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INCREASING PATHS ON THE ONE-SKELETON OF A CONVEX COMPACT SET IN A NORMED SPACE

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Let C be a convex compact set in a normed space E and let $\text{skel}_1 C$ be the subset of C that contains those boundary points of C which are not centres of 2-dimensional balls in C. When l is a continuous functional on E, we say that the path $P = g([\alpha, \beta])$ is l-strictly increasing if $l(g(t_1)) < l(g(t_2))$ for every t_1, t_2 such that $\alpha \leq t_1 < t_2 \leq \beta$. D. G. Larman proved the existence of an l-strictly increasing path on the one skeleton of C with $l(g(\alpha)) = \min_{x \in C} l(x)$ and $l(g(\beta)) = \max_{x \in C} l(x)$.

In this paper we prove a theorem concerning the number of l-strictly increasing paths on the one-skeleton of C, that are mutually disjoint and along each of which l assumes values in a range arbitrarily close to its range on C.

1. The results. We quote and prove the following theorem

THEOREM 1. *Let C be a compact convex set of infinite dimension in a normed space E and l be a continuous linear functional on E, which is non constant on C. Let $\varepsilon > 0$ be given, $M = \max_{x \in C} l(x)$ and $m = \min_{x \in C} l(x)$. Then, for every $n = 1, 2, 3, \ldots$ there exist n l-strictly increasing paths, $P_k = g_k([\alpha, \beta])$, $k = 1, 2, \ldots, n$ on the one-skeleton of C, such that* $\operatorname{rel\,int} P_i \cap \operatorname{rel\,int} P_j = \varnothing$ *with $i \neq j$, $l(g_k(\alpha)) = m + \varepsilon$ and $l(g_k(\beta)) = M - \varepsilon$ for $k = 1, 2, \ldots, n$.*

Proof. Consider the sets $K_0 = \{x \in C \colon l(x) = M - \varepsilon\}$ and $K_1 = \{x \in C \colon l(x) = m - \varepsilon\}$. These sets are of infinite dimension and lie on two parallel hyperplanes. We define

$$A = C \cap \{x \in E \colon l(x) \geq m + \varepsilon\} \cap \{x \in E \colon l(x) \leq M - \varepsilon\}$$

Then we may select n linearly independent vectors $e_1, e_2, \ldots, e_n$ and n linear functionals $l_1 = l, l_2, \ldots, l_n$ on E such that the following properties hold:

(i) $l_1(e_1) = 1$, $l_i(e_i) \neq 0$ for $i = 2, 3, \ldots, n$ and $l_i(e_j) = 0$ for $i \neq j$

(ii) Let $E_n = [e_1, e_2, \ldots, e_n]$ be the n-dimensional subspace of E spanned by $e_1, e_2, \ldots, e_n$ and π_0 be the projection map on E, defined by $\pi_0(x) = l_1(x)e_1 + \cdots + l_n(x)e_n$. Then $\dim \pi_0(K_0) = \dim \pi_0(K_1) = n - 1$.

From the previous, it follows that $C_n = \pi_0(A)$ is a convex body in E_n, $\pi_0(K_0) = \{x \in C_n\colon\ l(x) = M - \varepsilon\}$ and $\pi_0(K_1) = \{x \in C_n\colon\ l(x) = m + \varepsilon\}$.

Let $u \in E_n$ be a unit vector perpendicular to e_1. Then according to the results proved in [3] we may choose a unit vector $u' \in E_n$ orthogonal to e_1, as close as we please to u and such that there are no line segments in the direction u' on the boundary of $C_n - \operatorname{rel\,int} \pi_0(K_0) - \operatorname{rel\,int} \pi_0(K_1)$. Then the projection σ_{n-1} of E_n onto the hyperplane E_{n-1} perpendicular to u' has an inverse function from $\operatorname{bd} \sigma_{n-1}(C_n) - \operatorname{rel\,int} \sigma_{n-1}(\pi_0(K_0)) - \operatorname{rel\,int} \sigma_{n-1}(\pi_0(K_1))$ back to C_n.

If $\{e_1, u_2, \ldots, u_{n-1}, u\}$ is an orthogonal system in E_n then we can choose, using induction, unit vectors $u'_{n-1}, \ldots, u'_3$ orthogonal to e_1 and as close as we please in direction to the projections of the vectors $u_{n-1}, \ldots, u_3$ onto the subspaces $E_{n-1} \subseteq [u']^{\perp}$, $E_{n-2} \subseteq [u', u'_{n-1}]^{\perp}, \ldots,$ $E_3 \subseteq [u', u'_{n-1}, \ldots, u'_4]^{\perp}$ and in such a way the projections $\sigma_k\colon E_k \to E_{k-1}$, $k = n-2, \ldots, 3$ have unique inverses from

$$\operatorname{bd} \sigma_k \circ \sigma_{k+1} \circ \cdots \circ \sigma_{n-1}(C_n) - \operatorname{rel\,int} \sigma_k \circ \sigma_{k+1} \circ \cdots \circ \sigma_{n-1}(\pi_0(K_0)) \\ - \operatorname{rel\,int} \sigma_k \circ \sigma_{k+1} \circ \cdots \circ \sigma_{n-1}(\pi_0(K_1))$$

back to $\sigma_{k+1} \circ \cdots \circ \sigma_{n-1}(C_n)$. We complete the orthonormal system $u', u'_{n-1}, \ldots, u'_2, u'_1$ by taking $u'_1 = e_1$ and u'_2 to be the unit vector perpendicular to $u', u'_{n-1}, \ldots, u'_3$, $u'_1 = e_1$ and closest to u_2.

Write now $\omega_{u'} = \sigma_2 \circ \cdots \circ \sigma_{n-1}$ for the projection of E_n on the two dimensional subspace E_2. For each t such that $m + \varepsilon \le t \le M - \varepsilon$, we define by $\xi_0(t)$ the point on the line segment $\{x \in \omega_{u'}(C_n)\colon\ l_1(x) = t\}$ whose second coordinate attains its maximum value. On the other hand we may suppose, by making appropriate transformation of C, that there exists a cylinder B in the convex body C_n of E_n such that $B = \overline{\operatorname{con}}(S_0 \cup S_1)$, where S_0 and S_1 are $(n-1)$-dimensional balls of diameter δ with the property $S_i \subseteq \operatorname{rel\,int} \pi_0(K_i)$, $i = 0, 1$ and the axis of B in the direction of e_1.

Let ε_0 be such that $0 < \varepsilon_0 < \min\{d(\operatorname{bd} \pi_0(K_0), S_0), d(\operatorname{bd} \pi_0(K_1), S_1)\}$ where d is the usual distance between two sets. The convexity of C_n implies $d(\operatorname{bd} C_n - \pi_0(K_0) - \pi_0(K_1), B) > \varepsilon_0$. Then there exist a linear functional $l_{u'}$ on E_n such that $l_{u'}(u') = 0$, $l_{u'}(u'_2) = 1$ and $l_{u'}(\xi_0(t)) > \varepsilon_0 + \delta/2 > 0$.

Now let $\xi'_0(t)$ be the point on the line segment $\{x \in \omega_{u'}(C_n)\colon\ l_1(x) = t\}$ whose second coordinate attains its minimum value, then $l_{u'}(\xi'_0(t)) < -(\varepsilon_0 + \delta/2) < 0$. Because of the choice of $u', u'_{n-1}, \ldots, u'_3$ the inverse function $\omega_{u'}^{-1}$ is uniquely defined from the curves $\xi_0(t)$ and $\xi'_0(t)$

back to the one-skeleton of C_n. Consider now $x_0(t) = \omega_{u'}^{-1}(\xi_0(t))$ and $x'_0(t) = \omega_{u'}^{-1}(\xi'_0(t))$ where $m + \varepsilon \leq t \leq M - \varepsilon$. Then $x_0(t)$ and $x'_0(t)$ where $m + \varepsilon \leq t \leq M - \varepsilon$ are paths on the one-skeleton of C_n. By construction $l_1(x_0(t)) = t$, $l_1(x'_0(t)) = t$,

$$(1) \qquad l_{u'}(x_0(t)) > \varepsilon_0 + \frac{\delta}{2}, \quad l_{u'}(x'_0(t)) < -\left(\varepsilon_0 + \frac{\delta}{2}\right)$$

$$\text{for } m + \varepsilon \leq t \leq M - \varepsilon.$$

We say then that $\{x_0(t), m + \varepsilon \leq t \leq M - \varepsilon\}$ and $\{x'_0(t), m + \varepsilon \leq t \leq M - \varepsilon\}$ are paths on the one-skeleton of C_n "in the direction near u". Following the methods developed in Theorem 1 in [2] we construct two l-strictly increasing paths $z_0(t)$ and $z'_0(t)$, $m + \varepsilon \leq t \leq M - \varepsilon$ on the one-skeleton of A such that

$$(2) \qquad l_1(z_0(t)) = t, \quad l_1(z'_0(t)) = t \quad \text{and}$$

$$\|\pi_0(z_0(t)) - x_0(t)\| < \frac{\varepsilon_0}{3}, \quad \|\pi_0(z'_0(t)) - x'_0(t)\| < \frac{\varepsilon_0}{3}$$

$$\text{where } m + \varepsilon \leq t \leq M - \varepsilon.$$

From relations (1) and (2) it follows that

$$(3) \qquad l_{u'}(\pi_0(z_0(t))) > \frac{2}{3}\varepsilon_0 + \frac{\delta}{2}, \quad l_{u'}(\pi_0(z'_0(t))) < -\left(\frac{2}{3}\varepsilon_0 + \frac{\delta}{2}\right)$$

$$\text{and} \quad \begin{aligned} &\pi_0\{z_0(t)\colon m + \varepsilon \leq t \leq M - \varepsilon\} \cap B = \varnothing, \\ &\pi_0\{z'_0(t)\colon m + \varepsilon \leq t \leq M - \varepsilon\} \cap B = \varnothing. \end{aligned}$$

As (2) holds we may say that $z_0(t)$, $z'_0(t)$ are paths on the one-skeleton of A in the direction near u and we write $z_0 = z_u$ and $z'_0 = z'_u$.

Let S be the unit ball in E^n, lying on the hyperplane $l_1(x) = 0$ and let θ be a positive number such that $0 < \theta < (1/2d)(\delta/2 + \varepsilon_0/3)$ where $d = \operatorname{diam} C_n$. The compactness of S implies the existence of unit vectors $u_1, u_2, \ldots, u_m$ such that for every unit vector u in S, there exists $i_0 \in \{1, 2, \ldots, m\}$ with $\|u - u_{i_0}\| < \theta$. Let $Z_{u_i}\{z_{u_i}(t), m + \varepsilon \leq t \leq M - \varepsilon\}$ and $Z_{u_{m+i}} = \{z'_{u_i}(t),\ m + \varepsilon \leq t \leq M - \varepsilon\}$ where $i = 1, 2, \ldots, m$ be paths on the one-skeleton of A in the direction near u_i. Let $j(Z_{u_1}, Z_{u_2}, \ldots, Z_{u_i})$ be the junction set of the paths $Z_{u_1}, Z_{u_2}, \ldots, Z_{u_i}$. Suppose now that $\operatorname{card} j(Z_{u_1}, Z_{u_2}, \ldots, Z_{u_{\lambda-1}}) < +\infty$ and $\operatorname{card} j(Z_{u_1}, Z_{u_2}, \ldots, Z_{u_\lambda}) = +\infty$ for some λ such that $1 \leq \lambda \leq 2m$. Renaming, if necessary, the paths $Z_{u_1}, Z_{u_2}, \ldots, Z_{u_\lambda}$ we consider the greatest integer k such that $1 \leq k \leq \lambda - 1$, $\operatorname{card} j(Z_{u_i}, Z_{u_\lambda}) < \infty$ for $i = 1, 2, \ldots, k - 1$ and $\operatorname{card} j(Z_{u_i}, Z_{u_\lambda}) = +\infty$ for $i = k, k + 1, \ldots, \lambda - 1$.

Let

$$\alpha = \inf\{t: t \in [m+\varepsilon, M-\varepsilon] \text{ and } z_{u_k}(t) \in j(Z_{u_k}, Z_{u_\lambda})\}$$

and

$$\beta = \sup\{t: t \in [m+\varepsilon, M-\varepsilon] \text{ and } z_{u_k}(t) \in j(Z_{u_k}, Z_{u_\lambda})\}.$$

As z_{u_k} and z_{u_λ} are continuous functions, there is a finite number of closed subintervals $[a_i, b_i]$, $i = 1, 2, \dots, \nu$, of $[m+\varepsilon, M-\varepsilon]$ with the following properties:

(i) $z_{u_k}(a_i) = z_{u_\lambda}(a_i)$, $z_{u_k}(b_i) = z_{u_\lambda}(b_i)$

(ii) $z_{u_k}(t) \neq z_{u_\lambda}(t)$, $\alpha_i < t < b_i$

(iii) $\max_{a_i < t < b_i} \|z_{u_k}(t) - z_{u_\lambda}(t)\| > \varepsilon_0/3$ for $i = 1, 2, \dots, \nu$.

Then

$$z_{u_\lambda}(m+\varepsilon, a) \cup z_k(a, a_1) \cup \bigcup_{i=1}^{\nu} z_{u_\lambda}(a_i, b_i)$$

$$\cup \bigcup_{i=1}^{\nu-1} z_{u_k}(b_i, a_{i+1}) \cup z_{u_k}(b_\nu, b) \cup z_{u_\lambda}(b, M-\varepsilon)$$

is an l-increasing path, $Z^*_{u_\lambda}$ say, on the one-skeleton of C that is different from Z_{u_λ} on the set

$$\Gamma = z_{u_k}(a, a_1) \cup \bigcup_{i=1}^{\nu-1} z_{u_k}(b_i, a_{i+1}) \cup z_{u_k}(b_\nu, b).$$

By construction the set Γ is within distance $\varepsilon_0/3$ from Z_{u_λ}, hence we have

$$\|z_{u_\lambda}(t) - z^*_{u_\lambda}(t)\| < \varepsilon_0/3 \quad \text{for every } t \in [m+\varepsilon, M-\varepsilon] \tag{4}$$

As card $j(Z_{u_i}, Z^*_{u_\lambda}) < +\infty$ for $i = 1, 2, \dots, k$, we can replace Z_{u_λ} by $Z^*_{u_\lambda}$ for every $\lambda = 1, 2, \dots, 2m$ with card $j(Z_{u_1}, \dots, Z_{u_{\lambda-1}}) < +\infty$ and card $j(Z_{u_1}, \dots, Z_{u_\lambda}) = +\infty$. Then card $j(Z^*_{u_1}, \dots, Z^*_{u_{2m}}) < +\infty$ and using (3) and (4) we get $|l_{u'}(\pi_0(z^*_{u_\lambda}(t)))| > \delta/2 + \varepsilon_0/3$ where $u' \in S$, $\|u' - u_\lambda\| < \theta$.

Now we can define the graph G with vertex set $V = \{K_0\} \cup \{K_1\} \cup j(Z^*_{u_1}, \dots, Z^*_{u_{2m}})$, where an ordered pair of these nodes is said to form a directed subgraph of G if they are joined by an l-increasing arc from $\bigcup_{i=1}^{2m} Z^*_{u_i}$, which contains no other node of G. The required result now follows from Menger-Whitney theorem for the finite graph G, if we are able to show that the removal of $(n-1)$ vertices from $j(Z^*_{u_1}, \dots, Z^*_{u_{2m}})$ still allows an l-increasing path running from K_0 to K_1.

Let $y_1, y_2, \ldots, y_{n-1}$ be $(n-1)$ vertices from $j(Z^*_{u_1}, \ldots, Z^*_{u_{2m}})$. For the points $\pi_0(y_1), \pi_0(y_2), \ldots, \pi_0(y_{n-1})$ of E_n, there exists a linear functional l' on E_n such that $l'(\pi_0(y_i)) \geq 0$, $i = 1,2,\ldots,n-1$, $l'(e_1) = 0$ and $l'(v) = 1$ for some $v \in S$. Let now $u \in S$ be an arbitrary vector such that $l'(u) = 0$ and $l_1(u) = 0$. For the vector u there exists a vector $u_k \in S$ such that $\|u - u_k\| \leq \theta$. Let $Z^*_{u_{m+k}}$ be the path on the one-skeleton of C in the direction near u_k, with

$$(5) \qquad l_{u_k}\left(\pi_0\left(z^*_{u_{m+k}}(t)\right)\right) < -\left(\frac{\delta}{2} + \frac{\varepsilon_0}{3}\right), \qquad m + \varepsilon \leq t \leq M - \varepsilon$$

We can also select u in such a way that $l'(u) = 0$ and $l_1(u) = 0$ for which the corresponding l_{u_k} has the property $l_{u_k}(v') = 1$ for some $v' \in S$ with $\|v - v'\| < \theta$.

Now, we may suppose that

$$(6) \qquad \begin{array}{ll} l_{u_k}(\pi_0(y_i)) \geq 0 & \text{for } i = 1,2,\ldots,\mu \quad \text{and} \\ l_{u_k}(\pi_0(y_i)) < 0 & \text{for } i = \mu+1,\ldots,n-1 \end{array}$$

Relations (5) and (6) imply that

$$(7) \qquad \pi_0(y_i) \notin \pi_0\left(Z^*_{u_{m+k}}\right) \quad \text{for } i = 1,2,\ldots,\mu$$

We have that $l'(v) = 1$, $l_{u_k}(v') = 1$ with $\|v - v'\| < \theta$ and $l'(\pi_0(y_i)) \geq 0$, $l_{u_k}(\pi_0(y_i)) < 0$ for $i = \mu + 1,\ldots,n-1$. Hence

$$(8) \qquad l_{u_k}(\pi_0(y_i)) \geq -d\theta - \left(\frac{\delta}{2} + \frac{\varepsilon_0}{3}\right), \qquad i = \mu + 1,\ldots,n-1.$$

From (5) and (8) we have that $\pi_0(y_i) \notin \pi_0(Z^*_{u_{m+k}})$ for $i = \mu + 1,\ldots,n-1$. Hence, from (7) and (8) follows that $y_i \notin Z^*_{u_{m+k}}$, $i = 1,2,\ldots,n-1$ which completes the proof of the theorem.

From the above theorem one can deduce the following corollaries whose proofs are omitted as obvious.

COROLLARY 1. *Suppose that C and l are defined as in Theorem* 1, *the faces*

$$F_0 = \left\{x \in C\colon l(x) = \min_{y \in C} l(y)\right\} \quad \text{and} \quad F_1 = \left\{x \in C\colon l(x) = \max_{y \in C} l(y)\right\}$$

are such that the dimension of $F_0' \cap F_1'$ is infinite, where F_0' and F_1' are the corresponding subspaces translates of F_0 and F_1 correspondingly. Then for every $n = 1,2,\ldots$ there are n l-strictly increasing paths on the one-skeleton of C mutually disjoint that join F_0 to F_1.

COROLLARY 2. *Suppose that C a compact convex set of infinite dimension in a normed space E. Then the one-dimensional Hausdorff measure of the one-skeleton is infinite.*

We may remark that the n-dimensional Hausdorff measure of the n-skeleton of a set C as in Corollary 2 is infinite for every $n = 1, 2, \ldots$. For a direct proof of this result see [**1**].

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