PACIFIC JOURNAL OF MATHEMATICS
Vol. 130, No. 1, 1987

A GENERALIZATION OF A THEOREM OF ATKINSON TO NON-INVARIANT MEASURES

DANIEL ULLMAN

We prove that, if T is an ergodic, conservative, non-singular automorphism of a Lebesgue space (X,μ), then the following are equivalent for f in $L^1(\mu)$:

(1) If $\mu(B)>0$ and $\varepsilon>0$, then there is an integer $n\neq 0$ such that

$$\mu\left(B\cap T^{-n}B\cap\left\{x:\left|\sum_{j=0}^{n-1} f(T^jx)\cdot\frac{d\mu\circ T^j}{d\mu}(x)\right|<\varepsilon\right\}\right)>0.$$

(2) $$\liminf_{n\to\infty}\left|\sum_{j=0}^{n-1} f(T^jx)\cdot\frac{d\mu\circ T^j}{d\mu}(x)\right|=0\quad\textbf{for a.e. } x.$$

(3) $\int f\,d\mu=0.$

Our basic objects of study are a non-atomic Lebesgue space $(X,\mathscr{B},\mu)$ and a conservative, aperiodic, non-singular automorphism $T:X\to X$. Associated with any measurable function $f:X\to\mathbf{R}^n$ is a cocycle $f^*:\mathbf{Z}\times X\to\mathbf{R}^n$ defined by

$$f^*(n,x)=\begin{cases}\sum_{k=0}^{n-1} f(T^kx), & n>0,\\ 0, & n=0,\\ -f^*(-n,T^nx), & n<0.\end{cases}$$

f^* satisfies the so-called cocycle identity:

$$f^*(m+n,x)=f^*(m,x)+f^*(n,T^mx), \tag{1}$$

for all integers m and n and for a.e. $x\in X$.

The non-singularity of T permits us to define the Radon-Nikodym derivative

$$\omega_k(x)=\frac{d\mu\circ T^k}{d\mu}(x)\quad\text{for } k\in\mathbf{Z},\text{ a.e. } x\in X.$$

We can use this to build what we call an H-cocycle—after Halmos [**4**], Hopf [**5**], and Hurewicz [**6**]—defined by

$$f_*(n,x) = \begin{cases} \sum_{m=0}^{n-1} \omega_m(x) f(T^m x) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\omega_n(x) f_*(-n, T^n x) & \text{if } n < 0. \end{cases}$$

The quotient ergodic theorem [**3**] asserts that, for an integrable f, the rate of growth of $f_*(n,x)$ depends only on the integral $\int f\,d\mu$. Analogous to (1) is the H-cocycle identity:

$$f_*(m+n,x) = f_*(m,x) + \omega_m(x) f_*(n, T^m x). \tag{2}$$

When T is measure-preserving, the H-cocycle coincides with the usual cocycle.

Suppose $B \in \mathscr{B}$. A cocycle or an H-cocycle $f(n,x)$ is *recurrent on* B if, for all $\varepsilon > 0$,

$$\mu\Big(\bigcup_{n \neq 0} B \cap T^{-n}B \cap \{x \in X \ni |f(n,x)| < \varepsilon\}\Big) > 0.$$

A cocycle or an H-cocycle $f(n,x)$ is *recurrent* if it is recurrent on all sets of positive measure. We call a function $f: X \to \mathbf{R}^n$ recurrent if $f^*(n,x)$ is, and we call it H-recurrent if $f_*(n,x)$ is.

These definitions coincide with the classical notion of recurrence (or sometimes "persistence") of random walks, introduced by Polya [**8**], who proved that the Bernoulli random walk on $\mathbf{Z}^n$ is recurrent (that is, bound to return to zero) if and only if $n = 1$ or 2. Later, Chung and Fuchs [**2**] proved that a random walk on $\mathbf{R}$ based on an increment random variable X of finite mean is recurrent if and only if $EX = 0$. In 1976, Atkinson [**1**] discovered the following beautiful result, extending the theorem of Chung and Fuchs to random walks with non-independent increments.

THEOREM (ATKINSON). *If T is ergodic and preserves a finite measure μ and f is a real, integrable function on X, then f is recurrent if and only if $\int f\,d\mu = 0$.*

The following result further extends the theorem of Chung and Fuchs to the non-stationary case.

THEOREM. *If T is an ergodic, conservative, non-singular automorphism of a Lebesgue space $(X, \mathscr{B}, \mu)$ and if $f: X \to \mathbf{R}$ is integrable, then the following conditions are equivalent*:

(1) *f_* is H-recurrent,*

(2) $\liminf |f_*(n, x)| = 0$ *for a.e.* $x \in X$, *and*
(3) $\int f \, d\mu = 0$.

Proof. The first thing to notice is that once we know this theorem for a measure μ, we know it for all measures ν equivalent to μ. To see this, note that the H-cocycle f_* built from f under $(X, \mathscr{B}, \nu, T)$ is related to the H-cocycle f'_* built from $f' = f \cdot d\nu/d\mu$ under $(X, \mathscr{B}, \mu, T)$ by the equation

$$f'_*(n, x) = \frac{d\nu}{d\mu}(x) \cdot f_*(n, x).$$

This shows that f'_* gets small exactly when f_* gets small. Since $\int f \, d\nu = 0$ exactly when $\int f' \, d\mu = 0$, we inherit the result for f and ν from the result for f' and μ.

In particular, since this theorem reduces to Atkinson's theorem if T preserves μ, we have the result for any dynamical system $(X, \mathscr{B}, \mu, T)$ with an equivalent finite invariant measure. We also see that there is no loss of generality in assuming that $\mu X = 1$ and we proceed under this assumption.

(1) $\Rightarrow$ (2) Let $D = \{x \in X \ni \liminf |f_*(n, x)| > \varepsilon\}$ for some $\varepsilon > 0$. If $\mu D > 0$, then there would be an integer N so large that

$$C = \left\{x \in D \ni |f_*(n, x)| > \varepsilon \text{ for all } n \text{ with } |n| > N\right\}$$

would have positive measure. One could then find a set $B \subset C$ of positive measure disjoint from its first N forward and backward translates. (Just remove from C points that return too soon under T or T^{-1} and use Kac's recurrence theorem [7].) Then

$$\mu\left(B \cap T^{-n}B \cap \{x \ni |f_*(n, x)| < \varepsilon\}\right) = 0$$

for all integers $n \neq 0$, which contradicts the H-recurrence of f.

(2) $\Rightarrow$ (3) This implication is proved via a simple application of the quotient ergodic theorem [3]. Let g be the constant function 1. Since $g_*(n, x) > 1$ for every x and all positive n,

$$|f_*(n, x)| \geq \left|\frac{f_*(n, x)}{g_*(n, x)}\right| \overset{\text{a.e.}}{\to} \frac{|\int f \, d\mu|}{|\int g \, d\mu|} = \left|\int f \, d\mu\right|.$$

If $\int f \, d\mu \neq 0$, this last quantity is positive and so $\liminf |f_*(n, x)| > 0$ for a.e. $x \in X$.

(3) $\Rightarrow$ (1) This argument encompasses the remainder of the paper. Three important estimates are isolated as lemmas.

Assume f_* is transient—i.e., not recurrent. This means that there is a set $B \in \mathscr{B}$ with $\mu B > 0$ and a $\delta > 0$ such that

$$\mu\left(B \cap T^{-n}B \cap \{x \ni |f_*(n,x)| < \delta\}\right) = 0 \quad \forall n \neq 0. \tag{3}$$

Let A be a subset of B with $\mu A = \mu B$ and such that

$$A \cap T^{-n}A \cap \{x \ni |f_*(n,x)| < \delta\} = \varnothing \quad \text{for all } n \neq 0. \tag{4}$$

By χ we will mean χ_A, the characteristic function of the set A.

For all $\varepsilon > 0$ and a.e. x, the quotient ergodic theorem tells us that

$$\left|\frac{\chi_*(n,x)}{g_*(n,x)} - \mu A\right| < \varepsilon \quad \text{for sufficiently large } n. \tag{5}$$

Another way to write this is to define the "weight" $w(j,x)$ of the integer j, depending on x, by:

$$w(j,x) = \begin{cases} \omega_j(x) & \text{if } T^j x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

For the remainder of the proof, fix x such that (5) holds (for an ε to be specified later) and such that $f_*(n,x)/g_*(n,x) \to \int f\,d\mu$. Then (5) translates to

$$\left|\sum_{j=0}^{n-1} w(j,x) - \mu A \cdot g_*(n,x)\right| < \varepsilon \cdot g_*(n,x). \tag{6}$$

We call an integer j *good* if $T^j x \in A$. Note that the previous summation has non-zero contribution only from good indices j. For good m, let I_m be the interval on the real line centered at $f_*(m,x)$ and of radius (i.e., half-length) equal to $w(m,x)\delta$. Let λ be Lebesgue measure on the line.

LEMMA 1. *If m is good, $f_*(j,x) \in I_m$ only when $j = m$.*

Proof of Lemma. That m is good means that $T^m x \in A$, which implies that

$$|f_*(j-m, T^m x)| \geq \delta \quad \text{for any } j \neq m. \tag{7}$$

The H-cocycle identity (2) can be written

$$f_*(j-m, T^m x) = \frac{f_*(j,x) - f_*(m,x)}{w(m,x)}.$$

Hence equation (7) implies that $|f_*(j,x) - f_*(m,x)| > w(m,x)\delta$, which is what it means to say that $f_*(j,x) \notin I_m$. $\square$

The intervals I_m may be of widely varying size. Yet the following lemma assures us that no I_m for large m can be nearly as long as the sum of lengths of I_j for $0 \le j < m$.

LEMMA 2. *If m is good and sufficiently large, then*

$$w(m,x) < \frac{1}{10}\sum_{j=0}^{m-1} w(j,x).$$

Proof of Lemma. Choose n large enough so that equation (6) holds for all $m > n$. Write

$$w(m,x) = \sum_{j=0}^{m} w(j,x) - \sum_{j=0}^{m-1} w(j,x)$$

and

$$\mu A \cdot w(m,x) = \mu A \cdot g_*(m+1,x) - \mu A \cdot g_*(m,x).$$

Subtracting the last equation from the one before yields

$$\begin{aligned} w(m,x)[1-\mu A] &\le \varepsilon g_*(m+1,x) + \varepsilon g_*(m,x) \\ &= 2\varepsilon g_*(m,x) + \varepsilon w(m,x) \end{aligned}$$

if $m > n$, using (6).

Rearranging:

$$w(m,x)[1-\mu A - \varepsilon] \le 2\varepsilon g_*(m,x).$$

If ε is sufficiently small, the quantity in square brackets is positive, and so we get

$$\begin{aligned} w(m,x) &\le \frac{2\varepsilon}{(1-\mu A-\varepsilon)} g_*(m,x) \\ &\le \left[\frac{2\varepsilon}{(\mu A-\varepsilon)(1-\mu A-\varepsilon)}\right]\sum_{j=0}^{m-1} w(j,x) \end{aligned} \tag{8}$$

where the second inequality comes from (6). Simply choose ε small enough so that the quantity in (8) in square brackets is less than $1/10$ and the lemma is proved. □

Let J_n be the convex hull of $\{f_*(j,x) \ni 0 \le j < n\}$. J_n is the shortest interval on the real line containing the first n $f_*(j,x)$'s. Our goal now is to show that the intervals J_n have bounded weight density.

LEMMA 3. *For sufficiently large n*

$$\sum_{j=0}^{n-1} w(j,x) < \frac{4}{\delta}\lambda J_n.$$

Proof of Lemma. Let $\mathfrak{F} = \{I_m \ni m \text{ is good and } 0 \leq m < n\}$. $\mathfrak{F}$ is a collection of possibly overlapping intervals of varying sizes. Let $\mathfrak{F}'$ be a subset of $\mathfrak{F}$ whose union equals that of $\mathfrak{F}$ and which is minimal with respect to this property. Call *m select* if $I_m \in \mathfrak{F}'$. Then

$$\begin{aligned}4\lambda J_n &> 2\lambda\Big(\bigcup_{m \text{ select}} I_m\Big) > \sum_{m \text{ select}} \lambda I_m \\ &> \sum_{m \text{ select}} \sum_{j \ni f_*(n,x) \in I_m} \delta w(j,x) > \delta \sum_{j=0}^{n-1} w(j,x).\end{aligned}$$

The first inequality comes from Lemma 2. The second inequality holds because the choice of $\mathfrak{F}'$ forces all real numbers to lie in at most two I_m with select indices m. The third inequality is just Lemma 1, and the fourth expresses the fact that every $f_*(j,x)$ with $0 \leq j < n$ and j good lies in some select I_m. The lemma is proved. □

It is now a simple matter to complete the proof of the theorem. Equation (6) says that, for all $\varepsilon > 0$,

$$\sum_{j=0}^{n-1} w(j,x) > g_*(n,x)(\mu A - \varepsilon),$$

if n is large enough. Hence Lemma 3 tells us that

$$\lambda J_n > \frac{\delta}{4} g_*(n,x)(\mu A - \varepsilon).$$

This implies that

$$\sup_{0 \leq j < n} |f_*(j,x)| > \frac{\delta}{8}(\mu A - \varepsilon) g_*(n,x).$$

Thus, for infinitely many n,

$$\frac{|f_*(n,x)|}{g_*(n,x)} > \frac{\delta}{8}(\mu A - \varepsilon) > 0,$$

if ε is small enough.

But the left-hand-side of this expression approaches $|\int f\, d\mu|$, which is seen to be, as required, greater than zero. □

I would like to thank Jack Feldman for suggesting this problem and for many helpful discussions.

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Received May 3, 1986 and in revised form July 15, 1986. This research was supported in part by NSF grant DMS84-03182 and constituted part of the authors Ph.D. dissertation.

THE GEORGE WASHINGTON UNIVERSITY
WASHINGTON, DC 20052