PACIFIC JOURNAL OF MATHEMATICS
Vol. 132, No. 2, 1988

STABILITY OF UNFOLDINGS IN THE CONTEXT OF EQUIVARIANT CONTACT-EQUIVALENCE

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M. Golubitsky and D. Schaeffer introduced the notion of equivariant contact-equivalence between germs of C^∞ equivariant mappings, in order to study perturbed bifurcation problems having a certain symmetry property. The main tool used is the so-called "Unfolding Theorem" for the qualitative description of the symmetry-preserving perturbations of these problems. From the point of view of applications, a relevant notion is that of stability of unfoldings. In this paper we prove the equivalence of the universality and the stability of unfoldings in the context of equivariant contact-equivalence.

1. Universal Γ-unfolding. Let Γ be a compact Lie group acting orthogonally on $\mathbf{R}^n$ and $\mathbf{R}^p$. We write $\mathscr{E}_{n,p}^{\Gamma}$ for the space of C^∞ germs $f\colon (\mathbf{R}^n, 0) \to \mathbf{R}^p$ of Γ-equivariant mappings (i.e. $f(\gamma x) = \gamma f(x)$ for all $\gamma \in \Gamma$). The space of Γ-invariant C^∞-germs $h\colon (\mathbf{R}^n, 0) \to \mathbf{R}$ (i.e. $h(\gamma x) = h(x)$ for all $\gamma \in \Gamma$) is denoted by $\mathscr{E}_n^{\Gamma}$. In what follows we shall consider germs $G\colon (\mathbf{R}^n \times \mathbf{R}, 0) \to \mathbf{R}^p$ and $F\colon (\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^q, 0) \to \mathbf{R}^p$ and we shall assume that Γ acts trivially on $\mathbf{R}$ and $\mathbf{R}^q$.

The notion of equivariant contact-equivalence introduced by Golubitsky and Schaeffer [3] is the following:

DEFINITION 1.1. We say that G_1 and $G_2 \in \mathscr{E}_{n+1,p}^{\Gamma}$ are Γ-equivalent if

$$G_1(x, \lambda) = T(x, \lambda) G_2(X(x, \lambda), \Lambda(\lambda))$$

where

(1.1.1) $$T\colon (\mathbf{R}^n \times \mathbf{R}, 0) \to \mathrm{Gl}_p(\mathbf{R}) \quad \text{is } C^\infty.$$

(1.1.2) $$(X, \Lambda)\colon\ (\mathbf{R}^n \times \mathbf{R}, 0) \to (\mathbf{R}^n \times \mathbf{R}, 0) \quad \text{is } C^\infty,$$

$$\det(d_x X(0)) > 0 \quad \text{and} \quad \Lambda'(0) > 0.$$

(1.1.3) $$X(\gamma x, \lambda) = \gamma X(x, \lambda) \quad \text{for all } \gamma \in \Gamma.$$

(1.1.4) $$\gamma^{-1}T(\gamma x, \lambda)\gamma = T(x, \lambda) \quad \text{for all } \gamma \in \Gamma.$$

A q-parameter Γ-unfolding of $G \in \mathscr{E}^{\Gamma}_{n+1,p}$ is a germ $F \in \mathscr{E}^{\Gamma}_{n+1+q,p}$ such that $F(x, \lambda, 0) = G(x, \lambda)$.

DEFINITION 1.2. A q-parameter Γ-unfolding $F \in \mathscr{E}^{\Gamma}_{n+1+q,p}$ of $G \in \mathscr{E}^{\Gamma}_{n+1,p}$ is said to be a universal Γ-unfolding if every Γ-unfolding H of G is induced by F in the following way: assume that $H \in \mathscr{E}^{\Gamma}_{n+1+q',p}$; then there exist C^∞ germs $T\colon (\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^{q'}, 0) \to \mathrm{Gl}_p(\mathbf{R})$ and $(X, \Lambda, \alpha)\colon (\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^{q'}, 0) \to (\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^{q}, 0)$ such that:

(1.2.1) $$H(x, \lambda, \beta) = T(x, \lambda, \beta) \cdot F(X(x, \lambda, \beta), \Lambda(\lambda, \beta), \alpha(\beta)).$$

(1.2.2) $$X(\gamma x, \lambda, \beta) = \gamma X(x, \lambda, \beta) \quad \text{for all } \gamma \in \Gamma.$$

(1.2.3) $$\gamma^{-1}T(\gamma x, \lambda, \beta)\gamma = T(x, \lambda, \beta) \quad \text{for all } \gamma \in \Gamma.$$

(1.2.4) $$(X(x, \lambda, 0), \Lambda(\lambda, 0)) \equiv (x, \lambda).$$

(1.2.5) $$T(x, \lambda, 0) \equiv I_p \quad \text{where } I_p \text{ is the identity } p \times p\text{-matrix.}$$

Let $\mathscr{M}^{\Gamma}_{n+1,p} = \{T\colon (\mathbf{R}^{n+1}, 0) \to M_p(\mathbf{R}) \mid T \text{ is } C^\infty \text{ and satisfies (1.2.3)}\}$ where $M_p(\mathbf{R})$ is the space of real $p \times p$ matrices. For $G \in \mathscr{E}^{\Gamma}_{n+1,p}$ we define

$$\begin{aligned} M_G\colon \mathscr{M}^{\Gamma}_{n+1,p} \oplus \mathscr{E}^{\Gamma}_{n+1,n} &\to \mathscr{E}^{\Gamma}_{n+1,p} \\ (T, X) &\mapsto T \cdot G + (d_x G) \cdot X \end{aligned}$$

and

$$\begin{aligned} N_G\colon \mathscr{E}_1 &\to \mathscr{E}^{\Gamma}_{n+1,p} \\ \Lambda &\mapsto (d_\lambda G) \cdot \Lambda. \end{aligned}$$

Let

$$\tilde{\Gamma} G = M_G\left(\mathscr{M}^{\Gamma}_{n+1,p} \oplus \mathscr{E}^{\gamma}_{n+1,n}\right) \quad \text{and} \quad \Gamma G = \tilde{\Gamma} G + N_G(\mathscr{E}_1).$$

Roughly speaking, ΓG is the tangent space to the orbit $O_G = \{G' \in \mathscr{E}^{\Gamma}_{n+1,p} \mid G' \text{ is } \Gamma\text{-equivalent to } G\}$ at G.

If O_G has "finite codimension" that is $\dim_{\mathbf{R}} \mathscr{E}^{\Gamma}_{n+1,p}/\Gamma G < \infty$ we have the unfolding theorem:

THEOREM 1.3 (GOLUBITSKY-SCHAEFFER [3]). *Let $G \in \mathscr{E}^{\Gamma}_{n+1,p}$ be of finite codimension and let $F \in \mathscr{E}^{\Gamma}_{n+1+q,p}$ be an unfolding of G. Then F is a universal Γ-unfolding of G if and only if*

$$\frac{\partial F}{\partial \alpha_1}(x, \lambda, 0), \dots, \frac{\partial F}{\partial \alpha_q}(x, \lambda, 0)$$

(where $(x, \lambda, \alpha) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^q$) project onto a spanning set of $\mathscr{E}^{\Gamma}_{n+1,p}/\Gamma G$ i.e.

$$\mathscr{E}^{\Gamma}_{n+1,p} = M_G\left(\mathscr{M}^{\Gamma}_{n+1,p} \oplus \mathscr{E}^{\Gamma}_{n+1,n}\right) + N_G(\mathscr{E}_1) + \mathbf{R}\left\{\frac{\partial F}{\partial \alpha_i}(x, \lambda, 0)\right\}. \tag{1.3.1}$$

REMARK 1.4. In fact, Golubitsky and Schaeffer [3] indicated how to prove the sufficiency of the condition (1.3.1). The necessity of (1.3.1) is proved in the following way (see [4] p. 259): Let $h \in \mathscr{E}^{\Gamma}_{n+1,p}$ and consider the one-parameter Γ-unfolding $H \in \mathscr{E}^{\Gamma}_{n+1+1,p}$ defined by $H(x, \lambda, t) = G(x, \lambda) + th(x, \lambda)$. Since F is universal, there exist T, X, Λ and α as in 1.2 such that

$$H(x, \lambda, t) = T(x, \lambda, t) \cdot F(X(x, \lambda, t), \Lambda(\lambda, t), \alpha(t)).$$

We obtain

$$\begin{aligned} h(x, \lambda) &= \frac{\partial H}{\partial t}(x, \lambda, t)\,|_{t=0} \\ &= \frac{\partial}{\partial t} T(x, \lambda, t) \cdot F(X(x, \lambda, t), \Lambda(\lambda, t), \alpha(t))\,|_{t=0} \end{aligned}$$

which is easily seen to belong to $\Gamma G + \mathbf{R}\{\partial F(x, \lambda, 0)/\partial \alpha_i\}$.

2. Stability of Γ-unfoldings. Let U be a Γ-invariant open subset of $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^q$. We write $C^{\infty}_{\Gamma}(U, \mathbf{R}^p) = \{F \in C^{\infty}(U, \mathbf{R}^p) \mid F(\gamma x, \lambda, \alpha) = \gamma F(x, \lambda, \alpha)$ for each $\gamma \in \Gamma\}$ endowed with the topology induced by the Whitney C^{∞}-topology on $C^{\infty}(U, \mathbf{R}^p)$.

DEFINITION 2.1. Let U and V be Γ-invariant open subsets of $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^q$. Let $\overline{F} \in C^{\infty}_{\Gamma}(U, \mathbf{R}^p)$ and let $\overline{H} \in C^{\infty}_{\Gamma}(V, \mathbf{R}^p)$. We say that $\overline{F}$, at $(x_0, \lambda_0, \alpha_0) \in U^{\Gamma}$ is Γ-equivalent to $\overline{H}$ at $(x_1, \lambda_1, \alpha_1) \in V^{\Gamma}$ if there exist

C^∞ germs

$$\begin{aligned}
T&: (\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^q, (x_0, \lambda_0, \alpha_0)) \longrightarrow \mathrm{Gl}_p(\mathbf{R})\\
X&: (\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^q, (x_0, \lambda_0, \alpha_0)) \longrightarrow (\mathbf{R}^n, x_1)\\
\Lambda&: (\mathbf{R} \times \mathbf{R}^q, (\lambda_0, \alpha_0)) \longrightarrow (\mathbf{R}, \lambda_1)\\
\phi&: (\mathbf{R}^q, \alpha_0) \longrightarrow (\mathbf{R}^q, \alpha_1)
\end{aligned}$$

such that

$$F(x,\lambda,\alpha) = T(x,\lambda,\alpha)\cdot H(X(x,\lambda,\alpha), \Lambda(\lambda,\alpha), \phi(\alpha)), \tag{2.1.1}$$

$$(X, \Lambda, \phi) \quad \text{is a germ of a diffeomorphism,} \tag{2.1.2}$$

$$X(\gamma x,\lambda,\alpha) = \gamma X(x,\lambda,\alpha) \quad \text{and} \quad \gamma^{-1}T(\gamma x,\lambda,\alpha)\gamma = T(x,\lambda,\alpha) \tag{2.1.3}$$

for all $\gamma \in \Gamma$ where U^Γ and V^Γ are the sets of fixed points of U and V under the action of Γ.

DEFINITION 2.2. Let $G \in \mathscr{E}^\Gamma_{n+1,p}$ and let $F \in \mathscr{E}^\Gamma_{n+1+q,p}$ be a Γ-unfolding of G. We say that F is Γ-stable if, for every representative $\overline{F}$ of F defined on an Γ-invariant open neighbourhood U of $0 \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^q$, there is a neighbourhood $\mathscr{U}$ of $\overline{F}$ in $C^\infty_\Gamma(U, \mathbf{R}^p)$ such that, for every $\overline{H} \in \mathscr{U}$, there is a point $(x_0, \lambda_0, \alpha_0) \in U^\Gamma$ such that $\overline{F}$ at $(0,0,0)$ is Γ-equivalent to $\overline{H}$ at $(x_0, \lambda_0, \alpha_0)$.

The main result of this paper is:

THEOREM 2.3. *Let $G \in \mathscr{E}^\Gamma_{n+1,p}$ be such that the k-jet j^kG is Γ-sufficient. Then a Γ-unfolding $F \in \mathscr{E}^\Gamma_{n+1+q,p}$ of G is universal if and only if it is Γ-stable.*

Note. We say that the k-jet j^kG of G at 0 is Γ-sufficient if, for every $G_1 \in \mathscr{E}^\Gamma_{n+1,p}$ such that $j^kG_1 = j^kG$, G and G_1 are Γ-equivalent in the sense of Definition 1.1.

Before proceeding to the proof of Theorem 2.3 we shall give some transversality properties of universal Γ-unfoldings.

3. Transversality. Let $J^k_\Gamma(n+1,p) = \{$polynomial mappings on $\mathbf{R}^n \times \mathbf{R}$ into $\mathbf{R}^p$ which are Γ-equivariant and of degree $\leq k\}$. This is the space of k-jets of the elements of $\mathscr{E}^\Gamma_{n+1,p}$ i.e.

$$J^k_\Gamma(n+1,p) = \mathscr{E}^\Gamma_{n+1,p} / \left(\underline{m}^{k+1}_{x,\lambda} \cdot \mathscr{E}_{n+1,p}\right) \cap \mathscr{E}^\Gamma_{n+1,p}$$

where $\underline{m}_{x,\lambda}$ is the maximal ideal of $\mathscr{E}_{n+1} = \mathscr{E}_{x,\lambda}$. Let

$$\mathscr{G}^k = \{j^k(T, X, \Lambda) \mid T, X \quad \text{and } \Lambda \text{ are as in Definition 1.1}\}.$$

Then $\mathscr{G}^k$ is an analytic Lie group which acts analytically on $J^k_\Gamma(n+1,p)$ in the following way: for $\theta \in \mathscr{G}^k$ and $z \in J^k_\Gamma(n+1,p)$, put $\theta z = j^k((T,X,\Lambda)\cdot G)$ where $\theta = j^k(T,X,\Lambda)$, $z = j^k G$ and $((T,X,\Lambda)\cdot G)(x,\lambda) = T(x,\lambda)\cdot G(X(x,\lambda),\Lambda(\lambda))$. We shall write O^k_z for the orbit of z in $J^k_\Gamma(n+1,p)$ under the action of $\mathscr{G}^k$. As in [7, p. 41], we can prove

LEMMA 3.1. *The tangent space to O^k_z at z is*

$$T_z O^k_z = \pi_k\left[M_G\left(\mathscr{M}^\Gamma_{n+1,p} + (\underline{m}_{x,\lambda}\cdot\mathscr{E}_{n+1,p})\cap\mathscr{E}^\Gamma_{n+1,p}\right) + N_G(\mathscr{E}_1)\right]$$

where $\pi_k\colon \mathscr{E}^\Gamma_{n+1,p}\to J^k_\Gamma(n+1,p)$ is the natural projection.

An immediate consequence (see e.g. [1]) is

PROPOSITION 3.2. *Let $G\in\mathscr{E}^\Gamma_{n+1,p}$ be such that j^kG is Γ-sufficient. Then*

$$\Gamma G \supset \left(\underline{m}^{k+1}_{x,\lambda}\cdot\mathscr{E}_{n+1,p}\right)\cap\mathscr{E}^\Gamma_{n+1,p}.$$

3.3. For $\overline{F}\in C^\infty_\Gamma(U,\mathbf{R}^p)$ and $(x,\lambda,\alpha)\in U$ we define the germ

$$F^\alpha_{(x,\lambda)}\colon(\mathbf{R}^n\times\mathbf{R},0)\to\mathbf{R}^p$$
$$(y,\mu)\mapsto\overline{F}(x+y,\lambda+\mu,\alpha)$$

and we define

$$j^k_*\overline{F}\colon U\to\mathbf{R}^n\times\mathbf{R}\times J^k(n+1,p)$$
$$(x,\lambda,\alpha)\mapsto(x,\lambda,j^kF^\alpha_{(x,\lambda)})$$

where $J^k(n+1,p)$ is the space of k-jets of the elements of $\mathscr{E}_{n+1,p}$.

For $G\in\mathscr{E}^\Gamma_{n+1,p}$ we write S^k_z for the submanifold of $J^k(n+1,p)$ equal to $(\mathbf{R}^{n+1})^\Gamma\times O^k_z\times(J^k_\Gamma(n+1,p))^\perp$, where $z=j^kG$, $(\mathbf{R}^{n+1})^\Gamma$ is the set of fixed points under the action of Γ and $(J^k_\Gamma(n+1,p))^\perp$ is the orthogonal complement in $J^k(n+1,p)$ of the subspace $J^k_\Gamma(n+1,p)$.

LEMMA 3.3. *Let $F\in\mathscr{E}^\Gamma_{n+1+q,p}$ be a Γ-unfolding of $G\in\mathscr{E}^\Gamma_{n+1,p}$. Then j^k_*F is transverse to S^k_z at $(0,0,0)$ if and only if*

$$\Gamma G + \mathbf{R}\left\{\frac{\partial F}{\partial\alpha_i}(x,\lambda,0)\right\} + \left(\underline{m}^{k+1}_{x,\lambda}\cdot\mathscr{E}_{n+1,p}\right)\cap\mathscr{E}^\Gamma_{n+1,p} = \mathscr{E}^\Gamma_{n+1,p}. \tag{3.3.1}$$

Proof. The range of $d(j^k_*F)_{(0,0)}$ is

$$\mathbf{R}^n\times\mathbf{R}\times\pi_k\left[\mathbf{R}\left\{\frac{\partial F}{\partial x_i}(x,\lambda,0),\frac{\partial F}{\partial\lambda}(x,\lambda,0),\frac{\partial F}{\partial\alpha_j}(x,\lambda,0)\right\}\right].$$

Hence the above transversality condition is satisfied if and only if

$$\begin{aligned}\operatorname{Range} d(j^k_* F)_{(0,0,0)} &+ T_{(0,0)}(\mathbf{R}^{n+1})^\Gamma \times \{0\} + \{0\} \times T_z O^k_z \\ &+ \{0\} \times (J^k_\Gamma(n+1,p))^\perp \\ &= \mathbf{R}^{n+1} \times J^k(n+1,p);\end{aligned}$$

hence, by virtue of Lemma 3.1,

$$\pi_k\left[\Gamma G + \mathbf{R}\left\{\frac{\partial F}{\partial \alpha_i}(x,\lambda,0)\right\}\right] + (J^k_\Gamma(n+1,p))^\perp = J^k(n+1,p).$$

But

$$\pi_k\left[\Gamma G + \mathbf{R}\left\{\frac{\partial F}{\partial \alpha_i}(x,\lambda,0)\right\}\right] \subset J^k_\Gamma(n+1,p),$$

and the desired result follows.

4. Proof of Theorem 2.3. Let $G \in \mathscr{E}^\Gamma_{n+1,p}$ be such that $z = j^k G$ is Γ-sufficient and let $F \in \mathscr{E}^\Gamma_{n+1+q,p}$ be a Γ-unfolding of G.

4.1. *Universality $\Rightarrow$ stability.* Suppose that F is universal and let $\overline{F} \in C^\infty_\Gamma(U, \mathbf{R}^p)$ be a representative of F on an open Γ-invariant neighbourhood of $0 \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^q$. From the unfolding theorem and Lemma 3.3, we conclude that $j^k_* F$ is transverse to S^k_z at $(0,0,0)$. The Transversality Theorem (see [**8**, p. 321]) implies the existence of a neighbourhood $\mathscr{U}$ of $\overline{F}$ in $C^\infty(U, \mathbf{R}^p)$ such that, for every $\overline{H} \in \mathscr{U}$, $j^k_* \overline{H}$ intersects S^k_z transversally at at least one point $(x_0, \lambda_0, \alpha_0) \in U$. Put $\mathscr{U}_\Gamma = \mathscr{U} \cap C^\infty_\Gamma(U, \mathbf{R}^p)$. Then for each $\overline{H} \in \mathscr{U}_\Gamma$, there exists $(x_0, \lambda_0, \alpha_0) \in U$ such that $j^k_* \overline{H}(x_0, \lambda_0, \alpha_0) \in S^k_z$ and $j^k_* \overline{H}$ is transverse to S^k_z at $(x_0, \lambda_0, \alpha_0)$. We shall show that $\overline{F}$, at $(0,0,0)$, is Γ-equivalent to $\overline{H}$ at $(x_0, \lambda_0, \alpha_0)$. Let H be the germ at $(0,0,0)$ defined by $H(x,\lambda,\alpha) = \overline{H}(x_0 + x, \lambda_0 + \lambda, \alpha_0 + \alpha)$ and let h be the germ at $(0,0) \in \mathbf{R}^n \times \mathbf{R}$ given by $h(x,\lambda) = \overline{H}(x_0 + x, \lambda_0 + \lambda, \alpha_0)$; since $j^k_* \overline{H}(x_0, \lambda_0, \alpha_0) \in S^k_z$, we have $(x_0, \lambda_0) \in (\mathbf{R}^{n+1})^\Gamma$ and we deduce that $h \in \mathscr{E}^\Gamma_{n+1,p}$ since

$$\begin{aligned}h(\gamma x, \lambda) &= \overline{H}(x_0 + \gamma x, \lambda_0 + \lambda, \alpha_0) \\ &= \overline{H}(\gamma x_0 + \gamma x, \lambda_0 + \lambda, \alpha) = \gamma \overline{H}(x_0 + x, \lambda_0 + \lambda, \alpha_0) = \gamma h(x,\lambda)\end{aligned}$$

because $\overline{H} \in C^\infty_\Gamma(U, \mathbf{R}^p)$. Therefore $z_0 = j^k h \in O^k_z$; hence z_0 is Γ-sufficient since z is Γ-sufficient. Proposition 3.2 implies that

$$\Gamma h \supset \left(\underline{m}^{k+1}_{x,\lambda} \cdot \mathscr{E}_{n+1,p}\right) \cap \mathscr{E}^\Gamma_{n+1,p}. \tag{4.1.1}$$

On the other hand $O^k_z = O^k_{z_0}$, and so $j^k_* H$ is transverse at $(0,0,0)$ to

$S^k_{z_0}$, and this is equivalent, by virtue of Lemma 3.3, to the equality

$$\Gamma h + \mathbf{R}\left\{\frac{\partial H}{\partial \alpha_j}(x, \lambda, 0)\right\} + \left(\underline{m}^{k+1}_{x,\lambda} \cdot \mathscr{E}_{n+1,p}\right) \cap \mathscr{E}^{\Gamma}_{n+1,p} = \mathscr{E}^{\Gamma}_{n+1,p}.$$

From this equality and (4.1.1) we deduce that

$$\Gamma h + \mathbf{R}\left\{\frac{\partial H}{\partial \alpha_j}(x, \lambda, 0)\right\} = \mathscr{E}^{\Gamma}_{n+1,p},$$

and so, the unfolding theorem implies that H is a universal Γ-unfolding of h.

The germs h and G are Γ-equivalent (as in Definition 1.1) since the jets $z = j^k G$ and $z_0 = j^k h$ are Γ-sufficient and $O^k_z = O^k_{z_0}$. Thus, there exist T, X and Λ as in 1.1 such that

$$h(x, \lambda) = T(x, \lambda) G(X(x, \lambda), \Lambda(\lambda)).$$

Put $\tilde{F}(x, \lambda, \alpha) = T(x, \lambda) F(X(x, \lambda), \Lambda(\lambda), \alpha)$; then

$$\begin{aligned}\tilde{F}(x, \lambda, 0) &= T(x, \lambda) \cdot F(X(x, \lambda), \Lambda(\lambda), 0)\\ &= T(x, \lambda) \cdot G(X(x, \lambda), \Lambda(\lambda)) = h(x, \lambda),\end{aligned}$$

that is, $\tilde{F}$ is a q-parameter Γ-unfolding of h. But H is universal Γ-unfolding; we then easily deduce that H at $(0, 0, 0)$ is Γ-equivalent to $\tilde{F}$ at $(0, 0, 0)$. From there it is not difficult to see that $\overline{H}$ at $(x_0, \lambda_0, \alpha_0)$ is Γ-equivalent to F at $(0, 0, 0)$ (see e.g. [**2**, p. 173]). □

4.2. *Stability $\Rightarrow$ universality.* Suppose that F is Γ-stable but is not universal which, by virtue of the unfolding theorem, is equivalent to

$$\Gamma G + \mathbf{R}\left\{\frac{\partial F}{\partial \alpha_i}(x, \lambda, 0)\right\} \subsetneqq \mathscr{E}^{\Gamma}_{n+1,p}. \tag{4.2.1}$$

Since $j^k G$ is Γ-sufficient we have $\Gamma G \supset (\underline{m}^{k+1}_{x,\lambda} \mathscr{E}_{n+1,p}) \cap \mathscr{E}^{\Gamma}_{n+1,p}$, and so (4.2.1) is equivalent to

$$\Gamma G + \mathbf{R}\left\{\frac{\partial F}{\partial \alpha_j}(x, \lambda, 0)\right\} + \left(\underline{m}^{k+1}_{x,\lambda} \cdot \mathscr{E}_{n+1,p}\right) \cap \mathscr{E}^{\Gamma}_{n+1,p} \subsetneqq \mathscr{E}^{\Gamma}_{n+1,p};$$

hence Lemma 3.3 implies that $j^k_* F$ is not transverse to S^k_z at $(0, 0, 0)$.

We shall use the same method as S. Izumiya [**5**, p. 41]. By virtue of the foregoing there exists $w \in J^k(n+1, p)$ such that

$$w \notin \operatorname{Range} d(j^k_* F)_{(0,0,0)} + T_{(0,0,z)} S^k_z.$$

We may assume that $w \in J^k_\Gamma(n+1, p)$ and thus $w \notin T_z O^k_z$. Let U be a Γ-invariant neighbourhood of $(0, 0, 0)$ in $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^q$ and let $\overline{F} \in C^\infty_\Gamma(U, \mathbf{R}^p)$ and $\overline{w}$, defined on $U \cap \mathbf{R}^n \times \mathbf{R} \times \{0\}$, be representatives of F and w. For $t \in \mathbf{R}$, put $\overline{H}(x, \lambda, \alpha, t) = \overline{F}(x, \lambda, \alpha) + t\overline{w}(x, \lambda)$. Since F

is Γ-stable, there is $\varepsilon > 0$ such that, for every $t_0 \in [-\varepsilon, \varepsilon]$, there exists $(x_0, \lambda_0, \alpha_0) \in U^\Gamma$ such that $\overline{H}_{t_0}$ at $(x_0, \lambda_0, \alpha_0)$ is Γ-equivalent to F at $(0, 0, 0)$, where $\overline{H}_{t_0}(x, \lambda, \alpha) = \overline{H}(x, \lambda, \alpha, t_0)$. In particular,

$$\text{(4.2.2)} \qquad \dim \operatorname{Range} d(j_*^k \overline{H}_{t_0})_{(x_0, \lambda_0, \alpha_0)} = \dim \operatorname{Range} d(j_*^k F)_{(0,0,0)}.$$

On the other hand,

$$\text{(4.2.3)} \qquad \dim \operatorname{Range} d(j_*^k \overline{H})_{(0,0,0,0)} > \dim \operatorname{Range} d(j_*^k F)_{(0,0,0)}.$$

One easily sees (cf. [**5**, p. 41]) that there exists a submanifold Σ of $J^k(n+1, p)$ such that Σ contains a neighbourhood of z in O_z^k, $\operatorname{cod} \Sigma = \dim \operatorname{Range} d(j_*^k \overline{H})_{(0,0,0,0)}$, and $j_*^k \overline{H}$ is transverse to Σ at each point of $U \times [-\varepsilon, \varepsilon]$. But from Sard's Theorem it follows (see e.g. [**6**, p. 134]) that there exists $t_0 \in [-\varepsilon, \varepsilon]$ such that $j_*^k \overline{H}_{t_0}$ is transverse to Σ at every point of U. But, if ε is small enough, there exists $(x_0, \lambda_0, \alpha_0) \in U^\Gamma$ such that $\overline{H}_{t_0}$ at $(x_0, \lambda_0, \alpha_0)$ is Γ-equivalent to $\overline{F}$ at $(0, 0, 0)$. Thus $j_*^k \overline{H}_{t_0}(x_0, \lambda_0, \alpha_0) \in \{(x_0, \lambda_0)\} \times O_z^k \subset S_z^k$; we therefore have the equality (4.2.2). On the other hand, since $j_*^k H_{t_0}$ intersects Σ transversally at $(x_0, \lambda_0, \alpha_0)$ and $\operatorname{cod} \Sigma = \dim \operatorname{Range} d(j_*^k \overline{H})_{(0,0,0,0)}$ we have

$$\begin{aligned} \dim \operatorname{Range} d(j_*^k \overline{H})_{(0,0,0,0)} &= \dim \operatorname{Range} d(j_*^k \overline{H}_{t_0})_{(x_0, \lambda_0, \alpha_0)} \\ &= \dim \operatorname{Range} d(j_*^k \overline{F})_{(0,0,0)} \end{aligned}$$

in contradiction with (4.2.3). □

REMARK. As in the nonsymmetric context, one can consider the bifurcation parameter λ to be multi-dimensional and proves analogous results (see [**2**]).

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Received September 12, 1985. This work was partly supported by a grant from NSERC of Canada and a "subvention FCAC" from the Department of Education of Québec.

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