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FIXED POINT THEOREMS FOR SOME DISCONTINUOUS OPERATORS

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The purpose of this paper is to show the existence of fixed points for operators T defined on a subset K of a Banach space X and belonging to a class that the author calls $D(a, b)$ with $0 \leq a,\ b \leq 1$.

1. Introduction. Let T be a mapping of a set K into itself. An immediate question is whether some point is mapped onto itself; that is, does the equation

$$Tx = x \tag{1}$$

have a solution? If so, x is called a *fixed point* of T. This question generates a theory which began in 1912 with the work of L. E. J. Brouwer, who proved that any continuous mapping T of an n-ball into itself has a fixed point, and was followed in 1922 by S. Banach's Contraction Principle, which states that any mapping T of a complete metric space X into itself that satisfies, for some $0 < k < 1$, the inequality

$$d(Tx, Ty) \leq k\, d(x, y) \tag{2}$$

for all x and y in X, has a unique fixed point. Here d denotes the metric on X. J. Schauder **[13]**, Tychonoff **[16]**. S. Lefschetz **[10]**, F. Browder **[2]**, W. A. Kirk **[7]**, and many others have added to and generalized these basic results.

In 1969 and 1971, R. Kannan **[5]**, **[6]**, proved some fixed point theorems for operators T mapping a Banach space X into itself which, instead of the contraction property in (2), satisfy the condition:

$$\|Tx - Ty\| \leq \alpha\left[\|x - Tx\| + \|y - Ty\|\right], \tag{3}$$

for all x, y in X; where $0 < \alpha < 1/2$. G. Hardy and T. Rogers **[4]** generalized this result to continuous mappings T of a complete metric space X into itself that satisfy:

$$\begin{aligned} d(Tx, Ty) \leq{} & a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) \\ & + a_4 d(x, Ty) + a_5 d(y, Tx), \end{aligned} \tag{4}$$

for all x and y in X, where $a_i \geq 0$ and $a_1 + a_2 + a_3 + a_4 + a_5 < 1$. K. Goebel, W. A. Kirk, and T. N. Shimi **[3]**, extended the last result to continuous mappings of a nonempty bounded, closed and convex subset K of a uniformly convex Banach space into itself, where $a_i \geq 0$ and $a_1 + a_2 + a_3 + a_4 + a_5 \leq 1$.

In this article we will prove some fixed point theorems for operators T defined on a subset K of a Banach space X that satisfy the inequality.

$$\|Tx - Ty\| \leq a\|x - y\| + b[\|x - Tx\| + \|y - Ty\|] \tag{5}$$

for all x and y in K, where $0 \leq a, b \leq 1$. Any operator T satisfying condition (5) will be said to belong to class $D(a, b)$. A contraction operator is in class $D(k, 0)$ with $0 < k < 1$.

Note that although condition (2) implies the continuity of the operator T, condition (3) to (5) may hold even if the operator is not continuous. Indeed, any operator T is in $D(1, 1)$, since by the triangle inequality:

$$\|Tx - Ty\| \leq \|Tx - x\| + \|x - y\| + \|y - Ty\|.$$

Furthermore, inequality (5) is a direct consequence of (4) and the triangle inequality, provided we forego the upper bounds required in [**3**].

2. Discussion. If we carefully examine the statement and proof of Banach's Contraction Principle (see for example Kreyszig [**9**, pp. 300–302]) we observe that the main conclusions are

(i) There exists a unique fixed point.

(ii) A contraction mapping is an asymptotically regular operator for any point, that is, $\|T^{n-1}x - T^n x\| \to 0$ as $n \to \infty$.

(iii) The sequence $x_n = T^n x_0$, of Picard iterates converges to the unique fixed point.

Which of these conclusions hold for operators T in the class $D(a, b)$ with $0 \leq a, b < 1$?

First, observe that these classes are not empty: consider the discontinuous operator

$$Tx = \begin{cases} \gamma x, & 0 \leq x < 1/2, \\ \rho x, & 1/2 \leq x \leq 1, \end{cases}$$

with $0 < \gamma, \rho < 1$, $\gamma \neq \rho$. Then T is in $D(0, \mu/(1 - \mu))$ where $\mu = \max\{\gamma, \rho\}$ because

$$\frac{\gamma}{1 - \gamma}(x_i - Tx_i) = \gamma x_i, \quad \text{for } x_i \in [0, 1/2),$$

so that

$$\begin{aligned} |Tx_1 - Tx_2| &\leq \gamma(x_1 + x_2) = \frac{\gamma}{1 - \gamma}\{|x_1 - Tx_1| + |x_2 - Tx_2|\} \\ &\leq \frac{\mu}{1 - \mu}\{|x_1 - Tx_1| + |x_2 - Tx_2|\}. \end{aligned}$$

The same inequality holds if $x_i \in [1/2, 1]$. Now if $x_1 < 1/2 \leq x_2$; then

$$\frac{\gamma}{1-\gamma}(x_1 - Tx_1) = \gamma x_1; \ \frac{\rho}{1-\rho}(x_2 - Tx_2) = \rho x_2$$

and

$$|Tx_1 - Tx_2| \leq \gamma x_1 + \rho x_2 \leq \frac{\mu}{1-\mu}\{|x_1 - Tx_1| + |x_2 - Tx_2|\}.$$

Moreover, (i) as is obvious from its graph, T has a unique fixed point at $x = 0$,

$$\text{(ii)} \quad \frac{\gamma}{1-\gamma}(T^n x - T^{n+1}x) = \gamma T^n x \leq \mu^{n+1} x$$

for n sufficiently large, arbitrary x in $[0,1]$, and $0 < \mu < 1$, so that T is asymptotically regular at any point, and (iii) $\{x_n - T^n x\}_n$ converges to 0.

How many fixed points can an operator in the class $D(a, b)$, $0 \leq a$, $b < 1$, have? We shall show that the behaviour of the classes $D(a, b)$ is identical whether or not $b = 0$.

LEMMA 1. *Let T be in the class $D(a, b)$, $a, b \geq 0$, $a < 1$. If $F_T = \{x \in K | Tx = x\}$ is not empty, then F_T consists of a single point.*

Proof. Assume that x_i, $i = 1, 2$, are fixed point of T, and T satisfies $D(a, b)$. Then

$$\|x_1 - x_2\| = \|Tx_1 - Tx_2\| \leq a\|x_1 - x_2\|$$

which only holds if $x_1 = x_2$. □

When does T have a fixed point? Before answering this question we will need the following three interesting facts.

LEMMA 2. *If $T \in D(a, b)$, $a + 2b < 1$, then* $\operatorname{Inf}_{x \in K}\|x - Tx\| = 0$.

Proof. Define $x_n = T^n x_0$, with x_0 an arbitrary point. Then

$$\begin{aligned} \|x_n - x_{n+1}\| &= \|Tx_{n-1} - Tx_n\| \\ &\leq a\|x_{n-1} - x_n\| + b\big[\|x_{n-1} - x_n\| + \|Tx_{n-1} - Tx_n\|\big] \end{aligned}$$

so that

$$(1 - b)\|(I - T)x_n\| \leq (a + b)\|(I - T)x_{n-1}\|.$$

Hence

$$\|(I - T)x_n\| \leq \left(\frac{a+b}{1-b}\right)^n \|(I - T)x_0\|$$

and as $(a+b)/(1-b)<1$ it follows that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$; that is $\operatorname{Inf}_{x \in K}\|x - Tx\| = 0$. $\square$

Let X be a Banach space, T be a mapping of X into itself, and x be a point in X. The mapping T is *asymptotically regular in* x if $\|T^{n+1}x - T^n x\| \to 0$ as $n \to \infty$.

Because of the definition of x_n in the proof of Lemma 2 we observe that $T^n x_0 - T^{n+1}x_0 = x_n - x_{n+1} = (I - T)x_n \to 0$ as $n \to \infty$, so that we have

LEMMA 3. *If $T \in D(a,b)$, $a + 2b < 1$, then T is asymptotically regular at any point.*

From Banach's Contraction Principle, we know that for T in $D(a,0)$, the Picard iterates converge to the unique fixed point. Although we have not yet proved the existence of a fixed point, is there any property that a sequence of points $\{x_n\}$ in K could satisfy that would imply the existence of a fixed point and the convergence of that sequence to that point? The following result answers this question.

THEOREM 1. *Let K be a closed subset of a Banach space X and let $T \in D(a,b)$ with $0 \le a,b < 1$. Then the sequence $\{x_n\}_n$ contained in the set K, satisfies*

$$\lim_{n\to\infty}(x_n - Tx_n) = 0, \tag{6}$$

if and only if the sequence converges to the unique fixed point of T.

Proof. The condition is necessary because

$$\|Tx_n - Tx_m\| \le a\|x_n - x_m\| + B\{\|x_n - Tx_n\| + \|x_m - Tx_m\|\}$$

and applying the triangle inequality we have

$$\|Tx_n - Tx_m\| \le \frac{a+b}{1-a}\{\|x_n - Tx_n\| + \|x_m - Tx_m\|\}.$$

Thus, it follows, from the hypothesis that $\{Tx_n\}_n$ is a Cauchy sequence. Since X is complete and K is closed there exists $z \in K$ such that

$$\lim_{n\to\infty} Tx_n = z$$

and since $x_n - Tx_n \to 0$ as $n \to \infty$, then $x_n \to z$ as $n \to \infty$. Using the triangle inequality and the fact that $T \in D(a,b)$ with $b < 1$, we have:

$$\|z - Tz\| \le \frac{1+a}{1-b}\|z - x_n\| + \frac{1+b}{1-b}\|x_n - Tx_n\|$$

and since $x_n \to z$, and $x_n - Tx_n \to 0$ as $n \to \infty$ it follows that z is a fixed point under T. By Lemma 1 it is unique.

For the sufficiency part of the theorem we assume that there exists $z \in K$ such that $z = Tz$ and

$$\lim_{n \to \infty} x_n = z.$$

Since $T \in D(a, b)$ with $b < 1$, using the triangle inequality, we have:

$$\|Tx_n - x_n\| - \|x_n - z\| \le \|Tx_n - z\| \le a\|x_n - z\| + b\|x_n - Tx_n\|.$$

Thus,

$$(1 - b)\|Tx_n - x_n\| \le (1 + a)\|x_n - z\|,$$

by hypothesis, $Tx_n - x_n \to 0$ as $n \to \infty$. □

We have already seen that the Picard iterates of any point x in K satisfy equation (6). Hence we have proved

THEOREM 2. *Let K be a closed subset of a Banach space X, let $T \in D(a, b)$ with $a + 2b < 1$. Then T has a unique fixed point z in K.*

Moreover, the Picard iterates of any point x in K converge to z.

When $a + 2b < 1$ we can estimate the rate of convergence of the Picard iterates:

$$\begin{aligned}\|Tx - z\| = \|Tx - Tz\| &\le a\|x - z\| + b\|x - Tx\| \\ &\le (a + b)\|x - z\| + b\|z - Tx\|\end{aligned}$$

or

$$\|Tx - z\| \le \left(\frac{a + b}{1 - b}\right)\|x - z\|. \tag{7}$$

Hence

$$\|T^n x - z\| \le \left(\frac{a + b}{1 - b}\right)^n \|x - z\|$$

and $a + b < 1 - b$.

Here is an example of an operator T in $D(0, 1)$ which does not have a fixed point:

Consider the function

$$Tx = \begin{cases} x/4 + 19/50, & \text{if } 0 \le x < 1/2, \\ x/5 + 19/50, & \text{if } 1/2 \le x \le 1. \end{cases}$$

It is enough to see the case $x \in [0, 1/2)$, $y \in [1/2, 1]$ and to compare

$$|Tx - Ty| = \frac{|5x - 4y|}{20}$$

with

$$|x - Tx| + |y - Ty| = \frac{16y - 15x}{20}.$$

Analyzing the cases $5x - 4y \leq 0$ or $5x - 4y > 0$, we observe that $T \in D(0,1)$ but T does not have a fixed point in $[0,1]$.

The fixed point of the following operator T solves a differential equation which is not covered by the usual Picard Theorem, although the solution is found by the same iterative process.

Let

$$Tx(t) = \begin{cases} \gamma x(t) + \int_0^t k(s,t,x(s))\,ds, & 0 \leq x(t) \leq A, \\ \rho x(t) + \int_0^t k(s,t,x(s))\,ds, & x(t) > A, \end{cases}$$

where $k(s,t,x(s)) = ce^{-a(t-s)}x(s)$, $a, c > 0$, and $1 > \gamma > \rho > 0$. Let $0 \leq x(t) \leq A < y(t)$ such that $2x(s) \leq (1 + (1-\gamma)/(1-\rho))y(s)$ for all $0 \leq s \leq t$. Then

$$x(t) - Tx(t) = (1-\gamma)x(t) - \int_0^t k(s,t,x(s))\,ds$$

so that

$$\begin{aligned} |Tx(t) - Ty(t)| = \bigg| & \frac{\gamma}{1-\gamma}\left[(x(t) - Tx(t)) + \frac{1}{\gamma}\int_0^t k(s,t,x(s))\,ds\right] \\ & - \frac{\rho}{1-\rho}\left[(y(t) - Ty(t)) + \frac{1}{\rho}\int_0^t k(s,t,y(s))\,ds\right]\bigg| \end{aligned}$$

and

$$\begin{aligned} (9)\qquad & |Tx(t) - Ty(t)| \\ & \leq \frac{\gamma}{1-\gamma}\|x - Tx\| + \frac{\rho}{1-\rho}\|y - Ty\| \\ & \quad + \left|\frac{1}{1-\gamma}\int_0^t k(s,t,x(s))\,ds - \frac{1}{1-\rho}\int_0^t k(s,t,y(s))\,ds\right| \\ & \leq \frac{\gamma}{1-\gamma}[\|x - Tx\| + \|y - Ty\|] \\ & \quad + \frac{1}{1-\gamma}\left|\int_0^t ce^{-a(t-s)}\,ds\right|\|x - y\| \\ & \leq \frac{c}{a(1-\gamma)}\|x - y\| + \frac{\gamma}{1-\gamma}[\|x - Tx\| + \|y - Ty\|]. \end{aligned}$$

Inequality (9) also holds for any other positive choices of $x(t)$ and $y(t)$.

Thus

$$T \in D\left(\frac{c}{a(1-\gamma)}, \frac{\gamma}{1-\gamma}\right),$$

and if $0 < \gamma < \min(1/2, 1/3(1-c/a))$, the operator T will have a unique fixed point and any Picard iterates will converge to that fixed point.

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