PACIFIC JOURNAL OF MATHEMATICS
Vol. 137, No. 1, 1989

THREE QUAVERS ON UNITARY ELEMENTS IN C^*-ALGEBRAS

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Henry Dye in memoriam

Unitary polar decomposition of elements in C^*-algebras is discussed in relation to the theory of unitary rank; and characterizations of algebras admitting weak or unitary polar decomposition of every element are given.

Introduction. Let A be a unital C^*-algebra, and denote by $\mathrm{GL}(A)$ and $\mathscr{U}(A)$ the groups of invertible and unitary elements in A, respectively. The set

$$\mathscr{P}(A) = \mathscr{U}(A)A_+$$

consists of those elements that admit a unitary polar decomposition in A. The formulae $x = (x|x|^{-1})|x|$ and $x = u|x| = \lim u(|x|+n^{-1})$ show that $\mathrm{GL}(A) \subseteq \mathscr{P}(A)$ and that $\mathrm{GL}(A)$ is dense in $\mathscr{P}(A)$. Moreover, it was shown in **[12]** and **[16]** that each element in A has a canonical approximant in $\mathscr{P}(A)^=$.

We know from Mazur's theorem that $\mathrm{GL}(A) = A\backslash\{0\}$ only if $A = \mathbf{C}$. The corresponding question, when $\mathscr{P}(A) = A$, is more subtle, and will be addressed in the third of these short notes. In the first two we shall study certain phenomena in the unit ball A^1 of A. In particular we shall be concerned with the set

$$\mathscr{P}(A)^1 = \mathscr{U}(A)A^1_+.$$

(As usual we write S^1 for $S \cap A^1$, for any subset S of A.) It is quite easy to see that

$$\mathrm{GL}(A)^1 \subseteq \mathscr{P}(A)^1 \subseteq \tfrac{1}{2}(\mathscr{U}(A) + \mathscr{U}(A)),$$

and that these sets are dense in one another. By **[16**, Proposition 3.16] their common closure $(\mathscr{P}(A)^1)^=$ consists of those elements x in A such that for every $\varepsilon > 0$ there are unitary elements u_1, u_2 and u_3 with $x = \frac{1}{2}(1-\varepsilon)u_1 + \frac{1}{2}(1-\varepsilon)u_2 + \varepsilon u_3$.

1. Unitary rank revisited. Based on the Russo-Dye theorem **[17]**, the theory of unitary rank is the discussion of the least number of unitaries needed to express an element in A^1 as an element in $\mathrm{conv}(\mathscr{U}(A))$, cf. [7], [8], [16]. The point of departure is L. T. Gardner's observation, [2], that

$$(*)\qquad (A^1)^0 + \mathscr{U}(A) \subseteq \mathscr{U}(A) + \mathscr{U}(A).$$

Replacing the open unit ball $(A^1)^0$ with A^1, above, is usually not possible (unless A is a von Neumann algebra, see [8, Lemma 2.1]). Recently U. Haagerup [5] found that

$$(**)\qquad A^1 + 2\mathscr{P}(A)^1 \subseteq \mathscr{U}(A) + 2\mathscr{P}(A)^1,$$

and used this to verify the conjecture, [8, 3.5], that the unitary rank of an element x in A with $\|x\| \le 1 - 2/n$ cannot exceed n. We shall now show how the result $(**)$ may replace $(*)$, to give a slightly stronger theory.

Proposition 1.1. *For each x in A, let $\alpha = \mathrm{dist}(x, \mathrm{GL}(A))$. Then*

$$\mathrm{dist}(x, \mathscr{P}(A)^1) = \max\{\alpha, \|x\| - 1\}.$$

Moreover, if $x = v|x|$ is the polar decomposition of x in A'', and $f_0(t) = 1 \wedge (t-\alpha)_+$, then $x_0 = vf_0(|x|) \in (\mathscr{P}(A)^1)^=$, with $\|x - x_0\| =$ $\mathrm{dist}(x, \mathscr{P}(A)^1)$.

Proof. Put $\beta = \mathrm{dist}(x, \mathscr{P}(A)^1)$. Since $\mathrm{GL}(A)^1 \subseteq \mathrm{GL}(A)$ it is clear that $\beta \ge \alpha$. Since moreover $\mathrm{GL}(A)^1 \subseteq A^1$, it is also clear that $\beta \ge \|x\| - 1$. To show the inequality $\beta \le \max\{\alpha, \|x\| - 1\}$ take $\varepsilon > 0$ and define $f_\varepsilon(t) = 1 \wedge (t - (\alpha + \varepsilon))_+$. By [12, Theorem 5] there is a u_ε in $\mathscr{U}(A)$ such that

$$vf_\varepsilon(|x|) = u_\varepsilon f_\varepsilon(|x|),$$

and clearly this element belongs to $\mathscr{P}(A)^1$. It follows that $x_0 = vf_0(|x|) \in (\mathscr{P}(A)^1)^=$. Finally,

$$\begin{aligned}\|x - x_0\| &= \|v|x| - vf_0(|x|)\| = \|\,|x| - f_0(|x|)\| \\ &= \max\{t - f_0(t) | 0 \le t \le \|x\|\} = \max\{\alpha, \|x\| - 1\}.\end{aligned}$$

Theorem 1.2. *Given x in A^1, assume that*

$$\|\beta x - 2p\| \le \beta - 2$$

for some p in $\mathscr{P}(A)^1$ and some $\beta \ge 2$. Then with n the natural number such that $n - 1 < \beta \le n$, there are unitaries $u_1, \dots, u_n$ in $\mathscr{U}(A)$, such that

$$x = \beta^{-1}(u_1 + \cdots + u_{n-1}) + \beta^{-1}(\beta + 1 - n)u_n.$$

Proof. The case $\beta = 2$ easily reduces to the classical Murray-von Neumann result that $x = \frac{1}{2}(u + u^*)$ for every x in A^1_{sa}. If $\beta > 2$, put $y = (\beta - 2)^{-1}(\beta x - 2p)$. Then $\|y\| \leq 1$ and $\beta x = (\beta - 2)y + 2p$. By repeated application of Haagerup's result (**) we obtain unitaries u_k in $\mathscr{U}(A)$ and elements p_k in $\mathscr{P}(A)^1$ for $1 \leq k \leq n - 3$, such that

$$\begin{aligned}\beta x &= u_1 + 2p_1 + (\beta - 3)y = u_1 + u_2 + 2p_2 + (\beta - 4)y \\ &= \cdots = u_1 + \cdots + u_{n-3} + 2p_{n-3}(\beta + 1 - n)y.\end{aligned}$$

Since $0 \leq \beta + 1 - n < 1$ we can apply [8, Lemma 2.3] to obtain v_{n-3} and u_n in $\mathscr{U}(A)$ with

$$u_{n-3} + (\beta + 1 - n)y = v_{n-3} + (\beta + 1 - n)u_n.$$

Finally, by the classical case, $2p_{n-3} = u_{n-2} + u_{n-1}$ for some unitaries in $\mathscr{U}(A)$, and thus (relabeling v_{n-3} as u_{n-3}) we have the desired expression

$$\beta x = (u_1 + \cdots + u_{n-3}) + (u_{n-2} + u_{n-1}) + (\beta + 1 - n)u_n.$$

REMARK 1.3. Note that we actually obtain a slightly stronger decomposition

$$x = \beta^{-1}(u_1 + \cdots + u_{n-3}) + \beta^{-1}(\beta + 1 - n)u_n + 2\beta^{-1}p_0$$

for some p_0 in $\mathscr{P}(A)^1$.

PROPOSITION 1.4. *The infimum of those β for which Theorem* 1.2 *can hold is* $2(1 - \alpha)^{-1}$, *where* $\alpha = \operatorname{dist}(x, \mathrm{GL}(A))$.

Proof. By Proposition 1.1 we have

$$\begin{aligned}\operatorname{dist}(\beta x, 2\mathscr{P}(A)^1) &= 2\operatorname{dist}(\tfrac{1}{2}\beta x, \mathscr{P}(A)^1) \\ &= 2\max\left\{\tfrac{1}{2}\beta\alpha, \tfrac{1}{2}\beta\|x\| - 1\right\} = \max\{\beta\alpha, \beta\|x\| - 2\}.\end{aligned}$$

This maximum is $\leq \beta - 2$ precisely when $\beta\alpha \leq \beta - 2$, i.e. $\beta \geq 2(1 - \alpha)^{-1}$.

REMARK 1.5. Theorem 1.2 is closely patterned after [8, Proposition 3.1], with $\mathscr{P}(A)^1$ replacing $\mathscr{U}(A)$. The improvement is clear: even though $\|\beta x - u\| \leq \beta - 1$ for some u in $\mathscr{U}(A)$ we cannot conclude that $\beta x = u_1 + \cdots + u_{n-1} + (\beta + 1 - n)u_n$, simply because Gardner's result does not hold for the closed, but only for the open unit ball. Note also from Remark 1.3 that the result is best possible, because

$$\|\beta x - 2p_0\| = \|u_1 + \cdots + u_{n-3} + (\beta + 1 - n)u_n\| \leq \beta - 2.$$

2. Uniqueness of unitary means. Any non-zero complex number in the unit disk is the midpoint of a unique pair of unitary numbers. We show that the same fact is valid to a large extent, when **C** is replaced by an arbitrary unital C^*-algebra. This principle lies behind the arguments in [7, Remark 19] and [**13**]. Corollary 2.4 was obtained by R. V. Kadison and the author simultaneously (it rained a lot in Warwick this summer), and Proposition 2.7 was pointed out to me by M. Rørdam.

LEMMA 2.1. *If $x \in A$ and $x = \alpha u + \beta v$ for some unitaries u and v in $\mathscr{U}(A)$ and $0 < \alpha, \beta < 1$, $\alpha + \beta = 1$, then with $\gamma = \alpha^{1/2}\beta^{-1/2}$ we have $u = x + i\gamma^{-1}y$, $v = x - i\gamma y$, where $y \in A$ satisfying*

(i) $x^*x + y^*y = 1$, $xx^* + yy^* = 1$;

(ii) $i(x^*y - y^*x) = (\gamma - \gamma^{-1})y^*y$, $-i(xy^* - yx^*) = (\gamma - \gamma^{-1})yy^*$.

Conversely, if y satisfies (i) *and* (ii), *then with $u = x + i\gamma^{-1}y$ and $v = x - i\gamma y$ we have unitaries such that $x = \alpha u + \beta v$.*

Proof. The four equations expressing the unitarity of u and v are

$$\begin{aligned}
&x^*x + \gamma^{-2}y^*y + i\gamma^{-1}(x^*y - y^*x) = 1,\\
&xx^* + \gamma^{-2}yy^* - i\gamma^{-1}(xy^* - yx^*) = 1,\\
&x^*x + \gamma^2 y^*y - i\gamma(x^*y - y^*x) = 1,\\
&xx^* + \gamma^2 yy^* + i\gamma(xy^* - yx^*) = 1.
\end{aligned}$$

These are easily seen to be equivalent with the four equations contained in (i) and (ii).

PROPOSITION 2.2 (cf. [7, Remark 7]). *If $x = w|x|$ for some w in $\mathscr{U}(A)$ and $|\alpha - \beta| \le x \le 1$, then with*

$$y = \tfrac{1}{2}(\alpha\beta)^{-1/2} w|x|^{-1}(1 - |x|^2)^{1/2}[(|x|^2 - (\alpha - \beta)^2)^{1/2} - i(\alpha - \beta)(1 - |x|^2)^{1/2}]$$

we obtain unitaries u and v as in Lemmas 2.1 *such that $x = \alpha u + \beta v$.*

Proof. By straightforward computations we verify that y satisfies the conditions (i) and (ii) of Lemma 2.1 Note that when $\alpha = \beta = \frac{1}{2}$ we are back at the classical case $y = w(1 - |x|^2)^{1/2}$.

THEOREM 2.3. *If $x = \alpha u + \beta v$ for some x in* GL(A)*, where u, v are in $\mathscr{U}(A)$ and $0 < \alpha, \beta < 1$, $\alpha + \beta = 1$, then with y as in Lemma 2.1 we have*

$$y = \tfrac{1}{2}(\alpha\beta)^{-1/2} w|x|^{-1} z.$$

Here $w|x| = x$ is the unitary polar decomposition of x, and $z = h + ik$ is a normal element of A, commuting with $|x|$, such that

$$|h| = (1 - |x|^2)^{1/2}(|x|^2 - (\alpha - \beta)^2)^{1/2}, \qquad k = (\beta - \alpha)(1 - |x|^2).$$

Proof. We define

$$z = 2(\alpha\beta)^{1/2}|x|w^* y = 2(\alpha\beta)^{1/2} x^* y$$

(as we must), and compute, using (i), that

$$\begin{aligned} z^* z &= 4\alpha\beta y^* x x^* y = 4\alpha\beta y^*(1 - yy^*)y \\ &= 4\alpha\beta y^* y(1 - y^* y) = 4\alpha\beta(1 - x^* x)x^* x, \\ zz^* &= 4\alpha\beta x^* yy^* x = 4\alpha\beta x^*(1 - xx^*)x \\ &= 4\alpha\beta x^* x(1 - x^* x). \end{aligned}$$

Thus z is normal; and if $z = h + ik$, with h and k in A_{sa}, we have $h^2 + k^2 = z^* z = 4\alpha\beta|x|^2(1 - |x|^2)$.

From condition (ii) in Lemma 2.1 we have

$$\begin{aligned} k &= \tfrac{1}{2}i(z - z^*) = (\alpha\beta)^{1/2} i(x^* y - y^* x) \\ &= (\alpha\beta)^{1/2}(\gamma - \gamma^{-1})y^* y = (\alpha - \beta)(1 - |x|^2). \end{aligned}$$

With $a = 1 - |x|^2$ we then solve the equation for h^2:

$$\begin{aligned} h^2 &= |z|^2 - k^2 = 4\alpha\beta(1 - a)a - (\alpha - \beta)^2 a^2 \\ &= 4\alpha\beta a - (\alpha + \beta)^2 a^2 = (1 - |x|^2)(4\alpha\beta - 1 + |x|^2) \\ &= (1 - |x|^2)(|x|^2 - (\alpha - \beta)^2). \end{aligned}$$

To show, finally, that h, and therefore also z, commutes with $|x|$, we use the second part of (ii) to get

$$\begin{aligned} (\gamma - \gamma^{-1})|x|^2(1 - |x|^2) &= (\gamma - \gamma^{-1})x^*(1 - xx^*)x \\ &= (\gamma - \gamma^{-1})x^* yy^* x = -ix^*(xy^* - yx^*)x \\ &= \tfrac{1}{2}i(\alpha\beta)^{-1/2}(zx^* x - x^* x z^*). \end{aligned}$$

Multiplying with $2(\alpha\beta)^{1/2}$ and inserting $z = h + ik$ gives

$$2(\alpha - \beta)|x|^2(1 - |x|^2) = i(h|x|^2 - |x|^2 h) - 2k|x|^2.$$

Since $-k|x|^2 = (\alpha - \beta)|x|^2(1 - |x|^2)$ it follows that $h|x|^2 - |x|^2 h = 0$, as desired.

COROLLARY 2.4. *If* $x = \frac{1}{2}(u + v)$ *and* $x \in \mathrm{GL}(A)$, *then* $u = x + iy$, $v = x - iy$ *and* $y = w(1 - |x|^2)^{1/2} s$. *Here* $x = w|x|$ *is the polar decomposition, and* s *is a symmetry in* A'' *commuting with* $|x|$ *and multiplying* $1 - |x|^2$ *into* A.

Proof. By Theorem 2.3 we have $y = w|x|^{-1}h$, and we let e be the range projection of h_+ in A''. Then $s = 2e - 1$ is a symmetry commuting with $|x|$ and $s|h| = s(h_+ + h_-) = h_+ - h_- = h$. Since $|h| = (1 - |x|^2)^{1/2}|x|$ the result follows.

COROLLARY 2.5. *If* $x \in \mathrm{GL}(A)$ *such that* $|x|$ *is multiplicity-free (i.e. generates a maximal commutative* C^**-subalgebra of* A*) and has connected spectrum, then for each* α, β *there is at most one pair in* $\mathscr{U}(A)$ *such that* $x = \alpha u + \beta v$.

Proof. Put $B = C^*(|x|, 1)$, so that $B \sim C(\mathrm{sp}(|x|))$. If $x = \alpha u + \beta v$, let y and $z = h + ik$ be as in Theorem 2.3. It suffices to show that h is uniquely determined, up to a change of sign; because then the pair u, v will be unique. But

$$h \in B' \cap A = B,$$

so that $h = f(|x|)$ for some real function f in $C(\mathrm{sp}(|x|))$. We see that $f(\lambda)^2 = (1 - \lambda^2)(\lambda^2 - (\alpha - \beta)^2)$, whence

$$f(\lambda) = \pm(1 - \lambda^2)^{1/2}(\lambda^2 - (\alpha - \beta)^2)^{1/2}, \qquad \lambda \in \mathrm{sp}(|x|).$$

Since the spectrum is connected, exactly one of the signs must hold for all λ.

COROLLARY 2.6. *If* $x \in \mathscr{P}(A)$ *with* $|\alpha - \beta| < |x| < 1$, *and if the commutant of* $|x|$ *in* A *contains no non-trivial projections, then* $x = \alpha u + \beta v$ *for a unique pair of unitaries in* $\mathscr{U}(A)$.

Proof. As in the previous corollary it suffices to show uniqueness (modulo sign) of h. As $|\alpha - \beta| < |x| < 1$ we see that $|h| \in \mathrm{GL}(A)$ and thus $h = s|h|$ for some self-adjoint unitary s $(= h|h|^{-1})$ in the relative commutant of $|x|$. As $s = 2p - 1$ for some projection p, we see that $s = 1$ or $s = -1$.

PROPOSITION 2.7. *An element x in A with $\|x\| < 1$ belongs to $\frac{1}{2}\mathscr{U}(A) + \frac{1}{2}\mathscr{U}(A)$ if and only if $x = wa$ for some w in $\mathscr{U}(A)$ and some a in A^1_{sa}.*

Proof. Since $a = \frac{1}{2}(u + u^*)$ with $u = a + i(1 - a^2)^{1/2}$, the sufficiency is clear. To prove necessity, assume that $x = \frac{1}{2}(u + v)$ and take y as in Lemma 2.1 (with $\alpha = \beta = \frac{1}{2}$). Since $\|x\| < 1$ we see from (i) that both y^*y and yy^* are invertible, so that $y \in \mathrm{GL}(A)$ with $y = w|y|$ for some w in $\mathscr{U}(A)$. Put $a = w^*x$ and compute by (ii)

$$|y|a = |y|w^*x = y^*x = x^*y = x^*w|y| = a^*|y|.$$

Thus $|y|a$ is self-adjoint. On the other hand,

$$\begin{aligned}|y|a = y^*x = w^*|y^*|x &= w^*(1 - xx^*)^{1/2}x \\ &= w^*x(1 - x^*x)^{1/2} = a|y|,\end{aligned}$$

by (i), so that a and $|y|$ commute. Therefore

$$a = |y|^{-1}|y|a \in A_{sa}.$$

3. Unitary polar decomposition. We say that an element x in A admits a *weak polar decomposition* if $x = v|x|$ for some v in A with $\|v\| \leq 1$. Note that v is not assumed to be a partial isometry and, in particular, no uniqueness properties of the decomposition are expected. If a decomposition exists for every element we say that A has weak polar decomposition. Similarly we say that A has *unitary polar decomposition* if for every x in A there is a u in $\mathscr{U}(A)$ such that $x = u|x|$, i.e. $A = \mathscr{P}(A)$.

Recall from **[11]** that a unital C^*-algebra A is a SAW*-algebra if for each pair x, y of orthogonal elements in A_+ there is an element e in A_{sa} (which can then be assumed to satisfy $0 \leq e \leq 1$), such that $xe = 0$ and $(1 - e)y = 0$. We now say that A is an n-SAW*-algebra if $\mathbf{M}_n(A)$ is a SAW*-algebra. Clearly then $\mathbf{M}_m(A)$ is also a SAW*-algebra for each $m \leq n$. If the situation is stable, i.e. A is an n-SAW*-algebra for every n, we shall refer to A as a SSAW*-algebra.

One of the main difficulties with SAW*-algebras is that the definition, like the corresponding AW*-condition, only involves the commutative subalgebras of A. Therefore there is no compelling reason to believe that the SAW*-condition implies n-SAW* for $n > 1$. On the other hand, R. R. Smith and D. P. Williams show in **[20**, Theorem 3.4**]** that if A is a commutative SAW*-algebra (which means that $A = C(X)$ for some sub-Stonean space), then A is also SSAW*. The same happens when we investigate the natural source of SAW*-algebras: the *corona algebras*. These have the form $A = C(B)$, where B is a non-unital, but σ-unital C^*-algebra, and $C(B) = M(B)/B$. Clearly

$$\mathbf{M}_n(C(B)) = M(\mathbf{M}_n(B))/\mathbf{M}_n(B) = C(\mathbf{M}_n(B)),$$

so that all corona C^*-algebras are SSAW*.

PROPOSITION 3.1. *A C^*-algebra A is a SAW^*-algebra if and only if every self-adjoint element x admits a weak polar decomposition $x = v|x|$ with $v = v^*$.*

Proof. If A is a SAW*-algebra and $x \in A_{sa}$, consider the decomposition $x = x_+ - x_-$. Since $x_+x_- = 0$, there is an element e in A, $0 \le e \le 1$, such that $ex_- = 0$ and $(1-e)x_+ = 0$. Put $v = 2e - 1$ and note that $v = v^*$ and $-1 \le v \le 1$. Moreover,

$$v|x| = (2e-1)(x_+ + x_-) = x_+ - x_- = x.$$

Conversely, if A has weak polar decomposition in A_{sa}, consider an orthogonal pair x, y in A_+. By assumption

$$x - y = v|x-y| = v(x+y)$$

for some v in A_{sa} with $\|v\| \le 1$. Let $e = \frac{1}{2}(1+v)$, so that $1 - e = \frac{1}{2}(1-v)$, and use the facts $(1-v)x = (1+v)y = 0$ to verify that $(1-e)x = ey = 0$.

PROPOSITION 3.2. *If A is a 2-SAW^*-algebra, it has weak polar decomposition.*

Proof. We apply Proposition 3.1 to the self-adjoint element $\left(\begin{smallmatrix} 0 & x^* \\ x & 0 \end{smallmatrix}\right)$ in $\mathbf{M}_2(A)$, to obtain a self-adjoint matrix $w = \left(\begin{smallmatrix} y & v^* \\ v & z \end{smallmatrix}\right)$, satisfying the decomposition equation

$$\begin{aligned}\begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix} &= \begin{pmatrix} y & v^* \\ v & z \end{pmatrix} \left| \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix} \right| \\ &= \begin{pmatrix} y & v^* \\ v & z \end{pmatrix} \begin{pmatrix} |x| & 0 \\ 0 & |x^*| \end{pmatrix}.\end{aligned}$$

Direct computation shows that $x = v|x|$, and clearly $\|v\| \le 1$ since $\|w\| \le 1$.

PROPOSITION 3.3. *If A is a 4-SAW^*-algebra, there is for each pair x, y in A such that $x^*x \le y^*y$ an element w in A, with $\|w\| \le 1$, such that $x = wy$.*

Proof. Consider the elements

$$a = \begin{pmatrix} (|y|^2 - |x|^2)^{1/2} & 0 \\ x & 0 \end{pmatrix}, \quad b = \begin{pmatrix} |y| & 0 \\ 0 & 0 \end{pmatrix}$$

in $\mathbf{M}_2(A)$, and note that $a^*a = b^2$, i.e. $|a| = b$. Since $\mathbf{M}_2(A)$ is a 2-SAW*-algebra there is by Proposition 3.2 a matrix $c = (c_{ij})$ in $\mathbf{M}_2(A)$, with $\|c\| \leq 1$, such that $a = cb$. Multiplying the matrices we get

$$x = a_{21} = c_{21}|y|.$$

Since by the previous result, $y = u|y|$ for some u in A with $\|u\| \leq 1$, we have $|y| = u^*u|y| = u^*y$; and thus with $w = c_{21}u^*$ we get the desired result.

PROPOSITION 3.4. *If an element x in a C^*-algebra A admits a weak polar decomposition $x = v|x|$, such that*

$$\operatorname{dist}(v, \mathrm{GL}(A)) < 1,$$

then x has a unitary polar decomposition.

Proof. Put $\alpha = \operatorname{dist}(v, \mathrm{GL}(A))$. By [12, Corollary 8] we see that if $f \in C(\mathbf{R})$, such that $f(t) = 0$ for all $t \leq \alpha + \varepsilon$ for some $\varepsilon > 0$, then

$$vf(|v|) = u|v|f(|v|)$$

for some u in $\mathscr{U}(A)$. As $\alpha < 1$ we may choose f such that $f(1) = 1$. Since $v^*v|x| = |x|$, we have $(1-|v|)|x| = 0$, so that $(1-f(|v|))|x| = 0$. Consequently

$$u|x| = u|v|f(|v|)|x| = vf(|v|)|x| = v|x| = x.$$

THEOREM 3.5. *If a C^*-algebra A has unitary polar decomposition, then* $\mathrm{GL}(A)$ *is dense in A which is a SAW*-algebra. Conversely, if A is a 2-SAW*-algebra with* $\mathrm{GL}(A)$ *dense, then A has unitary polar decomposition.*

Proof. The first half of the theorem follows from Proposition 3.1 plus the fact that each element $u|x|$ in $\mathscr{P}(A)$ is the limit of $u(|x|+\varepsilon)$ in $\mathrm{GL}(A)$ as $\varepsilon \to 0$. The second half follows by combining Propositions 3.2 and 3.4.

COROLLARY 3.6. *A corona C^*-algebra has unitary polar decomposition if and only if the invertible elements are dense.*

Proof. As noted in the beginning of this section, corona algebras are SSAW*-algebras, so Theorem 3.5 takes on this simple form.

REMARK 3.7. In **[1]**, **[6]** and **[14]** M. J. Canfell, D. Handelman and A. G. Robertson prove (independently) that a compact Hausdorff space X is sub-Stonean (our terminology **[3]**, they talk about F-spaces) with $\dim X \leq 1$ if and only if $C(X)$ has unitary polar decomposition. Since $\dim X \leq 1$ is equivalent with $\mathrm{GL}(C(X))$ being dense in $C(X)$, the previous theorem represents a generalization to non-commutative C^*-algebras of their result.

Robertson also shows that the conditions above are equivalent with the equality

$$\tfrac{1}{2}(\mathscr{U}(C(X)) + \mathscr{U}(C(X))) = C(X)^1.$$

Presumably this also generalizes. At least Proposition 2.7 shows that if

$$\tfrac{1}{2}(\mathscr{U}(A) + \mathscr{U}(A)) = A^1$$

for some C^*-algebra A, then each element x in A has the form ua with u in $\mathscr{U}(A)$ and $a = a^*$. The problem is, of course, that a is not assumed to commute with $|x|$, so that we do not immediately obtain unitary polar decomposition.

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Received February 17, 1988.

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