PACIFIC JOURNAL OF MATHEMATICS
Vol. 129, No. 2, 1987

ANALYSIS OF INVARIANT MEASURES IN DYNAMICAL SYSTEMS BY HAUSDORFF MEASURE

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Hausdorff measure is a preliminary concept in the definition of Hausdorff dimension, which is one concept of the degree of singularity of a finite measure. In general, Hausdorff measure does not permit as detailed an analysis of an arbitrary natural invariant measure arising from a dynamical system as Lebesgue measure permits of an absolutely continuous measure. It is shown that even for a dynamical system as simple as a modified baker's transformation, the natural invariant measure has no representation as an indefinite integral with respect to any Hausdorff measure. However, Hausdorff measure can be used to compare different natural invariant measures according to degree of singularity even when their Hausdorff dimensions are identical.

1. Introduction. In this article we seek to illustrate some of the capabilities and limitations of Hausdorff measure for the analysis of invariant measures in dynamical systems in more detail than is possible with Hausdorff dimension alone.

Hausdorff dimension is a concept of the size of a set or the degree of singularity of a measure. In recent years it has often been used for the study of dynamical systems because many of the sets of interest which arise are of Lebesgue measure zero and many of the measures of interest whch arise are singular with respect to Lebesgue measure. Thus Lebesgue measure is of little or no aid in the analysis of these sets and measures. The Hausdorff dimension of a set of Lebesgue measure zero or a measure singular with respect to Lebesgue measure can, however, range over a wide spectrum of possible values; such sets and measures can thus be distinguished and classified by their Hausdorff dimensions.

Hausdorff measure is a necessary preliminary to the definition of Hausdorff dimension and in cases where the Hausdorff dimension fails to distinguish between two sets or between two measures (their Hausdorff dimensions being the same) it is sometimes possible to compare their sizes using Hausdorff measure.

Perhaps the greatest possible amount of information concerning a measure μ is given by a representation

$$\mu E = \int_E \delta(x)\, d\Lambda(x),$$

where δ is a density function and Λ is a "reference measure" which is uniform at each point, such as Hausdorff measure is. In several cases, an invariant measure arising from a dynamical system does have such a representation (we give some examples in §3) and in view of the natural relationship between the Hausdorff dimension of an invariant measure in a system and other properties of the system (for example, see **[12]**) and the relationship of Hausdorff measure to Hausdorff dimension, it seems reasonable to suppose that the invariant measure arising from a dynamical system might always have such a representation.

In this article we present a simple example of a dynamical system, namely, a modified baker's transformation, which shows that such is not the case. A by-product of the arguments used to establish this result is an illustration of the use of Hausdorff measure to provide a finer classification of measures than is possible by the use of Hausdorff dimension alone.

In §2 we review the definitions of Hausdorff measure and Hausdorff dimension. In §3 we give some examples of dynamical systems giving rise to invariant measures having representations as integrals of Hausdorff measure. In §4 we present our example of a system giving rise to an invariant measure without such a representation. In §5 we use Hausdorff measure to classify a family of measures akin to the example of §4 which have the same Hausdorff dimension and we discuss the extension of our methods to general invariant measures in dynamical systems.

2. Definition of Hausdorff measure and Hausdorff dimension. Let Ω be a metric space. Let λ be a function from some $(0, \eta)$ to $(0, \infty)$. The function λ need not be continuous or increasing. For a subset E of Ω we define the *Hausdorff measure* of E (associated with λ), denoted $m_\lambda E$, by

$$m_\lambda E = \sup_{0<\varepsilon<\eta} \inf_{\{A_n\}} \sum \lambda(\operatorname{diam} A_n),$$

where $\{A_n\}$ ranges over all coverings of E by open balls in Ω with $\operatorname{diam} A_n < \varepsilon$. Note that different choices of λ and different metrics on Ω can yield different measures m_λ. We call λ an *index function* for m_λ. The m_λ-measurable sets in Ω include all Borel sets. Many familiar measures can be obtained as special cases of Hausdorff measure. For example, if

$\Omega = \mathbf{R}^n$ and $\lambda(t) = t^n$, then m_λ is just a constant times n-dimensional Lebesgue measure. If we set $\lambda \equiv 1$, then m_λ is just counting measure; that is, $m_\lambda\{x\} = 1$ for each x in Ω.

If Ω is a linear space, and $\lambda(t) = t^d$ for some d, then the measure m_λ has an interesting scaling property. Let E be a set in Ω and a be a real positive number. Then

$$m_\lambda(aE) = a^d m_\lambda E,$$

where $aE = \{ax\colon\ x \in E\}$ is the set E magnified by the factor a. Compare this to the scaling property of n-dimensional Lebesgue measure. Also, m_λ is translation-invariant.

For the definition of the *Hausdorff dimension of a set*, we set $\lambda(t) = t^d$ for some unspecified value of d. Then $m_\lambda E$ is a function of d as well as E. If we fix E and vary d, we find that there is a critical value d_0 for d such that $m_\lambda E = \infty$ when $d < d_0$ and $m_\lambda E = 0$ when $d > d_0$. The number d_0 is the Hausdorff dimension of the set E, which we denote $\dim E$.

The *Hausdorff dimension of a finite measure* μ on Ω can be defined in terms of the Hausdorff dimension of subsets of Ω. Let

$$d_0 = \inf\{\dim E\colon\ \mu E > 0\};$$

then d_0 is the Hausdorff dimension of μ, denoted $\dim \mu$. Alternatively, $\dim \mu$ is the unique number such that, letting $\lambda(t) = t^d$, μ is absolutely continuous with respect to m_λ when $d < \dim \mu$ and μ is singular with respect to m_λ when $d > \dim \mu$.

For a set E and a Hausdorff measure m_λ, $m_\lambda E$ may be zero, or positive, or positive and finite. These are analogous to the range of possibilities for measures and, for a finite measure μ, correspond to μ being singular with respect to m_λ, absolutely continuous with respect to m_λ, or having a representation as an integral of m_λ, respectively.

Let μ and ν be finite measures. If $\dim \mu < \dim \nu$, then there exists a Hausdorff measure m_λ such that μ is singular with respect to m_λ while ν is absolutely continuous with respect to m_λ; for example, we may take $\lambda(t) = t^d$ with $d = (\dim \mu + \dim \nu)/2$. Thus we think of μ as being "more singular" than ν. Even in some cases where $\dim \mu = \dim \nu$, there exists m_λ such that μ is singular with respect to m_λ while ν is absolutely continuous with respect to m_λ; an example of this situation is given in §5. In this case, we could distinguish μ as being more singular than ν even though $\dim \mu = \dim \nu$. This is what was referred to in the introduction as Hausdorff measure providing a finer classification of measures than Hausdorff dimension alone. For a detailed treatment of this sort of classification of measures, see Rogers and Taylor [**8**].

3. Examples of invariant measures representable by integrals of Hausdorff measure.

A. The logistic map.

Let f: $[0,1] \to [0,1]$ be given by $f(x) = 4x(1-x)$. Let us define a sequence μ_n of measures on $[0,1]$ by setting μ_0 equal to Lebesgue measure on $[0,1]$ and defining $\mu_{n+1}E = \mu_n f^{-1}(E)$. The sequence μ_n converges weakly to a limit measure μ, which is the invariant measure for the system. Given an initial point x_0 chosen uniformly at random from $[0,1]$ and letting $x_{n+1} = f(x_n)$, μ is the probability distribution of x_n for n very large.

The measure μ has the integral representation

$$\mu E = \int_E \frac{dm(x)}{\pi\sqrt{x(1-x)}},$$

the notation $dm(x)$ denoting integration with respect to Lebesgue measure, which is a special case of Hausdorff measure. This representation gives a much better picture of μ than the bare fact that $\dim \mu = 1$; we can see, for example, that the density of μ goes to infinity at 0 and 1.

The logistic map may be the most widely-known example of a dynamical system; for an introduction to its theory see Collet and Eckmann [**2**]. The more general map $f_a(x) = ax(1-x)$ is more difficult to analyse, but in the work of Jakobson [**4**] and Benedicks and Carleson [**1**] it is shown that for a set of values of the parameter a of positive measure, the map f_a generates an invariant measure which is absolutely continuous and thus can be represented as an integral of Lebesgue measure.

B. For complex z, let $f(z) = z^2 - 2\bar{z}$. This map has as a chaotic attractor the region of $\mathbf{C}$ where

$$\rho(z) = 4(z^3 + \bar{z}^3) - (z\bar{z})^2 - 18z\bar{z} + 27 \geq 0.$$

Moreover, the invariant measure μ generated by f is absolutely continuous and has the integral representation:

$$\mu E = \int_E \frac{3\,dm_2(z)}{\pi^2\sqrt{\rho(z)}},$$

where $dm_2(z)$ denotes integration with respect to Lebesgue area measure in $\mathbf{C}$. Again, this representation tells us, for example, that the density goes to infinity on the boundary of the attractor and that the measure is invariant under multiplication of z by $-1/2 + i\sqrt{3}/2$.

This example and the mapping $f(x) = 4x(1 - x)$ above are members of a class of mappings with many special properties, among which is the existence of an absolutely continuous invariant measure with an algebraic density function. A more complete description is given in Withers [**11**].

C. A modified baker's transformation.

Let S be the square $[0,1] \times [0,1]$, endowed with the square metric, so that $\operatorname{dist}((x_1, y_1), (x_2, y_2)) = \sup\{|x_2 - x_1|, |y_2 - y_1|\}$. We define $f\colon S \to S$ by

$$f(x,y) = \begin{cases} (x/3, 2y) & \text{if } y < 1/2, \\ ((x+2)/3, 2y-1) & \text{if } y \geq 1/2. \end{cases}$$

Note $f(S) \neq S$. As before we define a sequence μ_n of measures on S by setting μ_0 equal to Lebesgue area measure on S and letting $\mu_{n+1}E = \mu_n f^{-1}(E)$. Then the sequence μ_n converges weakly to a limit measure μ on S. The measure μ is supported on the set $C \times [0,1]$, where C is the canonical Cantor set $(C = [0,1] - (1/3, 2/3) - (1/9, 2/9) - (7/9, 8/9) - \cdots)$, which has zero area. Thus μ cannot be represented as an integral of Lebesgue measure.

However, let us set $\lambda(t) = t^d$, where $d = \log_3 6 = \dim \mu$. Then μ has the representation

$$\mu E = \int_E \delta(x,y)\, dm_\lambda(x,y),$$

where $\delta(x,y) = 1$ if $x \in C$, $\delta(x,y) = 0$ otherwise, and the notation $dm_\lambda(x,y)$ denotes integration with respect to the measure m_λ. Again, from this representation we can see, for example, that μ is of uniform density on the set $C \times [0,1]$ and of zero density outside this set, which cannot be inferred just from the Hausdorff dimension of μ. A generalization of this example is the subject of the next section.

D. Let M be a Riemann manifold and $f\colon M \to M$ a $C^{1+\varepsilon}$ map which is *conformal*; i.e., its derivative is a scalar times an isometry. Let J be a compact subset of M with an open neighborhood V satisfying the following conditions:

(i) There exist $C > 0$ and $\alpha > 1$ such that $\|(f^n)'\| \geq C\alpha^n$ for all $x \in J$ and $n \geq 1$.

(ii) $J = \{x \in V\colon f^n(x) \in V \text{ for all } n > 0\}$.

(iii) For every nonempty open set U intersecting J there exists $n > 0$ such that $J \subset f^n(U)$.

These conditions make J a *repeller* for f and imply in particular that J is invariant under f. For a function ψ: $J \to \mathbf{R}$, we define the *pressure* $p(\psi)$ by

$$p(\psi) = \sup\left\{h(\mu) + \int \psi\, d\mu\right\},$$

where μ ranges over all probability measures on J invariant under f and h is the entropy of μ with respect to f. Let φ: $J \to \mathbf{R}$ be Hölder continuous. There is then a unique Radon measure μ on J such that $p(\varphi + \psi) - p(\varphi) \geq \int \psi\, d\mu$ for all ψ. The measure μ is a probability measure invariant under f and is called the *Gibbs measure* associated with φ.

In the case where $\varphi(x) = -\dim J \log\|f'\|$, it follows from a theorem of Ruelle [9] that μ has a representation as an integral:

$$\mu E = \int_E \delta(x)\, dm_\lambda(x),$$

where $\lambda(t) = t^{\dim J}$.

4. The modified baker's transformation. Our system has three parameters: p, q, and r, all positive numbers such that $p < 1$ and $q + r < 1$. Let us define g_0, g_1: $[0,1] \to [0,1]$ by $g_0(x) = qx$, $g_1(x) = rx + 1 - r$. Let S be the square $[0,1] \times [0,1]$. We define f: $S \to S$ by

$$f(x, y) = \begin{cases} (g_0(x), y/p) & \text{if } y \leq p, \\ (g_1(x), (y-p)/(1-p)) & \text{if } y > p. \end{cases} \tag{1}$$

As in example C of the previous section, f generates an invariant probability measure μ on S. It can be shown that the measure μ can be written as a product measure $\nu \times m$, where ν is a finite measure on the x-axis given by $\nu E = \mu(E \times [0,1])$ and m is Lebesgue measure on the y-axis. It can be further shown that μ has an integral representation as an integral of Hausdorff measure if and only if ν does. We can thus simplify our problem by considering the measure ν on $[0,1]$; we ask whether there exist an index function λ and a density δ such that

$$\nu E = \int_E \delta(x)\, dm_\lambda(x).$$

Our computations will be simplified if we restrict consideration to the following set Ξ of intervals. Each interval is indexed by indices $(\gamma_1, \gamma_2, \ldots, \gamma_n)$, each $\gamma_i \in \{0, 1\}$. We define

$$I(\gamma_1, \gamma_2, \ldots, \gamma_n) = g_{\gamma_1}\big(g_{\gamma_2}\big(\cdots g_{\gamma_n}([0,1]) \cdots\big)\big).$$

We call the number n of indices for $J = I(\gamma_1, \gamma_2, \ldots, \gamma_n)$ the *degree* of J and denote it $\deg J$. Note that with $q + r < 1$, if $\deg J = \deg K$ and $J \neq K$, then $J \cap K = \varnothing$. Let us define

$$F_n = \bigcup \{ J \in \Xi \colon \deg J = n \}.$$

Note $\nu([0,1] - F_n) = 0$. We further define

$$F = \bigcap_{n=0}^{\infty} F_n;$$

then $\nu([0,1] - F) = 0$ and we can ignore this part of the interval when seeking an integral representation. We also introduce a modification of Hausdorff measure.

4.1. Definition. Let $\lambda\colon (0, \eta) \to (0, \infty)$. Let $E \subset F$. We define the *modified Hausdorff measure* of E (associated with λ), denoted $M_\lambda E$, by

$$M_\lambda E = \sup_{0<\varepsilon<\eta} \inf_{\{I_n\}} \sum \lambda(\operatorname{diam} I_n),$$

where $\{I_n\}$ ranges over all coverings of E by elements of Ξ with $\operatorname{diam} I_n < \varepsilon$.

The only modification introduced is that the covering of E must be contained in the set Ξ.

The measure μ can be represented as an integral with respect to a modified Hausdorff measure M_λ if and only if it can be represented as an integral with respect to the ordinary Hausdorff measure m_λ. This follows from the next lemma, of which the proof is straightforward.

4.2. Lemma. *Suppose λ is such that for each $a > 0$, $\lambda(at)/\lambda(t)$ is bounded as t goes to 0. (On the real line, any Hausdorff measure of interest can be obtained from such an index function.) Let α be such that*

$$\lambda(t) < \alpha\left(\lambda\left(\frac{qt}{1-q-r}\right) + \lambda\left(\frac{rt}{1-q-r}\right)\right)$$

for t sufficiently small. Then for each $E \subset F$,

$$m_\lambda E \le M_\lambda E \le \alpha m_\lambda E.$$

Our main tool for seeking a representation for μ as an integral of modified Hausdorff measure M_λ is the following theorem.

4.3. THEOREM. *For* $\lambda\colon (0,\eta) \to (0,\infty)$ *and a finite measure* ν *on* F, *we define a function* $D^\lambda\nu\colon [0,1] \to (0,\infty)$ *as follows*:

$$D^\lambda \nu(x) = \inf_{0<\varepsilon<\eta} \sup_{x \in I \in \Xi} \frac{\nu I}{\lambda(\operatorname{diam} I)},$$

where $\operatorname{diam} I < \varepsilon$. *Let* $T_0 = \{x\colon D^\lambda\nu(x) = 0\}$, $T_+ = \{x\colon 0 < D^\lambda\nu(x) < \infty\}$, *and* $T_\infty = \{x\colon D^\lambda\nu(x) = \infty\}$. *Then*:

(i) ν *is absolutely continuous with respect to* M_λ *if and only if* $\nu T_\infty = 0$.

(ii) ν *is singular with respect to* M_λ *if and only if* $\nu T_0 = \nu T_+ = 0$.

(iii) ν *has a representation as an integral with respect to* M_λ *if and only if* $\nu T_0 = \nu T_\infty = 0$.

Proof. Rogers [7, Theorem 67] is exactly like this except using standard Hausdorff measure m_λ rather than modified Hausdorff measure M_λ. The proof of this theorem is a straightforward adaptation of his arguments.

We now make a symbolic analogy which is frequently used in the study of dynamical systems. Let us consider in more detail the properties of the intervals $I(\gamma_1, \gamma_2, \ldots, \gamma_n)$. Note first that if $k \le n$ then $I(\gamma_1, \gamma_2, \ldots, \gamma_n) \subset I(\beta_1, \beta_2, \ldots, \beta_k)$ if and only if $\gamma_i = \beta_i$ for $1 \le i \le k$. From the definition of $I(\gamma_1, \gamma_2, \ldots, \gamma_n)$ we see that

$$\operatorname{diam} I(\gamma_1, \gamma_2, \ldots, \gamma_n) = s_{\gamma_1} s_{\gamma_2} \cdots s_{\gamma_n},$$

where $s_0 = q$, $s_1 = r$. Using the fact that μ is uniform in the y-direction and invariant under f, we have that $\nu g_0(E) = p\nu E$ and $\nu g_1(E) = (1-p)\nu E$ for any ν-measurable set E. It follows that

$$\nu I(\gamma_1, \gamma_2, \ldots, \gamma_n) = t_{\gamma_1} t_{\gamma_2} \cdots t_{\gamma_n},$$

where $t_0 = p$, $t_1 = 1 - p$.

Now we consider $D^\lambda\nu(x)$. Recall that ν has a representation as an integral of the Hausdorff measure m_λ if and only if the set of points where $D^\lambda\nu(x) = 0$ or $D^\lambda\nu(x) = \infty$ is a ν-null set. We restate this in probabilistic terms. Let x be chosen at random, distributed according to ν. The measure ν has a representation as an integral of the Hausdorff measure m_λ if and only if $0 < D^\lambda\nu(x) < \infty$ with probability 1.

Let x be chosen at random, distributed according to ν. Consider the set Z of all intervals $I(\gamma_1, \gamma_2, \ldots, \gamma_n)$ in Ξ containing x. The set Z is totally ordered by inclusion. Recall that for $k \le n$, $I(\gamma_1, \gamma_2, \ldots, \gamma_n) \subset I(\beta_1, \beta_2, \ldots, \beta_k)$ if and only if $\gamma_i = \beta_i$ for $1 \le i \le k$. Thus corresponding to x there is an infinite sequence $(\gamma_1, \gamma_2, \ldots)$ of indices and Z contains just those intervals of the form $I(\gamma_1, \gamma_2, \ldots, \gamma_n)$, the indices being the first

n terms of the sequence. This gives a one-to-one correspondence between the points of F and infinite sequences $(\gamma_1, \gamma_2, \ldots)$ of indices. We note that the functions g_0 and g_1 are equivalent to right shifts on the set of sequences of indices.

Choosing the point x at random is equivalent to choosing the sequence $(\gamma_1, \gamma_2, \ldots)$ at random. We describe a method for choosing the sequence which makes ν the distribution of x. Choose the γ_n independently, each γ_n equalling s_i with probability t_i, $i = 0, 1$. We then have

$$P(x \in I(\gamma_1, \gamma_2, \ldots, \gamma_n)) = t_{\gamma_1} t_{\gamma_2} \cdots t_{\gamma_n},$$

as required.

We can now calculate $D^\lambda \nu(x)$ thusly:

$$D^\lambda \nu(x) = \limsup_{n \to \infty} \frac{\nu I(\gamma_1, \gamma_2, \ldots, \gamma_n)}{\lambda(\operatorname{diam} I(\gamma_1, \gamma_2, \ldots, \gamma_n))}$$

$$= \limsup_{n \to \infty} \frac{t_{\gamma_1} \cdots t_{\gamma_n}}{\lambda(s_{\gamma_1} \cdots s_{\gamma_n})}.$$

We take the logarithm of both sides:

$$\log D^\lambda \nu(x) = \limsup_{n \to \infty} \left[\sum_{i=1}^{n} \log t_{\gamma_i} - \Lambda\left(\sum_{i=1}^{n} \log s_{\gamma_i} \right) \right] \tag{2}$$

$$= \limsup_{n \to \infty} (u_n - \Lambda(v_n)),$$

where $\Lambda(u) = \log \lambda(e^u)$. Thus we can rephrase our problem in terms of a random walk on a lattice. In a plane with coordinates u and v, with initial position $(0, 0)$, we take randomly chosen steps. Each step is equal to the vector $(\Delta u = \log q,\ \Delta v = \log p)$ with probability p and equal to the vector $(\Delta u = \log r,\ \Delta v = \log(1 - p))$ with probability $(1 - p)$. If (u_n, v_n) is our position at the nth step, does there exist a function $\Lambda\colon (-\infty, a) \to (-\infty, \infty)$ such that

$$-\infty < \limsup_{n \to \infty} (v_n - \Lambda(u_n)) < \infty$$

with probability 1?

First suppose we try a function of the form $\Lambda(u) = ud$, equivalent to setting $\lambda(t) = t^d$. Thus we consider

$$\log D^\lambda \nu(x) = \limsup_{n \to \infty} (v_n - du_n) = \limsup_{n \to \infty} \sum_{i=1}^{n} (\Delta v_i - d\Delta u_i).$$

Note $\Delta v_i - d\Delta u_i$ equals $(\log p - d\log q)$ with probability p and equals $(\log(1-p) - d\log r)$ with probability $(1-p)$. The mean value of a step is therefore

$$\alpha = p\log p + (1-p)\log(1-p) - d(p\log q + (1-p)\log r).$$

If $\alpha > 0$ then this lim sup is $+\infty$ and $D^\lambda \nu(x) = \infty$ with probability 1; thus ν is singular with respect to m_λ by Theorem 4.3. If $\alpha < 0$ then this lim sup is $-\infty$ and $D^\lambda \nu(x) = 0$ with probability 1; thus ν is absolutely continuous with respect to m_λ by Theorem 4.3. We therefore have $\dim \nu = d_0$, where

$$d_0 = \frac{p\log p + (1-p)\log(1-p)}{p\log q + (1-p)\log r} \tag{3}$$

is the value of d which makes $\alpha = 0$. This is the formula dimension $= -$entropy/Lyapunov exponent as in **[12]**, p. 110. Also $\dim\mu = \dim\nu + 1 = d_0 + 1$.

In the case $d = d_0$ we apply the following theorem from statistics:

4.4. LAW OF THE ITERATED LOGARITHM (*Feller* [3]). *Let $\Delta a_1, \Delta a_2, \ldots$ be independent identically distributed random variables with mean a_0 and variance σ^2. Let*

$$a_n = \sum_{i=1}^n \Delta a_i.$$

Then with probability 1,

$$\limsup_{n\to\infty} \frac{a_n - na_0}{\sigma\sqrt{(2n\log\log n)}} = 1.$$

In the case at hand, we have $\Delta a_i = \Delta v_i - d_0\Delta u_i$, $a_0 = 0$, and $\sigma = 0$ if $\log p\ \log r = \log(1-p)\ \log q$; $\sigma > 0$ otherwise. Thus in the case $\sigma > 0$, we have

$$\begin{aligned} &\limsup_{n\to\infty} \frac{v_n - d_0 u_n}{\sigma\sqrt{(2n\log\log n)}} = 1, \quad \text{and} \\ &\limsup_{n\to\infty} (v_n - d_0 u_n) = \infty. \end{aligned} \tag{4}$$

Thus $D^\lambda\nu(x) = \infty$ with probability 1 and ν is singular with respect to m_λ by Theorem 4.3. In the case $\sigma = 0$, then $\Delta v_n - d_0\Delta u_n = 0$ with probability 1; thus

$$\log D^\lambda \nu(x) = \limsup_{n\to\infty} (v_n - d_0 u_n) = 0$$

with probability 1, and by Theorem 4.3, ν has a representation as an integral of the Hausdorff measure m_λ, where $\lambda(t) = t^d$, and $d = d_0$ reduces to the value $\log p/\log q = \log(1-p)/\log r$.

So we have a special case where ν (and hence μ) has a representation as an integral of Hausdorff measure. To treat the general case, we make use of an extended version of Theorem 4.4:

4.5. THEOREM (*Feller* [3]). *Let* $\Delta a_1, \Delta a_2, \ldots$ *be independent identically distributed random variables with mean* a_0 *and variance* σ^2, *each taking only two possible values. Let*

$$a_n = \sum_{i=1}^{n} \Delta a_i.$$

Let $\phi\colon (0,\infty) \to (0,\infty)$. *If the sum*

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n} \exp\left(-\frac{\phi^2(n)}{2}\right) \tag{5}$$

converges, then with probability 1,

$$a_n > na_0 + \sigma\sqrt{n}\,\phi(n) \tag{6}$$

for only finitely many n. *Conversely, if the sum* (5) *diverges, then with probability* 1, (6) *holds for infinitely many* n.

First we treat the special case $q = r$, so that $\Delta u = \log q$ with probability 1 and $u_n = n\log q$. We also assume without loss of generality that $p < 1/2$. We let $a_n = v_n$, so that $a_0 = p\log p + (1-p)\log(1-p)$ and $\sigma = \sqrt{p(1-p)}\,(\log(1-p) - \log p)$.

For our purposes, the crucial point of Theorem 4.5 is that it describes two alternatives, each of which in appropriate circumstances occurs with probability 1. Let z be the random variable

$$z = \limsup_{n\to\infty} v_n - \Lambda(u_n) - \Delta v_1,$$

so that $\log D^\lambda \nu(x) = z + \Delta v_1$. Suppose z is finite with positive probability. We show that this contradicts Theorem 4.5. Let

$$\varepsilon = (\log(1-p) - \log p)/3.$$

Then there exists an interval $(z_0, z_0 + \varepsilon)$ such that $z \in (z_0, z_0+\varepsilon)$ with positive probability δ. Recall that Δv_1 equals $\log p$ with probability p and equals $\log(1-p)$ with probability $(1-p)$. Thus the probability that $D^\lambda\nu(x) \in (z_0 + \log p,\ z_0 + \log p + \varepsilon)$ is at least $p\delta$ and the probability that $D^\lambda \nu(x) \in (z_0 + \log(1-p), z_0 + \log(1-p) + \varepsilon)$ is at least $(1-p)\delta$. Note that these two intervals are disjoint. Let

$$z_1 = z_0 + (\varepsilon + log p + log(1-p))/2.$$

The probability that $D^\lambda \nu(x) < z_1$ is at least $p\delta$ and the probability that $D^\lambda \nu(x) > z_1$ is at least $(1-p)\delta$. Let

$$\phi(n) = (\Lambda(n \log q) - na_0 + z_1)/\sigma\sqrt{n}.$$

Then

$$v_n > na_0 + \sigma\sqrt{n}\,\phi(n)$$

holds for only finitely many n with probability at least $p\delta$ and it holds for infinitely many n with probability at least $(1-p)\delta$. This contradicts Theorem 4.5.

Hence the probability that z and $\log D^\lambda \nu(x)$ are finite must be zero. Then, by Theorem 4.3, ν has no representation as an integral of Hausdorff measure m_λ. We state this result in a theorem.

4.6. THEOREM. *Suppose that λ is such that for each $a > 0$, $\lambda(at)/\lambda(t)$ is bounded as t goes to 0. If $q = r$ and $p \neq 1/2$, then the invariant measure μ generated by the modified baker's transformation* (1) *has no representation as an integral of Hausdorff measure.*

We note that any Hausdorff measure on the square S can be generated by an index function λ such that $\lambda(at)/\lambda(t)$ is bounded as t goes to 0, so that this hypothesis could be removed from the theorem.

For the more general case $q \neq r$, we introduce the variable

$$w = u + v\,\frac{\log q - \log r}{\log(1-p) - \log p}.$$

It is then true that

$$w_n = n\frac{\log q \log(1-p) - \log r \log p}{\log(1-p) - \log p} = n\Delta w.$$

We can then apply the techniques of this section to a random walk in the (w, v) plane to show that ν has an integral representation if and only if

$$\log p \log r = \log(1-p)\log q. \tag{7}$$

The modified baker's transformation is one of the simplest possible dynamical systems. That a system as simple as this gives rise to a natural invariant measure with no representation as an indefinite integral of Hausdorff measure suggests that generically the natural invariant measure arising from a dynamical system will have no such representation.

5. Classification of invariant measures by Hausdorff measures. We now consider the problem of comparing the degree of singularity of two invariant measures generated by different values of the parameters in the modified baker's transformation (1). Recall that we consider a finite measure μ more singular than a finite measure ν if there exists a Hausdorff measure m_λ such that ν is absolutely continuous with respect to m_λ while μ is singular with respect to m_λ. We first treat the restricted case $q = r$. Without loss of generality we assume $p < 1/2$.

In the previous section we calculated the Hausdorff dimension of the measure ν on $[0,1]$ generated by the modified baker's transformation (1) to be:

$$d_0 = \frac{p\log p + (1-p)\log(1-p)}{\log q}.$$

We will consider index functions of the form

$$\lambda(t) = t^{d_0}\exp\left(d\sqrt{\log(1/t)\log\log\log(1/t)}\right), \tag{8}$$

corresponding to a function $\Lambda(u) = \log\lambda(e^u)$ of the form

$$\Lambda(u) = d_0 u + d\sqrt{-u\log\log(-u)}\,.$$

Let us recall equation (4) from the previous section: with probability 1,

$$\limsup_{n\to\infty} \frac{v_n - d_0 u_n}{\sigma\sqrt{(2n\log\log n)}} = 1, \tag{9}$$

where $\sigma = \sqrt{p(1-p)}\,(\log(1-p) - \log p)$ is the standard deviation of $\Delta v_i - d_0\Delta u_i$ and $u_n = n\log q$. From equation (2), we have

$$\begin{aligned}\log D^\lambda\nu(x) &= \limsup_{n\to\infty} v_n - \Lambda(u_n)\\ &= \limsup_{n\to\infty} v_n - d_0u_n - d\sqrt{-u_n\log\log(-u_n)}\\ &= \limsup_{n\to\infty} v_n - d_0u_n - d\sqrt{-\log q}\,\sqrt{n\log\log n}\,.\end{aligned}$$

Let

$$d_1 = \sqrt{\frac{-2}{\log q}}\,\sigma = (\log(1-p) - \log p)\sqrt{\frac{-2p(1-p)}{\log q}}\,.$$

Then when $d > d_1$, then (9) implies $\log D^\lambda\nu(x) = -\infty$ with probability 1 and ν is absolutely continuous with respect to m_λ by Theorem 4.3. When $d < d_1$, then (9) implies $D^\lambda\nu(x) = \infty$ with probability 1 and ν is singular with respect to m_λ by Theorem 4.3.

Note the similarity between the determination of the appropriate value of d_1 and the definition of the Hausdorff dimension of a measure. With λ as in (8), $\dim \nu$ is the critical value such that ν is absolutely continuous with respect to m_λ when $d_0 > \dim \nu$ and singular with respect to m_λ when $d_0 < \dim \nu$. When $d_0 = \dim \nu$, we further have that d_1 is the critical value such that ν is absolutely continuous with respect to m_λ when $d > d_1$ and ν is singular with respect to m_λ when $d < d_1$. This criterion allows us to define the quantity d_1 for any measure, not just those arising from the modified baker's transformation (1). Moreover, for a set E, we can define a critical value d_1 such that $m_\lambda E = 0$ when $d > d_1$ and $m_\lambda E = \infty$ when $d < d_1$. Besides classifying measures according to degree of singularity and sets according to size, d_1 can be used to bound the errors involved in estimates using the dimension. We suggest the name *paradimension* for the quantity d_1.

The usefulness of paradimension for classifying invariant measures in dynamical systems depends on its taking a spectrum of values for a family of measures, as opposed to being uniformly zero, for example. The analysis in this paper shows that this is true for the family of measures arising from the modified baker's transformation. Hausdorff measures with index functions of the form (8) appear also in a theorem by Makarov **[5]** concerning the following situation. Let the unit disc be mapped conformally onto the interior of a Jordan curve Γ and let ν be the image under this mapping of Lebesgue measure on the boundary of the circle; thus ν is supported on Γ. Makarov's result, stated in our terminology, is that the dimension of ν is necessarily 1 while its paradimension may range from zero to some upper bound. Further work by Przytycki, Urbański, and Zdunik **[6]** ties this result in with the study of measures on repellers in dynamical systems and includes calculations of the paradimension of some such measures.

For an arbitrary ergodic invariant measure, the Hausdorff dimension can be obtained from the following formula for Hausdorff dimension at a point x:

$$\dim \nu = \liminf_{\varepsilon \to 0} \frac{\log \nu B(x, \varepsilon)}{\log \varepsilon},$$

where $B(x, \varepsilon)$ denotes the ball of radius ε centered at x. For an ergodic invariant measure in a dynamical system, the lim inf is independent of x except on a set of ν-measure zero. In a more general context, we must take the essential infimum over x. Proof of an essentially equivalent formula is given in Tricot **[10]**. This formula is often used in computations of the dimension as in **[12]**. We have found an analogous formula for the paradimension; the paradimension of a measure ν *at a point* x is given by

$$d_1(\nu) = \limsup_{\varepsilon \to 0} \frac{\log \nu B(x, \varepsilon) - \dim \nu \log \varepsilon}{\sqrt{(\log 1/\varepsilon \log \log \log 1/\varepsilon)}}. \tag{10}$$

Again, for an invariant ergodic measure, the value of the lim sup is independent of x except on a set of ν-measure zero. The proof of this formula is rather lengthy, so we do not present it here.

One application of (10) is to calculate d_1 in the case $q \neq r$. Equation (4) still holds; we thus have with probability 1:

$$\limsup_{n \to \infty} \frac{v_n - d_0 u_n}{\sqrt{(n \log \log n)}} = \sigma\sqrt{2},$$

where σ^2 is the variance of the random variable $\Delta v_i - d_0 \Delta u_i$; thus

$$\sigma = \frac{-|\log p \log r - \log(1-p)\log q|}{p \log q + (1-p)\log r} \sqrt{p(1-p)}.$$

We also have

$$\lim_{n \to \infty} \frac{-u_n \log \log(-u_n)}{n \log \log n} = -p \log q - (1-p)\log r$$

with probability 1. We can thus calculate from (10):

$$\begin{aligned} d_1(\nu) &= \limsup_{n \to \infty} \frac{\log \nu B(x, e^{u_n}/2) - d_0 u_n}{\sqrt{(-u_n \log \log(-u_n))}} \\ &= \limsup_{n \to \infty} \frac{v_n - d_0 u_n}{\sqrt{(-u_n \log \log(-u_n))}} \\ &= \frac{\sigma\sqrt{2}}{\sqrt{(-p \log q - (1-p) \log r)}} \\ &= \frac{|\log p \log r - \log(1-p)\log q| \sqrt{2p(1-p)}}{(-p \log q - (1-p) \log r)^{3/2}}. \end{aligned}$$

In going from the first to the second step above we have replaced $\log \nu B(x, e^{u_n}/2)$ by v_n, which is not quite the same thing, but this restriction to intervals in the collection Ξ does not affect the value of the lim sup.

6. Acknowledgments. This research was supported in part by a grant from the Naval Academy Research Council.

I am indebted to the referees for their conscientious work. The present form of this article owes much to their suggestions. Their advice was particularly helpful in the revision of §§3 and 5.

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Received April 8, 1986 and in revised form October 13, 1986.

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