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REDUCING THE ORDER OF THE LAGRANGEAN FOR A CLASSICAL FIELD IN CURVED SPACE-TIME

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We show how the Lagrangean L can be replaced by another, L^*, having the same extremals, but having only first order derivatives and being in fact a first degree polynomial in these derivatives.

1. Introduction. Whittaker [4] showed how to reduce the order and degree of a Lagrangean to 1 in the case of one-dimensional "space-time". As he points out, this leads instantly to Hamilton's canonical formalism. Rodrigues [3] showed how to do this without using coordinates in the configuration space. "Reducing the order" is not an adequate description of the construction since when the order is 1 (as it usually is) it still takes some work to make it of the first degree.

In [1] we treated the case of true (several dimensional) space-time. However, we took it to be $\mathbf{R}^4$, and we used coordinates in the field space.

Such a theorem is not usable when several coordinate systems must be used in space-time M. This is because the L^* doing the desired things is not unique.

Our construction of L^* here depends on the choice of an affine connection Γ and a volume element ω in M. The result is not independent of the Γ and ω chosen, but it is independent of coordinates.

2. π-manifolds. In this paper, a suitable order N of differentiability is assumed.

Let M be a manifold which we will call *space-time*. Let P be another manifold. We will call it a π-manifold if there is defined on it a *regular* map $\pi: P \to M$.

If P and Q are two π-manifolds, let $F(P, Q)$ be the class of all maps f of open sets in P into Q for which

$$\pi(f(p)) = \pi(p) \tag{2.1}$$

whenever $f(p)$ is defined. Let p be a point of P and let $f, g \in F(P, Q)$. Say $f \equiv g$ at p if f and g agree up to the Nth order at p. Let $J^N(P, Q)$ be the set of equivalence classes. An element C of $J^N(P, Q)$ may be represented by a pair (p, f) where C is an equivalence at p and $f \in C$. Define $\pi(C) = \pi(p) = \pi(f(p))$. Then $J^N(P, Q)$ is also a π-space.

Let $X^1,\ldots,x^m, y^1,\ldots,y^n$ be coordinates in P and Q respectively. Then $(y^i)_{\lambda\cdots\nu}$ shall be the function defined in $J^N(P, Q)$ be saying that

$$(y^i)_{\lambda\cdots\nu}(C) = \frac{\partial^k(y^i \circ f)}{\partial x^\lambda \cdots \partial x^\nu}(p). \tag{2.2}$$

Here k must not exceed N.

A particularly useful kind of coordinates $x^1,\ldots,x^m$ are coordinates *exponential at a point p*. For this one must select an affine connection Γ for P. Let $T^1(P)_p$ be the tangent space to P at p. Select a *linear* coordinate system in $T^1(P)p$ and transfer these to P using the exponential map defined by Γ (see [**2**].)

3. Lagrangeans. In classical mechanics, a Lagrangean is a function defined on $\mathbf{R} \times T^1(Q)$, the latter being the tangent bundle of configuration space. Now $\mathbf{R} \times T^1(Q)$ is naturally isomorphic with $J^1(M, M \times Q)$ where $M = \mathbf{R}$, and $M \times Q$ has the projection on M. Since we want to use 2.1 we define Lagrangeans in this *milieu.*

Let M be a manifold of dimension m. Let S be a π-manifold. If coordinates $t^1,\ldots,t^m$ are chosen for M, then $t^1 \circ \pi,\ldots,t^m \circ \pi$ are (independent, by the regularity of π) variables in S. We will abbreviate them to $t^1,\ldots,t^m$.

A *Lagrangean* (of order at most N) is an m-form Λ defined on $J^N(M, S)$ such that in terms of coordinates $t^1,\ldots,t^m$ in M,

$$\Lambda = L\, dt^1 \wedge \cdots \wedge dt^m. \tag{3.1}$$

Let the class of these Lagrangeans be called $\mathscr{L}^N(S)$.

3.2. THEOREM. *Let M and S be as above. Let Γ be an affine connection for M. Let ω be a volume element for M. Let K be the cotangent bundle of $J^N(M, S)$. Then Γ, ω define a mapping*

$$\mathscr{L}^N(S) \to \mathscr{L}^1(K), \qquad \Lambda \to \Lambda^*. \tag{3.3}$$

This mapping is linear and 1 : 1

3.4. *L^* is a polynomial of the first degree in the derivatives (necessarily only of the first order), and*

3.5. *Λ^* has the same extremals as Λ.*

The precise (and natural) meaning of 3.5 is given in §5 below, together with the proof of 3.2.

Property 3.4 is useful for the following construction.

Let $x^1,\dots,x^n$ (plus those $t^1,\dots,t^m$) be coordinates in S. $\Lambda^* = L^*\,dt^1 \wedge \cdots \wedge dt^m$ and by 3.4, L^* is a sum of a term $-H$ depending only on the x^i and t^λ, and a sum of terms like $a(x^i)_\lambda$. Thus Λ^* is a sum of $-H\,dt^1 \wedge \cdots \wedge dt^m$ and of terms like $a(x^i)_\lambda\,dt^1 \wedge \cdots \wedge dt^m$. This latter term is congruent *modulo* $dx^i - (x^i)_\lambda\,dt^\lambda$ to $a\,dt^1 \wedge \cdots \wedge dx^i \wedge \cdots dt^m$, dx^i being in the λth place. Thus Λ^* is congruent to an m-form involving only the x^i and t^λ, that is, an m-form on S.

4. A vector field U_Γ on $J^N(M, S)$.

4.1. THEOREM. *Let Γ be an affine connection for M. Using Γ one can construct a vector field U_Γ on $J^N(M, S)$. Let $A \in J^N(M, S)$ and let $t^1,\dots,t^m$ be exponential coordinates at $\pi(A)$ in M. Use $t^1,\dots,t^m$ together with some further coordinates $x^1,\dots,x^n$ as coordinates in S. Then at A,*

$$U_\Gamma = (x^i)_\lambda \frac{\partial}{\partial (x^i)_\lambda} + 2(x^i)_{\lambda\mu}\frac{\partial}{\partial (x^i)_{\lambda\mu}} + \cdots + N(x^i)_{\lambda\cdots\nu}\frac{\partial}{\partial (x^i)_{\lambda\cdots\nu}}. \tag{4.2}$$

A sum is intended in 4.2. For example, by

$$(x^i)_{\lambda\cdots\mu}\frac{\partial}{\partial (x^i)_{\lambda\cdots\mu}}$$

where there are k indices, we mean the sum over all sets of k indices such that $1 \le \lambda \le \cdots \le \mu \le m$.

We now prove 4.1. Let $A = (a, f)$ be a point of $J^N(M, S)$. Express f in terms of $t^1,\dots,t^m$:

$$f = \varphi(t^1,\dots,t^m).$$

For sufficiently small real s and $|t^\lambda|$, f_s can be defined by

$$f_s = \varphi(e^s t^1,\dots,e^s t^m).$$

Define the "moving" point A_s in $J^N(M, S)$ as (a, f_s). We define U_Γ at A to be the tangent to the curve $s \to A_s$ for $s = 0$.

We must show that its $(x^i)_{\lambda\cdots\mu}$ component is correctly represented in 4.2. Now

$$(x^i)_{\lambda\cdots\mu}(A_s) = \frac{\partial^k}{\partial t^\lambda \cdots \partial t^\mu}[x^i \circ f_s]\bigg|_{t=0}$$

$$= e^{ks}\frac{\partial^k}{\partial t^\lambda \cdots \partial t^\mu}[x^i \circ f]\bigg|_{t=0} = e^{ks}(x^i)_{\lambda\cdots\mu}(A).$$

Then we take d/ds of this for $s = 0$, obtaining

$$k(x^i)_{\lambda\cdots\mu}$$

as 4.2 asserts. The construction of A_s is clearly independent of the coordinates.

4.3. COROLLARY. *Let K be the cotangent bundle $T_1(J)$ of $J = J^N(M, S)$. Then Γ defines a real-valued function on K whose expression in terms of coordinates is*

$$p_i^\lambda\overline{(x^i)_\lambda} + 2p_i^{\lambda\mu}\overline{(x^i)_{\lambda\mu}} + \cdots + Np_i^{\lambda\cdots\nu}\overline{(x^i)_{\lambda\cdots\nu}}. \tag{4.4}$$

Here $p_i^{\lambda\cdots\mu}$ is the "momentum" coordinate dual to the configuration coordinate $(x^i)_{\lambda\cdots\mu}$ in J.

To prove 4.3, we need only show that 4.4 is independent of the coordinate representation. A point B is a pair (A, g) where A is a point of J as before, and g is a map from a neighborhood of A into $\mathbf{R}$, where $g(A) = 0$. The value of $p_i^{\lambda\cdots\mu}$ at B is

$$\frac{\partial g}{\partial (x^i)_{\lambda\cdots\mu}}\bigg|_A.$$

The value of $\overline{(x^i)_{\lambda\cdots\mu}}$ at B is just $(x^i)_{\lambda\cdots\mu}(A)$. Here the latter is defined in J, and the former is the coordinate defined in $T_1(J)$ as obtained from the latter and the projection ρ

$$\begin{array}{c} T_1(J) = K \\ \downarrow \\ J = J^N(M, S) \end{array} \tag{4.5}$$

Hence the value of 4.4 at B is $U_\Gamma[g]$, the result of applying the operator U_Γ to g, and is thus independent of coordinates.

4.6. COROLLARY. *Let J and K be as in 4.3. Let $P = J^1(M, K)$. Then Γ defines a real-valued function on P whose value at $C = (c, h)$ is*

$$\overline{p_i^\lambda} Z_\lambda^i + \overline{p_i^{\lambda\mu}} Z_{\lambda\mu}^i + \cdots + \overline{p_i^{\lambda\cdots\nu}} Z_{\lambda\cdots\nu}^i \tag{4.7}$$

where $Z_{\lambda\cdots\mu}^i$ is the sum of all

$$\left(\overline{(x^i)_{\lambda\cdots\hat{\sigma}\cdots\mu}}\right)_\sigma. \tag{4.8}$$

Several explanations are needed.

4.81. In forming the sum $Z_{\lambda\cdots\mu}^i$ we select each index appearing in $\lambda \cdots \mu$, form 4.8, and add the results. $\hat{\sigma}$ means delete 'σ' from the string $\lambda \cdots \mu$.

4.82. The $\overline{p_i^{\lambda\cdots\mu}}$ is $p_i^{\lambda\cdots\mu} \circ \xi$ where ξ is the projection of $P = J^1(M, K) \to K$. Hence, for an element $C = (c, h)$ of P, wherein h maps a neighborhood of $c \in M$ into K,

$$\overline{p_i^{\lambda\cdots\mu}}(C) = p_i^{\lambda\cdots\mu}(h(c)) \tag{4.83}$$

which is

$$\left.\frac{\partial g}{\partial (x^i)_{\lambda\cdots\mu}}\right|_A$$

if $h(c) = (A, g)$, $A \in K$.

4.84. The bar on the $(x^i)_{\lambda\cdots\mu}$ means that (see 4.5)

$$\overline{(x^i)_{\lambda\cdots\mu}} = (x^i)_{\lambda\cdots\mu} \circ \rho.$$

For any real variable z (and thus for $\overline{(x^i)_{\lambda\cdots}}$) on K, 2.2 with $N = 1$ gives sense to z_σ for $1 \le \sigma \le m$.

To begin the proof of 4.6, we evaluate at $C = (c, h)$ of P. By 2.2, the answer is

$$\left.\frac{\partial}{\partial t^\sigma}\left[\overline{(x^i)_{\lambda\cdots\mu}} \circ h\right]\right|_c.$$

Let us write ∂_σ for $\partial/\partial t^\sigma$. So

$$Z_{\lambda\cdots\mu}^i = \sum \partial_\sigma\left[(x^i)_{\lambda\cdots\hat{\sigma}\cdots\mu} \circ \rho \circ h\right].$$

We need a somewhat more abstract version of 2.2. Let $\pi_1(C) = p$, $\pi_2(C) = h$ in 2.2. Replace x^λ by t^λ and y^i by x^i, since that is the notation for the present instance. Then 2.2 says

$$(x^i)_{\lambda\cdots\mu} = \left\{\partial_\lambda \cdots \partial_\mu\left(x^i \circ \pi_2\right)\right\} \circ \pi_1.$$

Accordingly,

$$Z^i_{\lambda\cdots\mu} = \sum \partial_\sigma\left[\left\{\partial_{\lambda\cdots\hat{\sigma}\cdots\mu}\left(x^i \circ \pi_2\right)\right\} \circ \pi_1 \circ \rho \circ h\right]\Big|_c.$$

We assert that $\pi_1 \circ \rho \circ h$ is the identity map. First of all π_1 is the map π for the π-space J. Then ρ (4.5) is a π-space morphism, for that is the way in which K is made into a π-space. So $\pi_1 \circ \rho = \pi$. But h has to satisfy 2.1, so $\pi \circ h = \pi$. But the π for M itself is the identity (on some neighborhood). Therefore

$$Z^i_{\lambda\cdots\mu} = \sum \partial_\sigma\left\{\partial_{\lambda\cdots\hat{\sigma}\cdots\mu}\left(x^i \circ \pi_2\right)\right\} = k\partial_{\lambda\cdots\mu}\left(x^i \circ \pi_2\right)$$

where k is the length of the string $\lambda \cdots \mu$ with nothing deleted. So

$$Z^i_{\lambda\cdots\mu} = k\left\{\partial_{\lambda\cdots\mu}\left(x^i \circ \pi_2\right)\right\} \circ \pi_1 \circ \rho \circ h\Big|_c = k(x^i)_{\lambda\cdots\mu}(A)$$

since $h(c) = (A, g)$. Combine this with 4.83 and obtain that this Z term contributes

$$k(x^i)_{\lambda\cdots\mu}(A)\frac{\partial g}{\partial(x^i)_{\lambda\cdots\mu}}$$

to the sum 4.7. This sum is evidently $U_\Gamma[g]$ evaluated at A. Thus 4.7 is independent of the coordinates. Thus it defines the function for which 4.6 holds.

5. Proof of Theorem 3.2. Let Λ be given. Choose coordinates in M. Then 3.1 defines L, a function defined on $J^N(M, S)$. We have projections

$$(5.1) \qquad \begin{array}{c} J^1(M, K) = P \\ \xi \downarrow \\ K = T_1(J) \\ \rho \downarrow \\ J = J^N(M, S) \\ \downarrow \\ S \end{array}$$

Hence L defines a function on $J^1(M, K)$, which we denote by L also, for simplicity. Using ξ, we can lift the function 4.4 up to a function φ on P. The function defined by 4.6 may be called ψ.

Let

$$(5.2) \qquad \omega = \lambda\, dt^1 \wedge \cdots \wedge dt^m,$$

$\lambda \neq 0$, be the volume element postulated in 3.2.

We define

$$L^* = L + \lambda(\psi - \varphi) = L - \lambda\varphi + \lambda\psi,$$

and

$$\Lambda^* = L^*\, dt^1 \wedge \cdots \wedge dt^m.$$

We consider assertion 3.4. 4.8 is a derivative of first order. The coefficients in 4.7 are mere coordinates, and the Z's are linear in the derivatives 4.8. Hence 3.4 holds.

To show 3.5 we mention first that L^* has the form

$$\begin{aligned} L^* = L &+ P_i^{\lambda}\left[\overline{\left((X^i)\right)_{\lambda}} - \overline{(x^i)_{\lambda}}\right] \\ &+ P_i^{\lambda\mu}\left[\overline{\left((x^i)_{\lambda}\right)_{\mu}} + \overline{\left((x^i)_{\mu}\right)_{\lambda}} - 2\overline{(x^i)_{\lambda\mu}}\right] \\ &+ P_i^{\lambda\mu\nu}\left[\overline{\left((x^i)_{\mu\nu}\right)_{\lambda}} + \overline{\left((x^i)_{\lambda\nu}\right)_{\mu}} + \overline{\left((x^i)_{\lambda\mu}\right)_{\nu}} - 3\overline{(x^i)_{\lambda\mu\nu}}\right] + \cdots, \end{aligned}$$

where $P_i^{\lambda}, P_i^{\lambda\mu}, \ldots$ are λ (see 5.2) times the $p_i^{\lambda}, p_i^{\lambda\mu}, \ldots$ lifted up to $J^1(M, K)$ by the projections 5.1. It follows from [**1**, 3.4] that L^* has the "same" extremals as l. The meaning of *same* is as follows. An extremal for L^* gives us expressions

$$x^i = f^i(t^1, \ldots, t^m), \tag{5.3}$$

$$\begin{aligned} (x^i)_{\lambda} &= u_{\lambda}^i(t^1, \ldots, t^m) \\ (x^i)_{\lambda\mu} &= u_{\lambda\mu}^i(t^1, \ldots, t^m) \\ &\ \vdots \\ P_{\lambda}^i &= g_{\lambda}^i(t^1, \ldots, t^m) \\ &\ \vdots \end{aligned} \tag{5.4}$$

If we abandon 5.4 and those following it, then 5.3 gives an extremal for L in the usual sense. It is shown in [**1**] that

$$u_{\lambda}^i = \frac{\partial f^i}{\partial t^{\lambda}},$$

and so forth.

6. Correction to [1].

(a) Delete 6.5. (Prop. 6.6. remains true, with the φ of 6.7.)

(b) Delete 6.9. (Better results are in the author's "The dynamic differential forms of the Klein-Gordon field and the conformal group", Jour. Geometry and Physics, Vol. 1, 1983.)

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