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# ON THE BEHAVIOUR OF CAPILLARIES AT A CORNER

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**Consider the solution of the capillary surface equation over domains with a corner. It is assumed that the corner is bounded by lines. If the corner angle $2\alpha$ satisfies $0 < 2\alpha < \pi$ and $\alpha + \gamma < \pi/2$ where $0 \le \gamma < \pi/2$ is the contact angle between the surface and the container wall then it is shown that the leading term which was discovered by Concus and Finn is equal to the solution up to $O(r^\varepsilon)$ for an $\varepsilon > 0$ where $r$ denotes the distance from the corner.**

We consider the non-parametric capillary problem in presence of gravity over a bounded base domain $\Omega \subset \mathbb{R}^2$ with a corner. That means, we seek a surface $S\colon u = u(x)$, defined over $\Omega$, such that $S$ meets vertical cylinder walls over the boundary $\partial\Omega$ in a prescribed constant angle $\gamma$ such that the following equations are satisfied, see Finn [3],

$$\operatorname{div} Tu = \kappa u \quad \text{in } \Omega, \tag{1}$$

$$\nu \cdot Tu = \cos\gamma \quad \text{on the smooth parts of } \partial\Omega, \tag{2}$$

where

$$Tu = \frac{Du}{\sqrt{1 + |Du|^2}},$$

$\kappa = \text{const.} > 0$ and $\nu$ is the exterior unit normal on $\partial\Omega$.

Let the origin $x = 0$ be a corner of $\Omega$ with interior angle $2\alpha$ satisfying

$$0 < 2\alpha < \pi. \tag{3}$$

We assume that the corner is bounded by lines near $x = 0$, see Figure 1. Furthermore, we assume that the contact angle satisfies

$$0 \le \gamma < \frac{\pi}{2}. \tag{4}$$

Concus and Finn [2] have shown that $u$ is bounded near $x = 0$ if and only if $\alpha + \gamma \ge \pi/2$ is satisfied.

In the case $\alpha + \gamma > \pi/2$ there exists an asymptotic expansion of $u$ near the origin, cf. [4]. In the borderline case $\alpha + \gamma = \pi/2$ Tam [5]

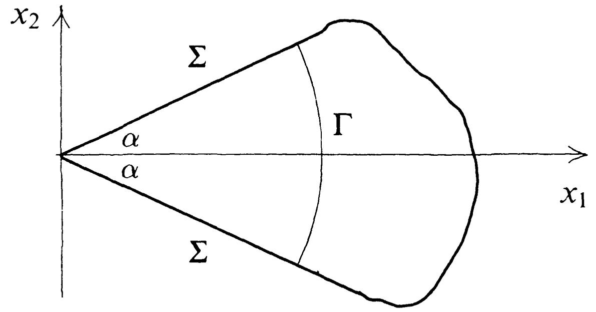


FIGURE 1

obtained that the normal vector to the surface $S$ is continuous up to the corner.

In this note we are interested in the case

$$\alpha+\gamma<\frac{\pi}{2}. \tag{5}$$

If (5) is satisfied, then a solution of (1), (2) is unbounded near the origin, see Concus and Finn [**2**] or Finn [**3**, Theorem 5.5]. Moreover, the leading term of a possible asymptotic expansion was given in these works. Here we show that this term is an approximation up to $O(r^{\varepsilon})$ for the solution itself.

The easy proof is based on a comparison principle of Concus and Finn [**1**], see also Finn [**3**, Theorem 5.1] and requires only some calculations with barrier functions which are not much different from comparison functions used by Concus and Finn [**2**], see also Finn [**3**, Proof of Theorem 5.5].

Let $r,\theta$ be polar coordinates centered at $x=0$, and set $k=\sin\alpha/\cos\gamma$.

THEOREM. *Let $u$ be a solution of* (1), (2). *Then, provided* (3), (4) *and* (5) *are satisfied, one has for an $\varepsilon>0$ the expansion*

$$u=\frac{\cos\theta-\sqrt{k^2-\sin^2\theta}}{k\kappa r}+O(r^{\varepsilon})$$

*near the corner.*

*Proof.* We set $B_\rho(0)=\{x\in\mathbb{R}^2; x_1^2+x_2^2<\rho^2\}$, $\rho>0$, and $\Omega_\rho=\Omega\cap B_\rho(0)$. Let

$$v=\frac{\cos\theta-\sqrt{k^2-\sin^2\theta}}{k\kappa r}$$

and set $w=v-Ar^{\lambda}$ where $A=\text{const.}>0$ and $\lambda=\text{const.}>0$.

Using polar coordinates, we obtain after some calculation that there are positive numbers $r_0$, $K_0$ and $\lambda_0$ such that for all $r$, $A$ and $\lambda$ with

$$(6) \qquad 0 < r \le r_0, \quad A > 0, \quad 0 < \lambda \le \lambda_0 \quad \text{and} \quad A\lambda \le K_0$$

we have

$$(7) \qquad \operatorname{div} Tw = \kappa w + A\kappa r^\lambda + \eta_1 + \eta_2$$

in $\Omega_{r_0}$, where

$$|\eta_1| \le c_1 r^3 \quad \text{and} \quad |\eta_2| \le c_2 A\lambda r^\lambda.$$

The constants $c_1$, $c_2$ do not depend on $r$, $\lambda$ and $A$. Moreover, we find after calculation that

$$(8) \qquad \nu \cdot Tw < \cos\gamma \quad \text{on } \Sigma_{r_0} \text{ if } \gamma > 0$$

and

$$(9) \qquad \nu \cdot Tw = \cos\gamma = 1 \quad \text{on } \Sigma_{r_0} \text{ if } \gamma = 0$$

for all $r$, $A$ and $\lambda$ satisfying (6). Here we set

$$\Sigma_\rho = [\partial\Omega \cap B_\rho(0)] \backslash \{0\}.$$

Now, we choose $A_1$ and $\lambda_1$ ($\lambda_1 > 0$ small and $A_1 > 0$ large) from the region defined by (6) such that

$$(10) \qquad A_1(\kappa - c_2\lambda_1) r^{\lambda_1} - c_1 r^3 > 0$$

is satisfied in $\Omega_{r_1}$, $r_1 > 0$ small enough, $r_1 \le r_0$. Set $\Gamma_\rho = \Omega \cap \partial B_\rho(0)$. We may choose $A_2 \ge A_1$ and $\lambda_2 \le \lambda_1$ satisfying (6) such that we have $v(r_1, \theta) - A_2 r_1^{\lambda_2} < u$ on $\Gamma_{r_1}$. The boundedness of $u$ on $\overline{\Gamma}_{r_1}$ is a consequence of a result of Concus and Finn [2], cf. also Finn [3, Proof of Theorem 5.5]. The inequality (10) remains valid for these $A_2$, $\lambda_2$ too. That means, we have obtained that $w = v - A_2 r^{\lambda_2}$ satisfies $\operatorname{div} Tw \ge \kappa w$ in $\Omega_{r_1}$, cf. (7) and (10), $\nu \cdot Tw \le \nu \cdot Tu$ on $\Sigma_{r_1}$, cf. (8) or (9) and (2), and $w \le u$ on $\Gamma_{r_1}$. The comparison principle of Concus and Finn, see for example Finn [3, Theorem 5.1], implies

$$v - A_2 r^{\lambda_2} \le u \quad \text{in } \Omega_{r_1}.$$

Setting $w = v + Ar^\lambda$, we obtain an upper bound for $u$ as follows. Again, by calculation we find in $\Omega_{r_0}$ for $r$, $A$, $\lambda$ satisfying (6) (we use the same notation for the constants $r_0, K_0, \ldots$, which may be different from the corresponding constants from above) that

$$(7') \qquad \operatorname{div} Tw = \kappa w - A\kappa r^\lambda + \eta_1 + \eta_2$$

where $\eta_1$, $\eta_2$ fulfill the same inequalities as above. If $\gamma > 0$, then for $r$, $A$ and $\lambda$ satisfying (6), we see after some calculation that

$$(8') \qquad \nu \cdot Tw \geq \cos\gamma + c_3 A\lambda r^{\lambda+1} - c_4 r^4$$

is true on $\Sigma_{r_0}$ with positive constants $c_3$ and $c_4$ not depending on $A$, $r$ and $\lambda$. Suppose that (6) and that for a positive constant $K_1$, $K_1 \leq K_0$, the inequality $K_1 \leq A\lambda$ is satisfied. In particular, we assume that

$$(11) \qquad K_1 \leq A\lambda \leq K_0$$

for $A$ and $\lambda$ from the region given by (6).

Now, inequality (8′) implies that there are positive constants $r_1$, $A_1$ and $\lambda_1$ such that one has

$$(12) \qquad \nu \cdot Tw \geq \cos\gamma \quad \text{on } \Sigma_{r_1}$$

for all $A$ and $\lambda$ with $A \geq A_1$ and $0 < \lambda \leq \lambda_1$ satisfying (11). We may choose an $A = A_2$ and a $\lambda = \lambda_2$ such that the inequality

$$(10') \qquad -A_2 r^{\lambda_2}(\kappa - c_2\lambda_2) + c_1 r^3 < 0$$

takes place in $\Omega_{r_2}$ for an $r_2 \leq r_1$. Now, we take $A_3$ large enough, $A_3 \geq A_2$, and $\lambda_3 > 0$ small enough, $\lambda_3 \leq \lambda_2$, so that (11) and the next inequality (13) are both satisfied,

$$(13) \qquad v + A_3 r_2^{\lambda_3} \geq u \quad \text{on } \Gamma_{r_2}.$$

Hence, since (10′) remains valid if $A_2$ is replaced by $A_3$ and $\lambda_2$ by $\lambda_3$, we obtain, see (7′) and (10′),

$$\operatorname{div} Tw \leq \kappa w \quad \text{in } \Omega_{r_2}.$$

From (13), this inequality and because (12) is true on $\Sigma_{r_2}$ it follows

$$v + A_3 r^{\lambda_3} \geq u \quad \text{in } \Omega_{r_2}$$

from the comparison principle of Concus and Finn.

If $\gamma = 0$, then the above considerations with respect to $\Sigma$ are superfluous since (9) takes place for this $w$ too. Thus, the theorem is proved.

REMARK. An inspection of the above proof shows that we have proven in fact a stronger result as formulated in the theorem: there exist positive constants $\rho_0$, $A$ and $\lambda$ only depending on $\alpha$, $\gamma$ and $\kappa$ and not on the particular solution $u$ considered such that

$$|u - v| \leq Ar^\lambda \quad \text{in } \Omega_{\rho_0}.$$

This follows because $r_1$ in the above proof concerning the lower bound for $u$ and $r_2$ which occurs in the proof of the upper bound do not depend on $u$. Then we use that there are bounds for $|u|$ on $\overline{\Gamma}_{r_1}$ and $\overline{\Gamma}_{r_2}$ which do not depend on $u$ itself, compare Finn [**3**, Proof of Theorem 5.5].

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