

NONCOINCIDENCE INDEX, FREE GROUP ACTIONS, AND THE FIXED POINT PROPERTY FOR MANIFOLDS

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Let M be a compact oriented connected topological manifold. We show that if the Euler characteristic $\chi(M) \neq 0$ and M admits no degree zero self-maps without fixed points, then there is a finite number r such that any set of r or more fixed-point-free self-maps of M has a coincidence (i.e. for two of the maps f and g there exists $x \in M$ so that $f(x) = g(x)$). We call r the noncoincidence index of M . More generally, for any manifold M with $\chi(M) \neq 0$ there is a finite number r (called the restricted noncoincidence index of M) so that any set of r or more fixed-point-free nonzero degree self-maps of M has a coincidence. We investigate how these indices change as one passes from a space to its orbit space under a free action. We compute the restricted noncoincidence index for certain products and for the homogeneous spaces SU_n/K , K a closed connected subgroup of maximal rank; in some cases these computations also give the noncoincidence index of the space.

1. Introduction. Let M be a connected topological manifold. In an earlier paper [10] the author introduced the noncoincidence index, defined as $1 + m$ if the maximum cardinality m of a set of noncoincident fixed-point-free self-maps of M is finite, and as ∞ otherwise. We shall denote the noncoincidence index of M by $NI(M)$. Two immediate consequences of the definition are:

1. If $NI(M) < \infty$, then any self-map of M has a periodic point of period at most $NI(M)$;
2. If a group G acts freely on M , then $|G| \leq NI(M)$.

In [10] the author established some sufficient conditions for $NI(M)$ to be ∞ and did explicit calculations of $NI(M)$ in some cases where $NI(M)$ is finite. The first major result of this paper characterizes the compact oriented manifolds of finite noncoincidence index.

THEOREM 1.1. *If M is a compact oriented manifold, then $NI(M) < \infty$ if and only if $\chi(M) \neq 0$ and M admits no fixed-point-free self-maps of degree zero. In this case*

$$(1) \quad NI(M) \leq \beta_1^2 + \beta_2^2 + \cdots + \beta_n^2,$$

where $\beta_1, \beta_2, \dots, \beta_n$ are the Betti numbers of M .

REMARK. The bound (1) may be compared to similar bounds for periodic points and free group actions. If X is a compact polyhedron with $\chi(X) \neq 0$ and f is a map from X to itself that induces an automorphism of $H^*(X; \mathbf{Q})$, then f has a periodic point of period at most

$$\max \left(\sum_{i \text{ even}} \beta_i, \sum_{i \text{ odd}} \beta_i \right),$$

where β_i is the i th Betti number of X (see [3] or [8]). Also, any group acting freely on such a polyhedron X has order at most $|\chi(X)|$, since it must divide $\chi(X)$.

It follows immediately from the definition that $NI(M) = 1$ if and only if M has the fixed point property. If $NI(M)$ is finite but not 1, there may be a free action by a group G so that the orbit space M/G has the fixed point property. In §3 we prove some results of this type: the simplest is the following one.

THEOREM 1.2. *Let M be a compact oriented simply connected manifold, and suppose a finite group G acts freely on M . Then $NI(M/G) \leq NI(M)/|G|$. (In particular, M/G has the fixed point property if $|G| = NI(M)$.)*

The usefulness of the noncoincidence index is limited by the fact that $NI(M) = \infty$ for so many manifolds M . For example, the noncoincidence index of a product is ∞ unless both factors have the fixed point property. In §4 we discuss the restricted noncoincidence index, which is always equal to the noncoincidence index when the latter is finite, but which is finite for a larger class of manifolds. We have the following result about products.

THEOREM 1.3. *Let M be a product of even-dimensional spheres, complex projective spaces, and quaternionic projective spaces \mathbf{HP}^n with n even. Then the restricted noncoincidence index of M is the product of the noncoincidence indices of its indecomposable factors.*

In §5 we compute the restricted noncoincidence of any generalized complex flag manifold (i.e. homogeneous space G/K , where G is a compact simple Lie group of type A_n and K is a closed connected subgroup of maximal rank). This can be improved to a computation of the noncoincidence index for some classes of such flag manifolds. In particular, we prove the following result.

THEOREM 1.4. *Let M be a generalized complex flag manifold: then M is homeomorphic to SU_n/K , where K is a subgroup of the form $SU_n \cap U_{n_1} \times \cdots \times U_{n_k}$ with $n_1 \leq n_2 \leq \cdots \leq n_k$, $n_1 + \cdots + n_k = n$. Suppose n is even or no more than one odd number occurs in the sequence n_1, n_2, \dots, n_k with odd multiplicity. Then there is a free action by a finite group G on M so that M/G has the fixed point property provided one of the following conditions holds:*

1. $k = 2$ and $n_1 \leq 3$ or $n_2 \geq 2n_1^2 - n_1 - 1$;
2. $k = 3$, $n_1 = 1$, and $n_3 \geq 2n_2^2 - 1$;
3. $n_1 = n_2 = \cdots = n_{k-1} = 1$.

2. The noncoincidence index. In this section we establish some general results on the noncoincidence index of a compact oriented n -manifold M . We also prove some results helpful in computing the noncoincidence index from information about the automorphisms of $H^*(M; \mathbf{Q})$.

For any self-map f of M , the diagram

$$M \xrightarrow{i_f} M \times M \xrightarrow{\text{pr}_1} M,$$

where $i_f(x) = (x, f(x))$ and pr_1 is projection onto the first factor, is a microbundle in the sense of Milnor [15]. In fact it is the pullback microbundle f^*t_M , where t_M is the tangent microbundle of M . The ‘zero section’ of f^*t_M (i.e., the image of i_f) is $\text{gr } f$, the graph of f . By a theorem of Kister [13], any sufficiently small neighborhood of $\text{gr } f$ in $M \times M$ is an \mathbf{R}^n -bundle over M (where \mathbf{R}^n is thought of as having a distinguished origin but no linear structure). We use this fact in proving the next result.

THEOREM 2.1. *For any sufficiently small neighborhood U of $\text{gr } f$ in $M \times M$, $\text{pr}_1: U - \text{gr } f \rightarrow M$ is an $(\mathbf{R}^n - 0)$ -bundle and the obstruction to sectioning it is $\text{deg } f\chi(M)$.*

Proof. Take U sufficiently small so that $U \xrightarrow{\text{pr}_1} M$ is an \mathbf{R}^n -bundle. Since M is oriented, so is any bundle realization of t_M or f^*t_M . Then the obstruction to sectioning $U - \text{gr } f \xrightarrow{\text{pr}_1} M$ is an element of $H^n(M; \mathbf{Z})$. For $f = \text{id}$, this obstruction is the Euler class $e(M)$: by naturality, the obstruction in the general case is $f^*e(M)$ (cf. [17]). Since $\langle f^*e(M), [M] \rangle = \text{deg } f\chi(M)$, the conclusion follows.

THEOREM 2.2. *If M admits a fixed-point-free self-map of degree zero, then $NI(M) = \infty$.*

Proof. Let $f: M \rightarrow M$ be a fixed-point-free map with $\deg f = 0$. Then $\text{gr } f \cap \Delta(M) = \emptyset$, where $\Delta(M) \subset M \times M$ is the diagonal. Take a neighborhood U_1 of $\text{gr } f$ small enough so that $U_1 \cap \Delta(M) = \emptyset$ and $U_1 \xrightarrow{\text{pr}_1} M$ is an \mathbf{R}^n -bundle: then the previous result implies that there is a section of $U_1 - \text{gr } f \xrightarrow{\text{pr}_1} M$. Such a section gives a map $f_1: M \rightarrow M$ which is fixed-point-free and noncoincident with f . Now repeat this procedure: take a neighborhood $U_2 \subset U_1$ of $\text{gr } f$ with $U_2 \cap \text{gr } f_1 = \emptyset$. Sectioning $U_2 - \text{gr } f \xrightarrow{\text{pr}_1} M$ gives a map $f_2: M \rightarrow M$ which is fixed-point-free and noncoincident with f and f_1 . In this way we get $NI(M) \geq k$ for any k .

REMARK. A similar argument (taking neighborhoods of $\Delta(M)$) shows that $NI(M) = \infty$ when $\chi(M) = 0$. This is proved without assuming orientability in [10].

Let $LZ(M) = \{f^*: H^*(M; \mathbf{Q}) \rightarrow H^*(M; \mathbf{Q}) \mid f \text{ is fixed-point-free}\}$. (This is a slight modification of the definition used in [10].) By the Lefschetz fixed point theorem $L(\phi) = 0$ for any $\phi \in LZ(M)$, where $L(\phi)$ is the Lefschetz number

$$L(\phi) = \sum_{i \geq 0} (-1)^i \text{Tr}(\phi_i: H^i(M; \mathbf{Q}) \rightarrow H^i(M; \mathbf{Q})).$$

If $LZ(M)$ consists of automorphisms of $H^*(M; \mathbf{Q})$ (i.e. endomorphisms of nonzero degree) and $\chi(M) \neq 0$, we call M *L-rigid*. By Theorem 2.2 (and the subsequent remark), $NI(M) = \infty$ unless M is *L-rigid*.

We call automorphisms ϕ, ψ of $H^*(M; \mathbf{Q})$ *compatible* if $L(\phi\psi^{-1}) = 0$. (Note that $L(\mu) = \deg \mu L(\mu^{-1})$, so the relation is symmetric; it need not be reflexive or transitive.) If M is *L-rigid*, every element of $LZ(M)$ is compatible with the identity. From [10] we have the following result, which is proved by using the Lefschetz coincidence theorem.

THEOREM 2.3. *If M is L-rigid and $NI(M) \geq k$, then $LZ(M)' = LZ(M) \cup \{\text{id}\}$ contains a set of k pairwise compatible elements that includes the identity.*

Theorem 2.3 has an immediate corollary in the case where $H^*(M; \mathbf{Q})$ is a truncated polynomial algebra on an even-dimensional generator. Suppose $H^*(M; \mathbf{Q})$ is the polynomial algebra on $x \in H^{2d}(M; \mathbf{Q})$ modulo the relation $x^{h+1} = 0$: we call h the *height* of x . Then any endomorphism ϕ of $H^*(M; \mathbf{Q})$ has the form $\phi(x) = ax$, and by direct

computation its Lefschetz number is

$$L(\phi) = 1 + a + a^2 + \dots + a^h.$$

If we denote the right-hand side by $P_h(a)$, then $P_h(1) = h + 1$ and

$$P_h(a) = \frac{1 - a^{h+1}}{1 - a}$$

if $a \neq 1$. It follows that $P_h(a) = 0$ if and only if h is odd and $a = -1$.

The rational cohomology algebras of an even-dimensional sphere S^{2n} , complex projective space CP^n , and quaternionic projective space HP^n are truncated polynomial algebras on even-dimensional generators of heights 2, n , and n respectively. Then $NI(S^{2n}) \leq 2$ by Theorem 2.3, and in fact $NI(S^{2n}) = 2$ since the antipodal map is fixed-point-free. Similarly $NI(CP^n) = 2$ if n is odd, and we have $NI(CP^n) = 1$ if n is even. The case for quaternionic projective space is a bit more complicated: we have $NI(HP^n) = 1$, for n even, but also $NI(HP^n) = 1$ for odd $n \neq 1$. This is because there is no self-map of HP^n which induces the endomorphism of $H^*(HP^n; \mathbf{Z})$ that sends the generator x to $-x$ for odd $n \neq 1$, as can be seen from mod 3 Steenrod operations [2]. We shall return to these examples in §4.

The next result completes the proof of Theorem 1.1 by putting an upper bound on the noncoincidence index of an L -rigid manifold.

THEOREM 2.4. *If M is L -rigid, then $NI(M) \leq \beta_1^2 + \beta_2^2 + \dots + \beta_n^2$, where $\beta_1, \beta_2, \dots, \beta_n$ are the Betti numbers of M .*

Proof. As in [1] we can find a class $\bar{u} \in H^n(M \times M; \mathbf{Q})$ so that $d^*(\bar{u}) \in H^n(M; \mathbf{Q})$ is the Euler class of M , i.e.

$$\langle d^*(\bar{u}), [M] \rangle = \chi(M),$$

where $d: M \rightarrow M \times M$ is the diagonal map. Further, if we choose a basis p_1, p_2, \dots, p_k for $H^*(M; \mathbf{Q})$ and let q_1, q_2, \dots, q_k be a dual basis (so $\langle p_i \cup q_j, [M] \rangle = \delta_{ij}$), then

$$\bar{u} = \sum_{i=1}^k (-1)^{\dim p_i} p_i \times q_i.$$

Then for any endomorphism ϕ of $H^*(M; \mathbf{Q})$, the element

$$(\phi \times \text{id})(\bar{u}) = \sum_{i=1}^k (-1)^{\dim p_i} \phi(p_i) \times q_i$$

has the property $\langle d^*((\phi \times \text{id})(\bar{u})), [M] \rangle = L(\phi)$ (cf. [1]).

Now suppose $NI(M) > \beta_1^2 + \beta_2^2 + \cdots + \beta_n^2 = \text{rank } H^n(M \times M; \mathbf{Q})$. Then there is a set $S = \{\phi_0, \phi_1, \dots, \phi_r\}$ of pairwise compatible elements of $LZ(M)$, where $\phi_0 = \text{id}$ and $r = \text{rank } H^n(M \times M; \mathbf{Q})$. The set $\{(\phi \times \text{id})(\bar{u}) \mid \phi \in S\}$ is linearly dependent: let

$$(1) \quad c_0 \bar{u} + c_1 (\phi_1 \times \text{id})(\bar{u}) + \cdots + c_r (\phi_r \times \text{id})(\bar{u}) = 0$$

be a relation of linear dependence. If $c_0 \neq 0$, then we can apply the operator $\langle d^*(\cdot), [M] \rangle$ to (1) to obtain $c_0 \chi(M) = 0$, contradicting the hypothesis. Otherwise, first choose $c_i \neq 0$ and apply $\phi_i^{-1} \times \text{id}$ to (1).

REMARK. The argument used here is essentially that of Fuller [3].

If S is a set of pairwise compatible automorphisms of $H^*(M; \mathbf{Q})$ that includes the identity, we call S a compatible set for M . In view of Theorem 2.3, we can bound $NI(M)$ by bounding the cardinality of compatible sets for M . The following results will be used in §§4–5.

PROPOSITION 2.5. *Let S be a compatible set for M contained in a subgroup G of $\text{Aut } H^*(M; \mathbf{Q})$, and suppose H is a subgroup of finite index in G such that $L(h) \neq 0$ for all $h \in H$. Then $|S| \leq [G : H]$.*

Proof. Let S be a compatible set for M . The conclusion follows if the cosets $H\phi$ and $H\psi$ are distinct for any two distinct elements $\phi, \psi \in S$. Suppose otherwise: then $\phi = h\psi$ for some $h \in H$, and $L(h) = L(\phi\psi^{-1}) = 0$, contradicting the hypothesis.

PROPOSITION 2.6. *Suppose S is a compatible set for M contained in a finite subgroup G of $\text{Aut } H^*(M; \mathbf{Q})$, and let R be a subset of G such that $\text{id} \in R$ and $L(\rho^{-1}\sigma) \neq 0$ for any $\rho, \sigma \in R$. Then $|S| \leq |G|/|R|$.*

Proof. It suffices to show that $\rho S \cap \sigma S = \emptyset$ for distinct elements ρ, σ of R . Suppose otherwise: then there are distinct elements s_1, s_2 of S so that $\rho s_1 = \sigma s_2$. But then $L(s_1 s_2^{-1}) = L(\rho^{-1}\sigma) \neq 0$ contradicting the compatibility of S .

3. Free group actions and the fixed point property. If M is a simply connected, compact oriented manifold that admits a free action by a finite group G , information about the relationship between the noncoincidence index of the orbit space M/G and that of M can be obtained by studying the finite covering $M \rightarrow M/G$. In this section we prove a number of theorems about this situation.

Suppose now that G is a finite group that acts freely on M , a compact oriented manifold. Then $H^*(M/G; \mathbf{Q}) = H^*(M; \mathbf{Q})^G$, the subalgebra of G -invariants. (Note that M/G is a manifold, but need not be orientable since we are not assuming that the elements of G preserve orientation.) We can assume $\chi(M) \neq 0$, since otherwise M and M/G have noncoincidence index ∞ . Let $LZ(M)' = LZ(M) \cup \{\text{id}\}$, and define $LZ(M)^G$ to be the set of elements of $LZ(M)'$ that preserve $H^*(M; \mathbf{Q})^G$.

If M is simply connected, then M is the universal cover of the manifold M/G . Given any map $f: M/G \rightarrow M/G$, there is a 'lift' $\tilde{f}: M \rightarrow M$ so that the diagram

$$(1) \quad \begin{array}{ccc} M & \xrightarrow{\tilde{f}} & M \\ \downarrow & & \downarrow \\ M/G & \xrightarrow{f} & M/G \end{array}$$

commutes. If \tilde{f}, \tilde{g} are lifts of noncoincident self-maps f, g of M/G , then $\tilde{f}(x)$ and $\tilde{g}(x)$ are in different G -orbits for every $x \in M$. Applying $H^*(\cdot; \mathbf{Q})$ to the diagram (1), we see that \tilde{f}^* takes the subalgebra $H^*(M; \mathbf{Q})^G$ into itself. In fact we have the following result.

THEOREM 3.1. *Let M be a compact oriented simply connected manifold of nonzero Euler characteristic. If a finite group G acts freely on M and $NI(M/G) \geq k$, then $LZ(M)^G$ contains a compatible subset of $k|G|$ elements that includes G .*

Proof. Let $\{f_1, f_2, \dots, f_k\}$ be a set of noncoincident self-maps of M/G with $f_1 = \text{id}$. For each f_i choose a lift $\tilde{f}_i: M \rightarrow M$. Then the set $\{g\tilde{f}_i \mid g \in G, 1 \leq i \leq k\}$ has the properties required by the conclusion.

Theorem 1.2 follows immediately from this result and Theorem 2.3. We give two other corollaries: in each M satisfies the hypotheses of Theorem 3.1 and G acts freely on M . (Note that G can be regarded as a subset of $LZ(M)'$.)

COROLLARY 3.2. *If $G \subset LZ(M)'$ is maximal as a set of pairwise compatible elements of $LZ(M)^G$, then M/G has the fixed point property.*

COROLLARY 3.3. *If $LZ(M)'$ is a group and $N(G)$ (the normalizer of G in $LZ(M)'$) acts freely on M , then $NI(M/G) = |N(G)/G|$.*

It is interesting to compare the preceding results to the following one, which does not use the idea of noncoincidence index.

THEOREM 3.4. *Suppose $H^*(M; \mathbf{Q})$ is concentrated in even dimensions and a group G of order $\chi(M)$ acts freely on M . Then M/G has the fixed point property.*

Proof. Since $\chi(M/G) = \chi(M)/|G| = 1$, the reduced rational cohomology $\tilde{H}^*(M/G; \mathbf{Q})$ is trivial. But then the Lefschetz fixed point theorem implies M/G has the fixed point property.

Both Theorem 1.2 and Theorem 3.4 can be used to prove that $\mathbf{R}P^n$ has the fixed point property for n even, since in that case $NI(S^n) = \chi(S^n) = 2$. But Theorem 1.2 shows that the orbit space of $\mathbf{C}P^n$, $n \geq 3$ odd, has the fixed point property for any free \mathbf{Z}_2 -action, while Theorem 3.4 does not apply since $\chi(\mathbf{C}P^n) = n + 1 \geq 4$.

4. Products and the restricted noncoincidence index. A difficulty with the noncoincidence index is that it is so often infinite. In fact, we have the following result.

PROPOSITION 4.1. *Let M be a compact manifold with $NI(M) < \infty$. Then any retract of M (other than M itself) has the fixed point property.*

Proof. Suppose $L \subset M$ is a retract admitting a fixed-point-free map f . Let $i: L \rightarrow M$ be the inclusion, $r: M \rightarrow L$ a retraction. If $L \neq M$, then $ifr: M \rightarrow M$ is a nonsurjective fixed-point-free map. But then $NI(M) = \infty$ by Theorem 2.2 of [10].

Thus, any product $M \times N$ has noncoincidence index ∞ unless both M and N have the fixed point property. If M is a compact oriented manifold, we define the restricted noncoincidence index of M , $RNI(M)$, by putting $RNI(M) \geq k + 1$ whenever M admits k self-maps of nonzero degree, each of which is fixed-point-free and no two of which have a coincidence. Evidently $RNI(M) \leq NI(M)$. We have the following results from the proofs of Theorems 2.3, 2.4 and 3.1, respectively.

PROPOSITION 4.2. *If M is a compact oriented manifold with $\chi(M) \neq 0$ and $RNI(M) \geq k$, then $LZ(M)'$ contains a compatible set for M with k elements.*

PROPOSITION 4.3. *For any compact oriented manifold M , $RNI(M) = NI(M)$ whenever the latter is finite. Further, $RNI(M) < \infty$ if and*

only if $\chi(M) \neq 0$, in which case

$$RNI(M) \leq \beta_1^2 + \beta_2^2 + \cdots + \beta_n^2,$$

where $\beta_1, \beta_2, \dots, \beta_n$ are the Betti numbers of M .

PROPOSITION 4.4. *Let M be a compact oriented simply connected manifold on which a group G acts freely by orientation-preserving homeomorphisms. Then $RNI(M/G) \leq RNI(M)/|G|$.*

We note that any group G acting freely on M has order $|G| \leq RNI(M)$, and that, if $\chi(M) \neq 0$, any self-map of M of nonzero degree has a periodic point of period at most $RNI(M)$. Also, $RNI(M) = 1$ if and only if M has the fixed point property for maps of nonzero degree.

It is evident that $RNI(M \times N) \geq RNI(M)RNI(N)$ for any two manifolds M and N . Equality holds in many cases (Theorem 1.3), but the inequality can be strict even for products of spheres and projective spaces (Proposition 4.7).

For many manifolds M , it is possible to classify the automorphisms of $H^*(M; \mathbb{Q})$ and use Proposition 2.5 or 2.6 together with Proposition 4.2 to compute $RNI(M)$. Let A be a graded algebra that is a tensor product of truncated polynomial algebras, with even dimensional generators x_1, \dots, x_n . Then from [11] we have the following results (ht x_i denotes the height of x_i).

THEOREM 4.5. *Any automorphism ϕ of A has the form $\phi(x_i) = a_i x_{\sigma(i)}$, where σ is an element of the symmetric group Σ_n such that $\dim x_i = \dim x_j$ and $\text{ht } x_i = \text{ht } x_j$ whenever $\sigma(i) = j$.*

THEOREM 4.6. *Let ϕ be the automorphism of A given by $\phi(x_i) = a_i x_{\sigma(i)}$. Then*

$$L(\phi) = \prod_{i=1}^k P_{h_i} \left(\prod_{j \in S_i} a_j \right),$$

where $\{1, 2, \dots, n\} = S_1 \cup \cdots \cup S_k$ is the partition corresponding to σ , $h_i = \text{ht } x_j$ for $j \in S_i$, and P_h is the polynomial introduced in §2.

As noted in §2, the cohomology algebras of even-dimensional spheres and complex and quaternionic projective spaces are truncated polynomial algebras on even-dimensional generators. We can use the algebraic results above to prove Theorem 1.3.

Proof of 1.3. The hypothesis insures that $H^*(M; \mathbf{Q})$ is a tensor product of truncated polynomial algebras on even-dimensional generators x_1, \dots, x_n . Suppose first that the x_i all have the same height and dimension. Then by Theorem 4.5 any automorphism of $H^*(M; \mathbf{Q})$ can be represented by $(a_1, \dots, a_n; \sigma)$, where the a_i are nonzero rational numbers and $\sigma \in \Sigma_n$. In fact, the automorphism group of $H^*(M; \mathbf{Q})$ is the semidirect product $(\mathbf{Q}^*)^n \rtimes \Sigma_n$, where \mathbf{Q}^* is the group of nonzero rational numbers and Σ_n acts on $(\mathbf{Q}^*)^n$ by permuting its factors. Now from Theorem 4.6 it follows that the Lefschetz number of $(a_1, \dots, a_n; \sigma)$ is always nonzero if the common height of the x_i is even, and zero exactly when the product of the a_i over some cycle of σ is -1 if the height is odd. In the first case $RNI(M) = 1$. In the second case there is a subgroup of index 2^n in $\mathbf{Q}^* \rtimes \Sigma_n$ given by the elements $(a_1, \dots, a_n; \sigma)$ with all the a_i positive (a complete set of coset representatives is given by those elements with $\sigma = \text{id}$ and $a_i = \pm 1$ for each i); every element of this subgroup has nonzero Lefschetz number. But then we can apply Propositions 2.5 and 4.2 to conclude that $RNI(M) \leq 2^n$.

For the general case, let $\{1, 2, \dots, n\} = S_1 \cup \dots \cup S_k$ be the coarsest partition such that all generators labelled by elements of a set S_i have the same dimension and height. Then $H^*(M; \mathbf{Q}) = A_1 \otimes \dots \otimes A_k$, where A_i is the subalgebra generated by $\{x_j | j \in S_i\}$. Theorem 4.4 implies that the automorphisms of $H^*(M; \mathbf{Q})$ leave the subalgebras A_i invariant, and the automorphism group of $H^*(M; \mathbf{Q})$ is just the direct product of those for A_1, \dots, A_k . By the argument of the preceding paragraph, $RNI(M)$ is bounded by the product of the numbers $2^{|S_i|}$ (for each i such that the generators of A_i have odd height) and 1 (for each i such that the generators of A_i have even height). But (by the remarks following 2.3) this is the product of the noncoincidence indices of the factors of M .

PROPOSITION 4.7. $RNI(S^2 \times \mathbf{HP}^3) = 3$.

Proof. By Theorem 4.5, all automorphisms of the cohomology of $S^2 \times \mathbf{HP}^3$ can be written as $\phi \otimes \psi$ for automorphisms ϕ, ψ of $H^*(S^2; \mathbf{Q})$ and $H^*(\mathbf{HP}^3; \mathbf{Q})$ respectively. Since $L(\phi \otimes \psi) = L(\phi)L(\psi)$ and $L(\psi) \neq 0$ for every automorphism ψ induced by a self-map of \mathbf{HP}^3 , the elements of $LZ(S^2 \times \mathbf{HP}^3)$ all have the form $a^* \otimes \psi$, where a is the antipodal map of S^2 . For two such maps to be noncoincident it is

necessary for their Lefschetz coincidence number to be zero, i.e.

$$\begin{aligned} 0 &= L(a^* \otimes \phi_1, a^* \otimes \psi_2) = -L((a^* \otimes \psi_1)^{-1} a^* \otimes \psi_2) \\ &= -L(\text{id} \otimes \psi_1^{-1} \psi_2) = -L(\psi_1^{-1} \psi_2). \end{aligned}$$

Now the only automorphism of $H^*(\mathbf{HP}^3; \mathbf{Q})$ with Lefschetz number zero is the one that multiplies the generator by -1 . This is not induced by any self-map of \mathbf{HP}^3 , but it can be obtained as $\psi_1^{-1} \psi_2$ for ψ_1, ψ_2 induced by self-maps of \mathbf{HP}^3 : for example, we can choose ψ_1 to multiply the generator by 360, and ψ_2 to multiply it by -360 (see [14]). Then the maps corresponding to ψ_1 and ψ_2 have Lefschetz coincidence number zero, and can be deformed to noncoincident maps f_1, f_2 [1], so $a \times f_1, a \times f_2$ are fixed-point-free noncoincident maps of the product: clearly this is the largest possible such set.

5. Complex flag manifolds. Manifolds of finite noncoincidence index occur ‘naturally’ as homogeneous spaces of the form G/K , where G is a simple compact connected Lie group and K is a closed connected subgroup of maximal rank. The simplest situation is when $K = T$, a maximal torus. Papadima [16], generalizing an earlier result of the author [9], has recently shown that G/T has noncoincidence index equal to the order of the Weyl group $W(G) = N_G T/T$ ($N_G T$ is the normalizer of T in G). Since $W(G)$ also acts freely on G/T , the orbit space $G/N_G T$ has the fixed point property by Theorem 1.2. (One can also use Theorem 3.4, since $\chi(G/T) = |W(G)|$ and the rational cohomology of G/T is concentrated in even dimensions.)

It appears that similar results hold if K has maximal rank but is not a torus. But even for the class of homogeneous spaces studied in [10], there were cases where the noncoincidence index was infinite. Here we study the case where G is of type A_n , where the most results on the cohomology endomorphisms of G/K seem to be available.

We consider homogeneous spaces of the form SU_n/K , where K is a closed subgroup of maximal rank: it is well-known that K is conjugate to a subgroup of the form $SU_n \cap U_{n_1} \times U_{n_2} \times \cdots \times U_{n_k}$, where $n_1 + \cdots + n_k = n$. Henceforth we shall assume without loss of generality that $K = SU_n \cap U_{n_1} \times \cdots \times U_{n_k}$ with $n_1 \leq n_2 \leq \cdots \leq n_k$. The space SU_n/K can be regarded as the space of orthogonal decompositions

$$(1) \quad \mathbf{C}^n = p_1 \oplus \cdots \oplus p_k, \quad \dim p_i = n_i.$$

The ‘Weyl group’ for this situation is $N_G K/K$, which is

$$N_{W(G)} W(K)/W(K) :$$

it can be identified with the group of permutations $\sigma \in \Sigma_k$ such that $n_i = n_j$ whenever $\sigma(i) = j$. This group acts on decompositions (1) by sending p_i to $p_{\sigma(i)}$, and this action is evidently free.

The cohomology of SU_n/K is generated by elements $x_{ij} \in H^{2j}(SU_n/K; \mathbf{Q})$, $1 \leq i \leq k$, $1 \leq j \leq n_i$, subject to the relations

$$(1 + x_{11} + \cdots + x_{1n_1}) \cdots (1 + x_{k1} + \cdots + x_{kn_k}) = 1.$$

The action of the group N_GK/K on cohomology is easily described: $\sigma \in N_GK/K$ sends x_{ij} to $x_{\sigma(i)j}$. More generally, for any rational number $\lambda \neq 0$ and $\sigma \in N_GK/K$ we can define an automorphism Φ_λ^σ of $H^*(SU_n/K; \mathbf{Q})$ by $\Phi_\lambda^\sigma(x_{ij}) = \lambda^j x_{\sigma(i)j}$: we call such an automorphism a permutation automorphism. Recently Shiga and Tezuka [18] have proved that every automorphism of $H^*(SU_n/K; \mathbf{Q})$ has this form. In order to find the restricted noncoincidence index of SU_n/K , we first compute the Lefschetz number of a permutation automorphism.

THEOREM 5.1. *Let λ be a rational number, $\sigma \in N_GK/K$ a permutation, and $\{1, 2, \dots, k\} = S_1 \cup \cdots \cup S_r$ the partition associated to σ . Then*

$$L(\Phi_\lambda^\sigma) = (1 - \lambda)(1 - \lambda^2) \cdots (1 - \lambda^n) \prod_{p=1}^r \prod_{j=1}^{n(S_p)} \frac{1}{1 - \lambda^j |S_p|},$$

where $n(S_p)$ is n_i for any (and thus every) $i \in S_p$.

Proof. This calculation is a straightforward generalization of those done in [9] and [10]. First note SU_n/K is homeomorphic to U_n/K' , where $K' = U_{n_1} \times \cdots \times U_{n_k}$, so we have a fibration

$$(2) \quad SU_n/K \rightarrow BU_{n_1} \times \cdots \times BU_{n_k} \rightarrow BU_n.$$

Now let

$$P = H^*(BU_{n_1} \times \cdots \times BU_{n_k}; \mathbf{Q}) = \mathbf{Q}[x_{ij} | 1 \leq i \leq k, 1 \leq j \leq n_i],$$

where x_{ij} is in dimension $2j$. Then N_GK/K acts on P by sending x_{ij} to $x_{\sigma(i)j}$, and we have an invariant decomposition

$$P = \bigotimes_{p=1}^r \bigotimes_{j=1}^{n(S_p)} A_{pj},$$

where $A_{pj} = \mathbf{Q}[x_{ij} | i \in S_p]$. Now

$$\sum_{i \geq 0} t^i \text{Tr}(\sigma : (A_{pj})_{2i} \rightarrow (A_{pj})_{2i}) = \frac{1}{1 - t^j |S_p|},$$

so the multiplicative property of trace on tensor products gives

$$(3) \quad \sum_{i \geq 0} t^i \operatorname{Tr}(\sigma : P_{2i} \rightarrow P_{2i}) = \prod_{p=1}^r \prod_{j=1}^{n(S_p)} \frac{1}{1 - t^{j|S_p|}}.$$

Now the spectral sequence of the fibration (2) collapses for degree reasons, so

$$P = H^*(SU_n/K; \mathbf{Q}) \otimes H^*(BU_n; \mathbf{Q})$$

additively. Together with (3) and the fact

$$\sum_{i \geq 0} t^i \dim H^{2i}(BU_n; \mathbf{Q}) = \frac{1}{(1-t) \cdots (1-t^n)},$$

this implies

$$\begin{aligned} \sum_{i \geq 0} t^i \operatorname{Tr}(\Phi_1^\sigma : H^{2i}(SU_n/K; \mathbf{Q}) \rightarrow H^{2i}(SU_n/K; \mathbf{Q})) \\ = (1-t) \cdots (1-t^n) \prod_{p=1}^r \prod_{j=1}^{n(S_p)} \frac{1}{1 - t^{j|S_p|}}, \end{aligned}$$

from which the conclusion follows by putting $t = \lambda$.

It will be convenient to introduce some further notation. For each positive integer q , let $M_q = \{i | n_i = q\}$ and $m_q = |M_q|$. We can now give the restricted noncoincidence index of SU_n/K as follows.

THEOREM 5.2. *Let $M = SU_n/K$ be a complex flag manifold. Then*

$$RNI(M) = \begin{cases} |N_G K/K|, & \text{if at most one of } m_1, m_3, m_5, \dots \text{ is odd;} \\ 2|N_G K/K|, & \text{otherwise.} \end{cases}$$

Proof. First we use 5.1 to determine the automorphisms Φ_λ^σ for which $L(\Phi_\lambda^\sigma) = 0$. Evidently this only happens if $\lambda = \pm 1$, so it follows at once that $RNI(M) \leq 2|N_G K/K|$. Since the automorphisms Φ_1^σ are induced by the free action of $N_G K/K$ on SU_n/K , we have $L(\Phi_1^\sigma) = 0$ for all $\sigma \neq \text{id}$. For $\lambda = -1$, we consider the limit in 5.1 as $\lambda \rightarrow -1$. This limit will be zero unless there are as many factors $1 - \lambda^i$, i even, in the denominator as there are in the numerator. There are $[n/2]$ such factors in the numerator, where brackets indicate the greatest integer function, while the denominator has

$$(4) \quad \begin{aligned} \sum_{|S_p| \text{ even}} n(S_p) + \sum_{|S_p| \text{ odd}} \left[\frac{n(S_p)}{2} \right] \\ = \sum_{|S_p| \text{ even}} \sum_{i \in S_p} \frac{n_i}{|S_p|} + \sum_{|S_p| \text{ odd}} \sum_{i \in S_p} \frac{1}{|S_p|} \left[\frac{n_i}{2} \right]. \end{aligned}$$

Since the n_i sum to n , (4) can equal $[n/2]$ only if $|S_p| \leq 2$ for all p and, with perhaps one exception if n is odd, $n(S_p)$ is even whenever $|S_p| = 1$. Since each set S_p must be contained in some M_q , there exists a σ whose partition $\{S_p\}$ satisfies these conditions only if at most one of $\{m_q | q \text{ odd}\}$ is odd. In this case, there exists σ with $\Phi_{-1}^\sigma \neq 0$, and we can use Proposition 2.6 (with $R = \{\text{id}, \Phi_{-1}^\sigma\}$) to conclude that $RNI(M) \leq |N_G K/K|$.

To complete the proof we need only show that $RNI(M) \geq 2|N_G K/K|$ in the case where two or more of $\{m_q | q \text{ odd}\}$ are odd: we do this by explicit construction. First we define a map $J: \mathbf{C}^n \rightarrow \mathbf{C}^n$ as follows. If $n = 2p$ is even, regard \mathbf{C}^n as the quaternionic vector space \mathbf{H}^p and define J to be multiplication by the quaternion j ; if $n = 2p + 1$ is odd, think of \mathbf{C}^n as $\mathbf{H}^p \oplus \mathbf{C}$ and let $J(v, z) = (jv, \bar{z})$. Then J is a conjugate-linear automorphism of \mathbf{C}^n that preserves orthogonality, and \mathbf{C}^n cannot have two orthogonal J -invariant subspaces of odd dimension (cf. Theorem 1 of [6]). Now regard SU_n/K as the space of decompositions of form (1), and define $2|N_G K/K|$ self-maps of SU_n/K by

$$(5) \quad (p_1, \dots, p_k) \rightarrow (p_{\sigma(1)}, \dots, p_{\sigma(k)}), \quad \sigma \in N_G K/K;$$

$$(6) \quad (p_1, \dots, p_k) \rightarrow (Jp_{\sigma(1)}, \dots, Jp_{\sigma(k)}), \quad \sigma \in N_G K/K.$$

Clearly there are no coincidences among the maps in (5) or those in (6). Now suppose there is a coincidence between a map in (5) and one in (6): then there is an element (p_1, \dots, p_k) of SU_n/K with

$$p_{\sigma(i)} = Jp_{\pi(i)}, \quad 1 \leq i \leq k,$$

for some $\sigma, \pi \in N_G K/K$. Then the subspaces

$$\bigoplus_{n_j=i} p_j, \quad m_i \neq 0,$$

are J -invariant and have dimension im_i . Since at least two of m_1, m_3, \dots are odd, this contradicts the properties of J . The conclusion then follows.

REMARK. If n is even, J has period 2 as a self-map of SU_n/K , and the group $N_G K/K \times \mathbf{Z}_2$ acts on SU_n/K via

$$(\sigma, \varepsilon) \cdot (p_1, \dots, p_k) = (J^\varepsilon p_{\sigma(1)}, \dots, J^\varepsilon p_{\sigma(k)}).$$

By the argument above, this action is free if two or more of m_1, m_3, \dots are odd. It is in general not orientation-preserving, however, since

(σ, ε) has degree $(-1)^{\varepsilon d} \operatorname{sgn} \hat{\sigma}$, where

$$d = \sum_{i < j} n_i n_j$$

is the complex dimension of SU_n/K and $\hat{\sigma}$ is the restriction of σ to $\{i | n_i \text{ is odd}\}$ (as follows from Theorem 2.3 of [12]).

If $n_1 = 1$, $H^*(SU_n/K; \mathbf{Q})$ may have nontrivial endomorphisms that factor through the cohomology of CP^{n-1} . These ‘projective’ endomorphisms are classified in [5] as follows. Let

$$1 - t^n = P_1(t)P_2(t) \cdots P_k(t)$$

be a factorization in $\mathbf{Z}[t]$ with $P_i(t)$ a polynomial of degree n_i and $P_r(t) = 1 - t$ for some r with $n_r = 1$. Then

$$\Psi(1 + x_{i1} + \cdots + x_{in_i}) = P_i(-\lambda x_{r1}), \quad 1 \leq i \leq k,$$

defines a projective endomorphism Ψ , and all projective endomorphisms have this form. We call λ the multiplier of Ψ .

THEOREM 5.3. *Let $M = SU_n/K$ be a complex flag manifold with no cohomology endomorphisms except permutation automorphisms and projective endomorphisms. Then $NI(M) = RNI(M)$ unless $k = 3$, $n_1 = n_2 = 1$, and n_3 is even.*

Proof. It is shown in [10] that SU_n/K has noncoincidence index ∞ when $k = 3$, $n_1 = n_2 = 1$, and n_3 is even, so we can exclude this case. In view of the preceding result, it is enough to show that $LZ(SU_n/K)$ contains no projective endomorphisms when $k > 2$. Now Theorem 3 of [7] requires that any projective endomorphism of $H^*(SU_n/K; \mathbf{Q})$ induced by a self-map have multiplier divisible by 2 or 3. But Proposition 5 of [6] says that a projective endomorphism of Lefschetz number zero has multiplier -1 .

At present there is no known flag manifold SU_n/K that fails to satisfy the hypothesis of Theorem 5.3. It seems reasonable to conjecture that this is always the case, but the existing evidence is somewhat limited. Full classifications of the endomorphisms of $H^*(SU_n/K; \mathbf{Q})$ are available only in the following cases:

1. $k = 2$ and $n_1 \leq 3$ or $n_2 \geq 2n_1^2 - n_1 - 1$ [4];
2. $k = 3$, $n_1 = 1$, and $n_3 \geq 2n_2^2 - 1$ [5];
3. $n_1 = n_2 = \cdots = n_{k-1} = 1$ [10].

In cases covered by 1–3 we can use Theorem 5.3 to conclude that $NI(SU_n/K) = RNI(SU_n/K)$, except when $k = 3$, $n_1 = n_2 = 1$, and n_3

is even. Theorem 1.4 now follows from 1.2 and the remark following 5.2, except in the latter case. In this case there are fixed-point-free self-maps of degree zero, but they do not preserve the subring of $N_G K/K$ -invariants in $H^*(SU_n/K; \mathbf{Q})$ (cf. Theorem 4.1 of [10]). Thus the orbit space has the fixed point property by 3.1.

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