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EIGENVALUE BOUNDS FOR THE DIRAC OPERATOR

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A natural question in the study of geometric operators is that of how much information is needed to estimate the eigenvalues of an operator. For the square of the Dirac operator, such a question has at least peripheral physical import. When coupled to gauge fields, the lowest eigenvalue is related to chiral symmetry breaking. In the pure metric case, lower eigenvalue estimates may help to give a sharper estimate of the ADM mass of an asymptotically flat spacetime with black holes. We use three tools to estimate the eigenvalues of the square of the (purely metric) Dirac operator: the conformal covariance of the operator, a patching method and a heat kernel bound.

I. A lower bound. Let V be a vector bundle associated to the $\mathrm{SO}(n)$ ($\mathrm{Spin}(n)$) frame bundle of a compact n-dimensional oriented (spin) Riemannian manifold X, with a positive-definite inner product $\langle\ ,\ \rangle$. For each metric g, let $T_g\colon C^\infty(V) \to C^\infty(V)$ be a geometric elliptic symmetric differential operator of order $j < n$. If $g' = e^{2\sigma}g$ is a conformally related metric, suppose that $T_{g'} = e^{-j\sigma}e^{-(n-j)\sigma/2}T_g e^{(n-j)\sigma/2}$. Let $\lambda_1^2(g)$ denote the lowest eigenvalue of T_g^2.

PROPOSITION 1. (i) *If T_g is invertible then $\exists c > 0$ s.t. $\forall g' \in [g]$, (the conformal class of g),*

$$\lambda_1^2(g') \geq c^{-2}(\mathrm{Vol}\, g')^{-2j/n}. \tag{1}$$

(ii) *Suppose that a multiple mV of V contains a trivial subbundle of real dimension $> n$. Then the best constant $\tilde{c}$ in (1) is*

$$d \equiv \sup_{f\neq 0}\left|\int \left\langle f, T_g^{-1}f\right\rangle d\,\mathrm{vol}\right| \Big/ \|f\|^2_{2n/(n+j)}.$$

Proof. (i) Let ψ range through $C^\infty(V)$. Then

$$\lambda_1^{-1}(g') = \sup_{\psi\neq 0}\left|\int \left\langle \psi, T_{g'}^{-1}\psi\right\rangle d\,\mathrm{vol}'\right| \Big/ \int \langle \psi,\psi\rangle\, d\,\mathrm{vol}'$$

$$= \sup_{\psi\neq 0}\left|\int e^{n\sigma}\left\langle \psi, e^{-(n-j)\sigma/2}T_g^{-1}e^{(n+j)\sigma/2}\psi\right\rangle d\,\mathrm{vol}\right| \Big/ \int e^{n\sigma}\langle \psi,\psi\rangle\, d\,\mathrm{vol}$$

$$= \sup_{f\neq 0}\left|\int \left\langle f, T_g^{-1}f\right\rangle d\,\mathrm{vol}\right| \Big/ \int e^{-j\sigma}\langle f,f\rangle\, d\,\mathrm{vol}.$$

By Hölder's inequality,

$$\left(\int |f|^{2n/(n+j)}\, d\,\mathrm{vol}\right)^{(n+j)/n} \leq \left(\int e^{-j\sigma}|f|^2\, d\,\mathrm{vol}\right)(\mathrm{Vol}\, g')^{j/n}.$$

Then

$$\lambda_1^{-1}(g')(\mathrm{Vol}\, g')^{j/n} \leq \sup_{f\neq 0}\left|\int \left\langle f, T_g^{-1}f\right\rangle d\,\mathrm{vol}\right| \Big/ \left(\int |f|^{2n/(n+j)}\, d\,\mathrm{vol}\right)^{(n+j)/n}$$

$$\leq \sup_{f\neq 0}\left(\|f\|_2 \Big/ \left\| |T_g|^{1/2} f\right\|_{2n/(n+j)}\right)^2.$$

Because $(I + \nabla^+\nabla)^{j/4}|T_g|^{-1/2}$ is bounded on $L_{2n/(n+j)}$ and $\mathscr{L}^{j/2}_{2n/(n+j)} \hookrightarrow L_2$, [5], the RHS is finite.

(ii) Consider $T_{g'} = mT_{g'}$ acting on mV with a C^∞ section $\tilde{\psi}$. Then

$$\tilde{c} = \sup_{\tilde{\psi}\neq 0}\left|\int \left\langle \tilde{\psi}, \tilde{T}_g^{-1}\tilde{\psi}\right\rangle d\,\mathrm{vol}'\right| \Big/ (\mathrm{Vol}\, g')^{j/n}\left(\int \langle \tilde{\psi}, \tilde{\psi}\rangle\, d\,\mathrm{vol}'\right)$$

$$\leq \sup_{\tilde{f}\neq 0}\left|\int \left\langle \tilde{f}, \tilde{T}_g^{-1}\tilde{f}\right\rangle d\,\mathrm{vol}\right| \Big/ \|\tilde{f}\|^2_{2n/(n+j)} = d.$$

With the hypothesis, the generic section of mV has no zeroes. Let $\{\tilde{f}_i\}$ be a sequence in $C^\infty(mV)$ approaching the $\sup d$. By perturbing each $\tilde{f}_i$ arbitrarily little in the C^∞ topology, we can assume that each $\tilde{f}_i$ has no zeroes. Define $g_i' = |\tilde{f}_i|^{4/(n+j)}g$ and $\tilde{\psi}_i = \tilde{f}_i/|\tilde{f}_i|$. Then

$$d = \lim_i \left|\int \left\langle \tilde{f}_i, \tilde{T}_g^{-1}\tilde{f}_i\right\rangle d\,\mathrm{vol}\right| \Big/ \|\tilde{f}_i\|^2_{2n/(n+j)}$$

$$= \lim_i \left|\int \left\langle \tilde{\psi}_i, \tilde{T}_{g_i'}^{-1}\tilde{\psi}_i\right\rangle d\,\mathrm{vol}_i'\right| \Big/ \left(\left(\int \langle \tilde{\psi}_i, \tilde{\psi}_i\rangle\, d\,\mathrm{vol}_i'\right)(\mathrm{Vol}\, g_i')^{j/n}\right) \leq \tilde{c}. \quad \square$$

II. The Dirac operator. For background on the Dirac operator we refer to [4]. X is a spin manifold with a fixed spin structure. The spinor bundle V is associated to the principal $\mathrm{Spin}(n)$ bundle over X. The Dirac operator is the composition $\not{D}$: $C^\infty(V) \xrightarrow{\nabla} C^\infty(V) \otimes \Lambda^1(X) \to C^\infty(V)$, the last map being Clifford multiplication.

PROPOSITION 2. *Take $g' = e^{2\sigma}g$. Then*

$$\not{D}_{g'} = e^{-\sigma}e^{-(n-1)\sigma/2}\not{D}_g e^{(n-1)\sigma/2}.$$

Proof. Let $\{e_j\}_{j=1}^n$ be an orthonormal frame for g, with dual frame $\{\tau_j\}_{j=1}^n$. Locally, $\not{D}_g = -i\sum_{i=1}^n \gamma^j\nabla_{e_j}$, with $\{\gamma_j\}_{j=1}^n \in \mathrm{End}(C^{2^{[n/2]}})$ satisfying $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$ and $\nabla_{e_j} = e_j + \frac{1}{4}\langle \omega_{ab}, e_j\rangle\gamma^a\gamma^b$. The new orthonormal frame for g' is $\{e'_j\}_{j=1}^n = \{e^{-\sigma}e_j\}_{j=1}^n$. The new connection is $\omega'_{ab} = \omega_{ab} - (e_a\sigma)\tau_b + (e_b\sigma)\tau_a$. Then

$$\begin{aligned} \not{D}_{g'} &= -i\sum \gamma^j \nabla_{e'_j} = -i\sum \gamma^j\left(e'_j + \tfrac{1}{4}\left\langle \omega'_{ab}, e'_j\right\rangle \gamma^a\gamma^b\right) \\ &= -ie^{-\sigma}\sum \gamma^j\left(\nabla_j + \tfrac{1}{4}(e_b\sigma)[\gamma^j, \gamma^b]\right) \\ &= -ie^{-\sigma}\sum \gamma^j\left(\nabla_j + \frac{n-1}{2}e_j\sigma\right) = e^{-\sigma}e^{-(n-1)\sigma/2}\not{D}_g e^{(n-1)\sigma/2}. \quad \square \end{aligned}$$

Thus $\not{D}$ is conformally covariant with $j = 1$. This differs from the corresponding equation in **[4]**, which has an additional line bundle tensored, by the factor $e^{-\sigma}$, but does not change the conclusion of **[4]** that the dimension of the harmonic spinor space is conformally invariant. The two Dirac operators can be compared because the conformal change in the metric does not affect the spinor bundle; only the soldering form on the Spin(n) bundle is changed, not the bundle itself.

Equation (1) implies, in particular, that on S^2, $\exists c > 0$ s.t. $\forall g$, $\lambda_1^2(g) \geq c^{-2}(\mathrm{Vol}\, g)^{-1}$. On the standard S^2, $\lambda_1 = 1$. Thus the best constant d satisfies $d \geq 1/\sqrt{4\pi}$. It appears that $d = 1/\sqrt{4\pi}$, although we have no proof.

The conformal covariance can also be used to get upper bounds on λ_1^2.

PROPOSITION 3. *Given a conformal glass* $[g]$, $\exists b > 0$ s.t. $\forall g' \in [g]$ *with* $R(g') < 0$, $\lambda_1^2(g') \leq -bR_{\min}(g')$.

Proof. Fix a g in the conformal class s.t. $R(g) < 0$ and write $g' = e^{2\sigma}g$. For any $\psi \in C^\infty(V)$,

$$\begin{aligned} \lambda_1^2(g) &\leq \int e^{n\sigma}\left|\not{D}_{g'}\psi\right|^2 d\,\mathrm{vol}\Big/\int e^{n\sigma}|\psi|^2\, d\,\mathrm{vol} \\ &= \int e^{-\sigma}\left|\not{D}_g e^{(n-1)\sigma/2}\psi\right|^2 d\,\mathrm{vol}\Big/\int e^{\sigma}\left|e^{(n-1/2)\sigma}\psi\right|^2 d\,\mathrm{vol}. \end{aligned}$$

Take $e^{(n-1/2)\sigma}\psi = \psi_0$, a lowest eigenfunction of $\not{D}_g$. Then

$$\lambda_1^2(g') \leq \lambda_1^2(g)\left(\sup|\psi_0|^2/\inf|\psi_0|^2\right)\int e^{-\sigma}\, d\,\mathrm{vol}\Big/\int e^{\sigma}\, d\,\mathrm{vol}.$$

For $n \geq 3$,

$$-4\frac{n-1}{n-2}e^{-n\sigma/2}\nabla^2 e^{(n-2)\sigma/2} + R(g)e^{-\sigma} = R(g')e^{\sigma}.$$

Then

$$\left(R_{\max}(g)/R_{\min}(g')\right)\left(\int e^{-\sigma}\,d\,\mathrm{vol}\Big/\int e^{\sigma}\,d\,\mathrm{vol}\right)$$
$$\le \int R(g)e^{-\sigma}\,d\,\mathrm{vol}\Big/\int R(g')e^{\sigma}\,d\,\mathrm{vol}$$
$$= 1 + n(n-1)\int e^{-\sigma}|\nabla\sigma|^2\,d\,\mathrm{vol}\Big/\int R(g')e^{\sigma}\,d\,\mathrm{vol} \le 1$$

and

$$\lambda_1^2(g') \le \lambda_1^2(g)\left(\sup|\psi_0|^2/\inf|\psi_0|^2\right)\left(R_{\min}(g')/R_{\max}(g)\right).$$

For $n = 2$, $-e^{-\sigma}\nabla^2\sigma + R(g)e^{-\sigma} = R(g')e^{\sigma}$,

$$\int Re^{-\sigma}\,d\,\mathrm{vol}\Big/\int R(g')e^{\sigma}\,d\,\mathrm{vol}$$
$$= 1 + \int e^{-\sigma}|\nabla\sigma|^2\,d\,\mathrm{vol}\Big/\int R(g')e^{\sigma}\,d\,\mathrm{vol} \le 1$$

and the same result holds. □

III. A patching method. We give an upper bound on S^2 using the method of [**1**].

PROPOSITION 4. *Let M_l be the set of metrics on S^2 with Gaussian curvature K satisfying $0 \le K \le l$. Then $\exists\alpha > 0$ s.t. $\forall l \in R^+$ and $\forall g \in M_l$, $\lambda_1^2(g) \le \alpha l$.*

Proof. First we solve for the lowest eigenfunction of the Dirichlet problem for $\not{D}^2$ on the unit disk. Take

$$\gamma^r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^\theta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad e_r = \frac{\partial}{\partial r} \quad \text{and} \quad e_\theta = \frac{1}{r}\frac{\partial}{\partial\theta}.$$

Then

$$\not{D}^2 = -\left(e_r + \frac{1}{2r}\right)^2 - e_\theta^2 + \gamma^r\gamma^\theta\frac{1}{r}e_\theta.$$

Take

$$\psi = \left(\eta_1(r)e^{i(m_1+1/2)\theta}, \eta_2(r)e^{i(m_2-1/2)\theta}\right),$$
$$\eta_1(1) = \eta_2(1) = 0,\ m_1, m_2 \in Z.$$

Then

$$\not{D}^2\psi = \left(-\left(\frac{\partial}{\partial r} + \frac{1}{2r}\right)^2\eta_1 + \frac{1}{r^2}\left(m_1^2 - \frac{1}{4}\right)\eta_1,\right.$$
$$\left. -\left(\frac{\partial}{\partial r} + \frac{1}{2r}\right)^2\eta_2 + \frac{1}{r^2}\left(m_2^2 - \frac{1}{4}\right)\eta_2\right).$$

WLOG, we can assume $\eta_2 = 0$. The lowest eigenfunction is $\psi_0 = (J_0(zr)e^{i\theta/2}, 0)$ with eigenvalue $\lambda_1^2 = z^2$, z being the first zero of J_0.

Take normal coordinates around a point x in S^2 and write g as $dr^2 + f(r,\theta)\,d\theta^2$. With $e_r = \partial/\partial r$ and $e_\theta = f^{-1/2}\partial/\partial\theta$,

$$\not{D}_g = -i\gamma^r\left(e_r + \frac{1}{4f}f_{,r}\right) - i\gamma^\theta e_\theta.$$

WLOG, we can assume $l = \pi^2$. Put $D = \{y \in S^2\colon \exists!$ minimal geodesic from x to y and $d(x,y) < 1\}$, a contractible domain. We wish to patch the Dirichlet solution onto X. Define $\psi \in L^2(V)$ by

$$\psi(y) = \begin{cases} \left(J_0(zd(x,y))e^{i\theta/2}, 0\right) & \text{if } y \in D, \\ 0 & \text{if } y \notin D. \end{cases}$$

Then ψ is C^1 a.e. and

$$\lambda_1^2 \le \int |\not{D}\psi|^2\,d\,\mathrm{vol}\Big/\int |\psi|^2\,d\,\mathrm{vol}$$
$$= \frac{\displaystyle\int_{S_1}\int_0^{a(\theta)} f^{1/2}\left(\partial_r J_0(zr) + \left(\frac{1}{4f}f_{,r} - \frac{1}{2r}\right)J_0(zr)\right)^2 dr\,d\theta}{\displaystyle\int_{S^1}\int_0^{a(\theta)} f^{1/2}J_0^2(zr)\,dr\,d\theta},$$

where $a(\theta)$ is min(distance to the cut locus of x along angle $\theta, 1$). Now

$$\int_0^{a(\theta)} f^{1/2}\left(\partial_r J_0(zr) + \left(\frac{1}{4f}f_{,r} - \frac{1}{2r}\right)J_0(zr)\right)^2 dr$$
$$= \int_0^{a(\theta)} f^{1/2}\left(r^{1/2}f^{-1/4}\partial_r\left(r^{-1/2}f^{1/4}J_0(zr)\right)\right)^2 dr$$
$$= \left[r^{1/2}f^{1/4}J_0(zr)\partial_r\left(r^{-1/2}f^{1/4}J_0(zr)\right)\right]_{r=0}^{a(\theta)}$$
$$- \int_0^{a(\theta)} f^{1/2}J_0(zr)\left(r^{-1/2}f^{-1/4}\partial_r\left(r\partial_r\left(r^{-1/2}f^{1/4}J_0(zr)\right)\right)\right) dr$$
$$= [\partial \text{ term}] - \int_0^{a(\theta)} f^{1/2}J_0(zr)\left[\partial_r\partial_r J_0(zr) + \frac{1}{2}f_{,r}f^{-1}\partial_r J_0(zr)\right.$$
$$\left. + \left(\frac{1}{4r^2} - \frac{3}{16}(f_{,r})^2f^{-2} + \frac{1}{4}f_{,rr}f^{-1}\right)J_0(zr)\right] dr.$$

Put $\nabla_{\partial_r}\partial_\theta \equiv c\partial_\theta$. Then $f,_r = 2cf$ and $f,_{rr} = 2(c^2 - K)f$. By Rauch's comparison theorem, there are no conjugate points in D and $(1/\pi^2)\mathrm{Sin}^2\pi r \leq f \leq r^2$, $\pi \,\mathrm{Cot}\,\pi r \leq c \leq 1/r$ in D. Thus $[\partial \text{ term}] \leq 0$ and

$$-\frac{1}{\int_0^{a(\theta)} f^{1/2}J_0^2(zr)\,dr}$$

$$\cdot \int_0^{a(\theta)} f^{1/2}J_0(zr)\Big(\partial_r\partial_r J_0(zr) + \frac{1}{2}f,_r f^{-1}\partial_r J_0(zr)$$

$$+\left(\frac{1}{4r^2} - \frac{3}{16}(f,_r)^2 f^{-2} + \frac{1}{4}f,_{rr}f^{-1}\right)(J_0(zr))\Big)\,dr$$

$$\leq -\frac{1}{\int_0^{a(\theta)} f^{1/2}J_0^2(zr)\,dr}$$

$$\cdot \int_0^{a(\theta)} f^{1/2}J_0(zr)\Big(\partial_r\partial_r J_0(zr)$$

$$+\frac{1}{r}\partial_r J_0(zr) + \left(\frac{1}{4r^2} - \frac{1}{4}c^2 - \frac{1}{2}K\right)J_0(zr)\Big)\,dr$$

$$= z^2 + \int_0^{a(\theta)} f^{1/2}J_0^2(zr)\left(\frac{1}{4}\left(c^2 - \frac{1}{r^2}\right) + \frac{1}{2}K\right)dr\Big/\int_0^{a(\theta)} f^{1/2}J_0^2(zr)\,dr$$

$$\leq z^2 + \frac{\pi^2}{2} + \frac{\frac{1}{4}\int_0^{a(\theta)} f^{1/2}J_0^2(zr)\left(\max\left(\pi^2\,\mathrm{Cot}^2\,\pi r, \frac{1}{r^2}\right) - \frac{1}{r^2}\right)dr}{\int_0^{a(\theta)} f^{1/2}J_0^2(zr)\,dr}$$

$$\leq z^2 + \frac{\pi^2}{2} + \frac{\frac{1}{4}\int_0^{a(\theta)} J_0^2(zr)\left[r\max\left(\pi^2\,\mathrm{Cot}^2\,\pi r, \frac{1}{r^2}\right) - \frac{1}{\pi}(\mathrm{Sin}\,\pi r)\frac{1}{r^2}\right]dr}{\int_0^{a(\theta)} \pi^{-1}(\mathrm{Sin}\,\pi r)J_0^2(zr)\,dr}.$$

Thus $\lambda_1^2 \leq \alpha l$ with

$$\alpha = \frac{\frac{1}{\pi^2}\left[z^2 + \frac{\pi^2}{2} + \frac{1}{4}\sup_{0\leq a\leq 1}\int_0^a J_0^2(zr)\left[r\max(\pi^2\,\mathrm{Cot}^2\,\pi r, r^{-2}) - \pi^{-1}(\mathrm{Sin}\,\pi r)r^{-2}\right]dr\right]}{\int_0^a \frac{1}{\pi}(\mathrm{Sin}\,\pi r)J_0^2(zr)\,dr}$$

$$< \infty. \qquad \square$$

IV. Heat kernel estimates. The higher eigenvalues of the Dirac operator can be estimated from below via upper bounds on the heat kernel of $D\!\!\!/^2$.

PROPOSITION 5. *If* $n > 1$ *then*

$$\forall\alpha > 0,\ je^{-\alpha} \leq 2^{[n/2]}\int_X e^{-(\alpha/4\lambda_j)R(x)}\left(\operatorname{Vol}(X)^{-1} + 4\left(\frac{2C_1}{n}\frac{\alpha}{\lambda_j}\right)^{-n/2}\right)$$

with

$$(2) \qquad C_1 = \inf_{\substack{\int f = 0 \\ f\in H_1(x)}} \int|\nabla f|^2\Big/\left(\int f^2\right)^{(2+n/n)}\left(\int|f|\right)^{-4/n}.$$

Proof. We have that both $\not{D} = \nabla^+\nabla + R/4$ and $R/4$ are self-adjoint on the unique closed extension of $\not{D}^2|_{C^\infty(V)}$ [7]. By the Golden-Thomson inequality,

$$\operatorname{Tr} e^{-T\not{D}^2} \leq \operatorname{Tr} e^{-TR/8}e^{-T\nabla^+\nabla}e^{-TR/8} = \int_X e^{-TR(x)/4}\tau\left(e^{-T\nabla^+\nabla}\right)(x,x),$$

with τ being the local fiber trace.

We write $e^{-T\nabla^+\nabla}(x,x)$ as a Feynman-Kac path integral. This is given as a limit of approximations, each of which can be estimated.

Let r be the cut radius of X, $K_T(x,y)$ be the kernel of $e^{-T\Delta}$ and $\rho(a)$ be a bump function which is 1 near 0 and 0 for $|a| > r/2$. Define the operator L_T on $L^2(V)$ by

$$L_T(x,y) = K_T(x,y)\left(P\exp\left\{-\int_y^x \frac{1}{2}\omega_{ab}\sigma^{ab}\right\}\right)\rho(d(x,y)),$$

the path-ordered integral being taken along the unique minimal geodesic from x to y. Then for $\psi \in L^2(V)$,

$$\begin{aligned}\lim_{T\to 0}\frac{d}{dT}L_T\psi(x) &= \lim_{T\to 0}\int\left(-\Delta_y K_T(x,y)\right)P \\ &\qquad\times\exp\left\{-\int_y^x\frac{1}{2}\omega_{ab}\sigma^{ab}\right\}\rho(d(x,y))\psi(y)\,dy \\ &= \lim_{T\to 0} -\int K_T(x,y)\Delta_y\left(P\exp\left\{-\int_y^x\frac{1}{2}\omega_{ab}\sigma^{ab}\right\}\rho(d(x,y))\psi(y)\right)dy \\ &= -\int\delta(x-y)\Delta_y P\exp\left\{-\int_y^x\frac{1}{2}\omega_{ab}\sigma^{ab}\right\}\psi(y)\,dy.\end{aligned}$$

Choose normal coordinates around x and a synchronous frame τ^i. Then

$$P\exp\left\{-\int_y^x\frac{1}{2}\omega_{ab}\sigma^{ab}\right\} = 1$$

and

$$-\Delta_y P\exp\left\{-\int_y^x \frac{1}{2}\omega_{ab}\sigma^{ab}\right\}\psi(y)\Big|_{y=x} = \partial^2\psi(x) = -(\nabla^+\nabla\psi)(x).$$

Thus

$$\lim_{n\to\infty} n\left\|\left(e^{-T\nabla^+\nabla/n} - L_{T/n}\right)\psi\right\| = 0$$

and

$$\begin{aligned}
&\left\|\left(e^{-T\nabla^+\nabla} - L^n_{T/n}\right)\psi\right\| \\
&= \left\|\sum_{i=0}^{n-1} L^i_{T/n}\left(e^{-T/n\nabla^+\nabla} - L_{T/n}\right)\left(e^{-T/n\nabla^+\nabla}\right)^{n-i-1}\psi\right\| \\
&\leq n \sup_{0\leq s\leq T}\left\|\left(e^{-T/n\nabla^+\nabla} - L_{T/n}\right)e^{-s\nabla^+\nabla}\psi\right\| \to 0,
\end{aligned}$$

showing that $e^{-T\nabla^+\nabla} = s\lim_{n\to\infty} L^n_{T/n}$.

Let $d\mu_{T,x,y}$ denote the Wiener measure on paths γ going from y to x in time T. Then for ψ, $\eta \in L^2(V)$,

$$\begin{aligned}
\left|\left\langle\psi, {}^{-T\nabla^+\nabla}\eta\right\rangle\right| &= \lim_{n\to\infty} |\langle\psi, L^n_{T/n}\eta\rangle| \\
&= \lim_{n\to\infty}\Bigg|\int\int \psi^+(x)P\exp\left\{-\int_{\tilde{\gamma}_n}\frac{1}{2}\omega_{ab}\sigma^{ab}\right\}\eta(y) \\
&\quad\times \prod_{i=0}^{n-1}\rho\left(d\left(\gamma\left(\frac{iT}{n}\right), \gamma\left(\frac{(i+1)T}{n}\right)\right)\right) d\mu_{T,x,y}(\gamma)\,dx\,dy\Bigg|,
\end{aligned}$$

$\tilde{\gamma}_n$ being the broken geodesic connecting the points $\{\gamma(iT/n)\}^n_{i=0}$. Because $P\exp\{-\int_{\tilde{\gamma}_n}\frac{1}{2}\omega_{ab}\sigma^{ab}\}$ is in Spin, this is

$$\leq \int |\psi(x)|\,|\eta(y)|\,d\mu_{T,x,y}(\gamma)\,dx\,dy.$$

Letting ψ and η approach V-valued δ-functions with support at x and values $\psi_0$and η_0,

$$\left\langle\psi_0, e^{-T\nabla^+\nabla}(x,x)\eta_0\right\rangle \leq |\psi_0|\,|\eta_0|\int d\mu_{T,x,y}(\gamma) = |\psi_0|\,|\eta_0|K_T(x,x).$$

Thus

$$\tau\left(e^{-T\nabla^+\nabla}(x,x)\right) \leq (\dim V)K_T(x,x).$$

Now

$$K_T(x,x) \le \mathrm{vol}(X)^{-1} + 4(2C_1T/n)^{-n/2} \qquad [\mathbf{2}].$$

Thus

$$\mathrm{Tr}\, e^{-tD^2} \le (\dim V) \int e^{-TR(x)/4}\left[\mathrm{vol}(X)^{-1} + 4\left(\frac{2C_1}{n}T\right)^{-n/2}\right].$$

Putting $T = \alpha/\lambda_j$ gives the desired result. □

COROLLARY. *For* $j \ge 2^{[n/2]}e^{n/2}$,

$$\lambda_j \ge c_1(4\,\mathrm{Vol}(X))^{-2/n}(2^{-[n/2]}e^{-n/2}j - 1)^{2/n} + \frac{1}{4}R_{\min}.$$

Proof. From (2)

$$je^{-\alpha} \le 2^{[n/2]}e^{-(\alpha/4\lambda_j)R_{\min}}\left(1 + 4\left(\frac{2c_1}{n}\frac{\alpha}{\lambda_j}\right)^{-n/2}\mathrm{Vol}(X)\right). \tag{3}$$

Putting $\alpha = \beta\lambda_j$,

$$\lambda_j \ge \frac{1}{\beta}\ln(j/2^{[n/2]}) - \frac{1}{\beta}\ln\left(1 + 4\left(\frac{2C_1}{n}\right)^{-n/2}\beta^{-n/2}\mathrm{Vol}(X)\right) + \frac{1}{4}R_{\min}.$$

Thus it suffices to assume $R_{\min} = 0$, pick β to estimate λ_j and then add $\frac{1}{4}R_{\min}$. Putting $R_{\min} = 0$ in (3),

$$\lambda_j \ge \alpha\left(\frac{1}{4}\,\mathrm{Vol}(X)\right)^{2/n}(2C_1/n)(j2^{-[n/2]}e^{-\alpha} - 1)^{2/n}.$$

This gives lower bounds whenever $j > 2^{[n/2]}$, but to get the best power law behaviour take $\alpha = n/2$ and $j \ge 2^{[n/2]}e^{n/2}$. Then

$$\lambda_j \ge C_1(4\,\mathrm{Vol}(X))^{-2/n}(2^{-[n/2]}e^{-n/2}j - 1)^{2/n}.$$

We note that C_1 can be estimated from below in terms of Diam(g), Vol(g) and Ric(g) [**2**]. □

REFERENCES

[1] S. -Y. Cheng, *Eigenvalue comparison theorems and its geometric applications*, Math. Z., **143** (1975), 289.

[2] S. -Y. Cheng and P. Li, *Heat kernel estimates and lower bound of eigenvalues*, Comment. Math. Helvetici, **53** (1981), 327.

[3] G. Gibbons, S. Hawkins, G. Horowitz and M. Perry, *Positive mass theorems for black holes*, Comm. Math. Phys., **88** (1983), 295.

[4] N. Hitchin, *Harmonic spinors*, Adv. in Math., **14** (1974), 1.
[5] M. Taylor, *Pseudodifferential Operators*, Princeton University Press, 1981
[6] C. Vafa and E. Witten, *Eigenvalue inequalities for Fermions in Gauge Theories*, Princeton Univ. Physics Preprint, 1984
[7] J. Wolf, *Essential self-adjointness for the Dirac operator and its square*, Indiana Univ. Math. J., **22** (1972), 611.

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