be the inclusion. Then j is now a map of Serre fibrations which is a weak homotopy equivalence on the fiber and the identity on the base. The exact homotopy sequences of the fibrations and the 5-lemma show that j is a weak homotopy equivalence. By [**6**, p. 135, Cor. 5.6] X has the homotopy type of a CW complex so it follows from [**16**, p. 405] that j_*: $[X, G(m)] \to [X, X \times Y]$ is a bijection, where $[A, B]$ is the set of homotopy classes of maps from A to B. Let $g \in \alpha \in [X, Y]$ and choose w: $X \to G(m)$ with $jw \simeq (\mathrm{id}, g)$. Thus $pw = \pi jw \simeq \pi(\mathrm{id}, g) = \mathrm{id}$: $X \to X$. But, by Prop. 1.2, X is a retract of a CW complex and it follows easily that X has the covering homotopy property with respect to Serre fibrations. So w: $X \to G(m)$ with $pw \simeq \mathrm{id}$ gives w': $X \to G(m)$ with $w \simeq w'$ and $pw' = \mathrm{id}$ (a section). Define $f = \pi_2 jw'$: $X \to Y$. Then $f(x) \in \pi_2 j(x \times m(x)) = m(x)$ so f is a selection for m and $f = \pi_2 jw' \simeq \pi_2 jw \simeq \pi_2(\mathrm{id}, g) = g$, so $f \in \alpha$.

2.2. THEOREM. *Suppose that m: $X \to Y$ is an r-open graph multifunction with infinitely connected values and X is a compact finite dimensional ANR. Then m has a selection.*

Proof. Prop. 1.5 (with $E = G(m)$, $T = X \times Y$, and $B = X$) shows that $G(m) \to X$ is a Serre fibration with infinitely connected fiber. Let r: $Z \to X$ be a retraction of a CW complex onto X (by Prop. 1.2). Then there is a map g: $Z \to G(m)$ with $pg = r$. It follows from the proposition below that there is a map g: $Z \to G(m)$ with $pg = r$.

Now define s: $X \to G(m)$ by $s = gi$ (i: $X \subset Z$). Then $ps = pgi = ri = \mathrm{id}$. The section s gives a selection f as in the last paragraph of the proof of Theorem 2.1.

The following proposition is well known. For completeness sake, however, a deduction from the results of [**16**] is given. The proposition can also be proved more directly by an elementary skeleton-by-skeleton argument.

PROPOSITION. *Suppose that p: $E \to B$ is a surjective Serre fibration with infinitely connected fiber, Z a CW complex, and f: $Z \to B$ a map. Then there is a map g: $Z \to E$ with $pg = f$.*

Proof. It follows from the exact sequence of a Serre function (see [**16**, p. 377]) that $E \to B$ is a weak homotopy equivalence. Then [**16**, p. 404, Th. 22] gives g': $Z \to B$ with pg homotopic to f. The covering homotopy property then gives g: $Z \to B$ with $pg = f$.

The next definition will allow us to go somewhat beyond the weak homotopy equivalence restriction in Theorem 2.1. For a multifunction m: $X \to Y$ let $M_i(x, y) = \text{Image}(\pi_2 j)_*$: $\pi_i(G(m), (x, y)) \to \pi_i(Y, y)$. The map $m(x) \hookrightarrow Y$ will be the inclusion.

2.3. DEFINITION. m: $X \to Y$ is *level* if $\pi_i(m(x), y) \to \pi_i(Y, y)$ is monic with image $M_i(x, y)$ for all $x \in X, y \in m(x)$, all $i \geq 0$.

It is easily checked that if $m(x) \to Y$ is a weak homotopy equivalence (as in Theorem 2.1) then m is level. In the event that X is infinitely connected (e.g., in 2.4 below) it is easy to check that m is level iff $\pi_i(m(x), y) \to \pi_i(Y, y)$ is monic and $\pi_i(m(x), y) \to \pi_i(G(m), (x, y))$ is epic for all x in X, y in $m(x)$, all i.

2.4. THEOREM. *Let m: $X \to Y$ be a level r-open graph multifunction with X a contractible compact finite dimensional ANR. Then m has a selection.*

Proof. It follows from Prop. 1.4 that $G(m) \to X$ is a Serre fibration. By Prop. 1.2 X is a retract of a CW complex so it has the covering homotopy property for $G(m) \to X$. X is contractible so we get a section s: $X \to G(m)$, $ps = \text{id}$. [If H_t: $X \to X$, $H_0(X) = x_0$, $H_1 = \text{id}$, then pick $(x_0, y_0) \in G(m)$ and let $s'(x) = (x_0, y_0)$ all x so that H is a homotopy of ps'. If G is the covering homotopy then $s = G_1$ is the section.] As in 2.1, $f = \pi_2 js$ is the desired selection.

So far only very elementary methods have been used to get cross sections of $G(m) \to X$. The next definition will allow us to invoke stronger results.

2.5. DEFINITION. A multifunction m: $X \to Y$ is *obstruction free* if each $m(x)$ is path-connected with abelian fundamental group and $H^{i+1}(X; \pi_i(m(x))) = 0$, $i \geq 1$.

2.6. THEOREM. *Let m: $X \to Y$ be a level, obstruction free, r-open graph multifunction, with X a compact finite dimensional ANR. Then m has a selection.*

Proof. Prop. 1.4 shows $G(m) \to X$ is a Serre fibration. Let $r: Z \to X$ be a retraction of a CW complex Z onto X (Prop. 1.2). It follows from, e.g., [**8**] that there is a lifting $g: Z \to G(m)$ with $pg = r$. (With a small extra assumption the existence of g also follows from [**17**, p. 302, Th. 6.11] or [**16**, Ch. 8].) Now $s = gi$ is a section of $G(m) \to X$ and $f = \pi_2 js$ is the desired selection.

The conditions in the definition of "obstruction free' can be weakened. The approach in [**8**] shows that it suffices to assume that $m(x)$ has a solvable fundamental group if $\pi_1(m(x))$ is replaced by its abelian subquotients. Also if it is already known that $G(m) \to X$ is a fibration there are obstruction sets $\mathrm{ob}_i(r) \subset H^i(X; \pi_{i-1}(m(x)))$ and one need only show that each contains zero.

3. Fixed point theorems. Recall that x is a fixed point of a multifunction $m: X \to X$ if $x \in m(x)$.

3.1. THEOREM. *Let $m: X \to X$ be an r-open graph multifunction. Suppose $m(X) \subset Y \subset X$ with $m(x) \subset Y$ a weak homotopy equivalence for all x in X and Y a compact finite dimensional ANR. Then m has a fixed point.*

Proof. Let $t: Y \to Y$ be the multifunction defined by $t(y) = m(y)$. Suppose $r: U \to G(m)$ is an X-retraction for $G(m)$. It is easy to check that $G(t) = G(m) \cap (Y \times Y)$ and $r': U' \to G(t)$ works for t where $U' = U \cap (Y \times Y)$ and r' is the restriction of r. Thus t is r-open graph. $t(y) \to Y$ is a weak homotopy equivalence and Y is a compact finite dimensional ANR. By Theorem 2.1 t has a selection $f: Y \to Y$ which is null homotopic. By the classical Lefschetz theorem (e.g., [**3**, Chap. 3]) f has a fixed point since $L(f) = 1 \neq 0$. So $x = f(x) \in t(x) = m(x)$, proving the theorem.

Note that if we remove the hypotheses on Y in 3.1 but assume instead that X is a compact finite dimensional ANR then the proof is simpler: Theorem 2.1 gives a null homotopic selection and the Lefschetz theorem gives the fixed point.

3.2. THEOREM. *Let $m: X \to X$ be an r-open graph multifunction with infinitely connected values. Suppose $m(X) \subset Y \subset X$ where Y is an acyclic compact finite dimensional ANR. Then m has a fixed point.*

Proof. Let t: $Y \to Y$ be the multifunction defined by m. As in the proof of 3.1, t is r-open graph and here t has infinitely connected values (since $t(y) = m(y)$). By Theorem 2.2 t has a selection f: $Y \to Y$. By the Lefschetz theorem [**3**] f has a fixed point so m does also.

Theorem 3.2 generalizes [**11**, Cor. 3.3] and its predecessors [**7, 9**]. In the next theorem some of the above hypotheses on Y are shifted to X. Here m^n will mean the composition of m with itself $n \geq 1$ times.

3.3. THEOREM. *Let m: $X \to X$ be an r-open graph multifunction with infinitely connected values and X a compact finite dimensional ANR. Suppose for some $n \geq 1$, $m^n(X) \subset Y \subset X$ where Y is acyclic. Then m has a fixed point.*

Proof. By Theorem 2.2 m has a selection f: $X \to X$. By a lemma of F. Browder [**2**] f has a fixed point so m does also.

In Theorem 3.3 it is sufficient to assume that i: $Y \subset X$ is zero on homology.

Finally, two results for level multifunctions are given. Interpret "m is level to Y" to mean that the multifunction $X \to Y$ defined by m is level.

3.4. THEOREM. *Let m: $X \to X$ be an r-open graph multifunction with $m(X) \subset Y \subset X$. Suppose m is level to Y and both X and Y are compact finite dimensional ANR's and X is contractible. Then m has a fixed point.*

Proof. m defines t: $X \to Y$ which is level and r-open graph. By Theorem 2.4 t has a selection g: $X \to Y$. The composite $f = gi$: $Y \subset X \to Y$ is null homotopic since X is contractible so it has a fixed point (by the Lefschetz theorem) and so does m.

3.5. THEOREM. *Let m: $X \to X$ be an obstruction free r-open graph multifunction with $m(X) \subset Y \subset X$. Suppose m is level to Y, i: $Y \subset X$ is null homotopic, and X is a compact finite dimensional ANR. Then m has a fixed point.*

The proof uses Theorem 2.6 and is otherwise like that of 3.4. Note that if more specific information is available about the spaces then the list of hypotheses in 3.5 can be shortened. For example, suppose m: $S^n \to S^n$, $n \geq 3$, is an r-open graph multifunction, level to Y, where $m(S^n) \subset Y \subset S^n$: and Y and $m(x)$ are homotopically equivalent to S^1. Then the other hypotheses of 3.5 are easily checked and m must have a fixed point.

4. Some comments on generalizations.

(1) The hypothesis "X is a compact finite dimensional ANR" is usually needed in §3 (see the example in [**9**]) but not in §2 — there "X is a retract of a CW complex" would suffice.

(2) The hypothesis "r-open graph" can be replaced by the more general (but less visual) "r-subopen". Here m: $X \to Y$ is defined to be r-subopen if the projection $G(m) \to X$ can be locally embedded as an X-retract of an open subset of some Serre fibration. This shows that Theorem 3.2 is a genuine generalization of the results of [**1**, Cor. 3.3] and [**10**, Cor. 2.3].

(3) The propositions of §1 can be stated with hypotheses for $i \leq N$ and conclusion "N-fibration" (see [**12**]). The other results of the paper can then be suitably generalized along the same lines.

(4) In his work on fixed points of upper semi-continuous multifunctions [**5**] Gorniewicz described a technique for extending results. This technique can be adapted to the present setting. Say that a multifunction m: $X \to Y$ is *Serre representable* if there is a Serre fibration q: $E \to X$ and a map u: $E \to G(m)$ with $pu = q$. Then "Serre representable" plus either a condition on the fibers of E or a relation between the fibers of E and $m(x)$ will give selection and fixed point theorems. The problem then is to find reasonable sufficient conditions (other than r-subopen) for a multifunction to have a representation with the needed extra conditions.

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UNIVERSITY OF KANSAS
LAWRENCE, KS 66045