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ON MEANS OF DISTANCES ON THE SURFACE OF A SPHERE (LOWER BOUNDS)

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Given N points $x_1, x_2, \ldots, x_N$ on a unit sphere S in Euclidean d space $(d \geq 3)$, we investigate the α-sum $\sum |x - x_j|^\alpha$, $\alpha > 1 - d$, of their distances from a variable point x on S. We obtain an essentially best possible lower bound for the L^1-norm of its deviation from the mean value. As an application, we prove similar bounds for the α-sums $\sum |x_j - x_k|^\alpha$ of mutual distances.

Introduction. Let $S = S^{d-1}$ be the surface of the unit (hyper)sphere in d-dimensional Euclidean space $(d \geq 3)$. Denote by $|x - y|$ the Euclidean distance between two points x and y on S^{d-1}. Let $\omega_N = (x_1, x_2, \ldots, x_N)$ be a fixed set of N points on S, and let $x \in S$ be a variable point. With each value of a parameter α $(1-d < \alpha < \infty)$ we associate a distance function $U_\alpha(x, \omega_N)$ on S^{d-1}, which we define as follows:

$$(1) \qquad U_\alpha(x, \omega_N) = \sum_{j=1}^{N} |x - x_j|^\alpha - N \cdot m(\alpha, d) \qquad (\alpha \neq 0),$$

$$U_0(x, \omega_N) = \sum_{j=1}^{N} \log |x - x_j| - N \cdot m(0, d) \qquad (\alpha = 0).$$

Here $m(\alpha, d)$ is the mean value of $|x - x_j|^\alpha$ on S, which means

$$m(\alpha, d) = \frac{1}{\sigma(S)} \int_S |x - x_j|^\alpha \, d\sigma(x) \qquad (\alpha \neq 0),$$

$$m(0, d) = \frac{1}{\sigma(S)} \int_S \log |x - x_j| \, d\sigma(x) \qquad (\alpha = 0),$$

where σ is the $(d-1)$-dimensional area measure on S.

We give two interpretations of the functions U_α. First, the sums $\sum |x - x_j|^\alpha$ are related to the classical α-means $(\frac{1}{N} \sum |x - x_j|^\alpha)$ of distances from the point x to the points of ω_N, which contain as special cases the arithmetic $(\alpha = 1)$, geometric $(\alpha = 0)$, and harmonic $(\alpha = -1)$ mean. Second, the sums $\sum |x - x_j|^\alpha$ can be considered as Riesz potentials (see **[3]**) of a discrete charge distribution with an atom of unit weight at each point x_j. The logarithmic $(\alpha = 0)$ and the Newtonian $(\alpha = 2 - d)$ potential are special Riesz potentials.

The problem we are going to discuss is a problem of irregularities of distribution. If we replace the discrete distribution ω_N in (1) by the continuous uniform distribution $N \cdot \sigma$ on S^{d-1}, the corresponding integrals vanish identically on S^{d-1}. The fact that uniform distribution can be approximated by an N point distribution to a certain degree of accuracy only implies the existence of certain lower bounds for the L^1-norm

$$\|U_\alpha(x, \omega_N)\|_1 = \frac{1}{\sigma(S)} \int_S |U_\alpha(x, \omega_N)| \, d\sigma(x).$$

We prove

THEOREM 1. *For each $N \geq 1$ and each $\alpha \neq 2, 4, \ldots, 1 - d < \alpha < \infty$, the following inequality holds*:

$$\|U_\alpha(x, \omega_N)\|_1 \geq c(d, \alpha) \cdot N^{-\alpha/(d-1)}. \tag{2}$$

Here $c(d, \alpha)$ is a positive constant depending on d and α only.

It will be proved in a later paper **[10]** that the result of Theorem 1 is best possible apart from the value of the constant $c(d, \alpha)$. Note that inequality (2) is false for $\alpha = 2, 4, \ldots$. In these exceptional cases, one can construct a point set ω_N for each $N \geq N_0(d, \alpha)$, such that $U_\alpha(x, \omega_N) \equiv 0$ on S^{d-1}. In the classical harmonic case $\alpha = 2 - d$, Theorem 1 is already contained in a paper by P. Sjögren **[6]**. After suitable choice of the parameters involved, his Theorem 1 implies that

$$-\min_{x \in S} U_{2-d}(x, \omega_N) \geq c(d) \cdot N^{(d-2)/(d-1)},$$

but his proof also applies to the case of the L^1-norm without any change. For $d = 3$ and $\alpha = -1$, our result has the following physical interpretation: Suppose we place N electrons (each of unit charge) on the surface S^2. The function $U_{-1}(x, \omega_N)$ measures the difference between the actual potential of ω_N at the point x, and its mean value which is equal to N. By Theorem 1, there exist points $x \in S$ at which the actual potential is by at least $c \cdot N^{1/2}$ below the mean value.

We also consider distance functionals $E_\alpha(\omega_N)$ which have the physical meaning of energy sums. Let

$$(3)\qquad \begin{aligned} E_\alpha(\omega_N) &= \sum_{j,k}(|x_j - x_k|^\alpha - m(\alpha, d)) \quad \text{for } 0 < \alpha < 2, \\ E_0(\omega_N) &= \sum_{j\neq k}(\log|x_j - x_k| - m(0, d)), \quad \text{and} \\ E_\alpha(\omega_N) &= \sum_{j\neq k}(|x_j - x_k|^\alpha - m(\alpha, d)) \quad \text{for } 1 - d < \alpha < 0. \end{aligned}$$

Note that when dealing with energy sums, we restrict ourselves to values of α satisfying $1 - d < \alpha < 2$.

If we replace the energy sums in (3) by the corresponding energy integrals with respect to uniform distribution $N \cdot \sigma$, we obtain the value zero. The fact that we approximate uniform distribution by a discrete distribution again gives rise to certain lower bounds for $E_\alpha(\omega_N)$. We prove

THEOREM 2. *For each $N \geq 2$, the following energy inequalities hold:*

(a) $E_\alpha(\omega_N) \leq -c(\alpha, d) \cdot N^{1-\alpha/(d-1)}$ $\quad (0 < \alpha < 2)$,

(b) $E_\alpha(\omega_N) \geq -c(\alpha, d) \cdot N^{1-\alpha/(d-1)}$ $\quad (1 - d < \alpha < 3 - d)$,

(c) $E_\alpha(\omega_N) \geq -c(\alpha, d) \cdot N^{1-\alpha/(2-\alpha)}$ $\quad (3 - d \leq \alpha < 0, d \geq 4)$,

(d) $E_0(\omega_N) \leq \frac{N}{2} \log N + O(N)$.

Let us make a few remarks. Theorem 2 is probably not best possible in the case (c), and in case (d) for $d \geq 4$. For $d = 3$, the logarithmic case has already been handled in the author's paper [**9**].

The sum $E_1(\omega_N)$ was studied by K. B. Stolarsky ([**7**], [**8**]). He discovered a beautiful identity between the sum $E_1(\omega_N)$, and the L^2-norm of a function that measures discrepancy of the point set ω_N with respect to spherical caps on S^{d-1}. Using W. M. Schmidt's lower bounds for the discrepancy of an N point set on S^{d-1} with respect to spherical caps (see [**5**]), Stolarsky was able to obtain nontrivial bounds for $E_1(\omega_N)$ in dimension $d \geq 5$. J. Beck [**1**], using his method of Fourier transforms, finally proved the (best possible) estimate

$$E_1(\omega_N) \leq -c(d) \cdot N^{(d-2)/(d-1)}.$$

The method we shall use in order to prove Theorem 2 is independent of Beck's method.

For $d = 3$ and $\alpha = -1$, Theorem 2 contains the following result of physical interest. The energy $\sum_{j\neq k} |x_j - x_k|^{-1}$ of a distribution of N electrons on S^2 satisfies the inequality

$$\sum_{j\neq k} |x_j - x_k|^{-1} \geq N^2 - c \cdot N^{3/2}.$$

For some basic facts on potential theory, we refer to the beautiful paper [4] by Polya and Szegö, and to Landkof's book [3]. The theory of spherical harmonics on S^{d-1} is treated f.e. in [2].

2. Proof of Theorem 1. The proof of Theorem 1 is based on the construction of appropriate test functions $T(x)$ on S^{d-1}, and the use of the inequality

$$(4)\quad \|U_\alpha(x, \omega_N)\|_1 \geq \frac{1}{\sigma(S)} \left| \int_S U_\alpha(x, \omega_N) T(x)\, d\sigma(x) \right| \Big/ \sup_{x\in S} |T(x)|.$$

Step 1. We introduce spherical coordinates $\theta = (\theta_1, \theta_2, \dots, \theta_{d-2})$ $(0 \leq \theta_p \leq \pi)$ and ϕ $(0 \leq \phi < 2\pi)$ on S^{d-1}. Let Δ be the spherical Laplace operator on S^{d-1}, and consider the differential equation

$$\Delta^l h_l(\cos\theta_1) = \left(\sin^{2-d}\theta_1 \frac{d}{d\theta_1}\left(\sin^{d-2}\theta_1 \cdot \frac{d}{d\theta_1}\right)\right)^l h_l(\cos\theta_1) = 1, \qquad l = 1, 2, \dots.$$

This equation has a solution on the interval $(0, \pi]$, which behaves like $(\sin(\theta_1/2))^{l-d+2}$ near the point $\theta_1 = 0$ for $l - d + 2 \neq 0, 2, 4, \dots$, and like $(\sin(\theta_1/2))^{l-d+2} \cdot \log\sin(\theta_1/2)$ in the remaining cases. The expansion of h_l into ultraspherical polynomials $P_n^{(\lambda)}(\cos\theta_1)$, $\lambda = \frac{d}{2} - 1$, is given by

$$(5)\qquad h_l(\cos\theta_1) \sim c(\lambda, l) \cdot \sum_{n=1}^{\infty} \frac{n+\lambda}{(n(n+2\lambda))^l} P_n^{(\lambda)}(\cos\theta_1).$$

The expansion (5), although not necessarily convergent, is known to be Poisson summable, which means

$$h_l(\cos\theta_1) = \lim_{r\to 1} c(\lambda, l) \sum_{n=1}^{\infty} \frac{n+\lambda}{(n(n+2\lambda))^l} r^n P_n^{(\lambda)}(\cos\theta_1)$$

for $0 < \theta_1 \leq \pi$.

For the given point set ω_N, consider the function

$$H_l(x, \omega_N) = \sum_{j=1}^{N} h_l(\cos\gamma_j(x)),$$

where $2\sin(\frac{1}{2}\gamma_j(x)) = |x - x_j|$ and $x \in S^{d-1}$. For the function H_l, the inequality $\|H_l(x, \omega_N)\|_1 \gg N^{1-2l/(d-1)}$ is easily proved in the following way.

Consider the subdomain $D \subset S^{d-1}$ determined by the relations $0 \le \frac{\pi}{2} - \theta_\rho \le \frac{\pi}{6}$ $(\rho = 1, 2, \dots, d-2)$, and $0 \le \phi \le \frac{\pi}{6}$. Let $r = r(N)$ be the integer satisfying

$$2N \le 2^{(d-1)r} < 2^d \cdot N.$$

We partition D into "cubes" $B_\mu = B_{\mu_1\mu_2\cdots\mu_{d-1}}$ $(1 \le \mu_\rho \le 2^r)$, where B_μ is determined by the inequalities

$$(\mu_\rho - 1) \cdot \frac{\pi}{6} \cdot 2^{-r} \le \theta_\rho \le \mu_\rho \cdot \frac{\pi}{6} \cdot 2^{-r} \qquad (\rho = 1, 2, \dots, d-2) \quad \text{and}$$
$$(\mu_{d-1} - 1) \cdot \frac{\pi}{6} \cdot 2^{-r} \le \phi \le \mu_{d-1} \cdot \frac{\pi}{6} \cdot 2^{-r}.$$

The set of subdomains B_μ containing none of the points x_j in their interior will be denoted by Λ. By the choice of r, we have

$$\sum_\Lambda \sigma(B_\mu) \gg 1.$$

For $x = (\theta, \phi) \in B_\mu \in \Lambda$, let

$$\tau_\mu(x) = 4^{-lr} \prod_{\rho=1}^{d-2} \sin^{2l} 6 \cdot 2^r \theta_\rho \cdot \sin^{2l} 6 \cdot 2^r \phi.$$

Define a test function $T(x)$ on S^{d-1} by putting

$$T(\theta, \phi) = \Delta^l \tau_\mu(\theta, \phi)$$

for $(\theta, \phi) \in B_\mu \in \Lambda$, and $T(\theta, \phi) = 0$ elsewhere. Note that $\sup_{x\in S} |T(x)| \ll 1$ holds. Multiplying $H_l(x, \omega_N)$ by $T(x)$, and integrating over S^{d-1}, we obtain, using Green's second formula:

$$(6) \quad \left|\int_S H_l(x, \omega_N) T(x)\, d\sigma(x)\right| = \left|\sum_\Lambda \int_{B_\mu} H_l(x, \omega_N) \Delta^l \tau_\mu(x)\, d\sigma(x)\right|$$
$$= \left|\sum_\Lambda \int_{B_\mu} \Delta^l H_l(x, \omega_N) \cdot \tau_\mu(x) \cdot d\sigma(x)\right|$$
$$\gg N \cdot \sum_\Lambda \int_{B_\mu} \tau_\mu(x)\, d\sigma(x) \gg N \cdot 4^{-lr} \gg N \cdot N^{-2l/(d-1)}.$$

Here we use the fact that the normal derivatives of $\Delta^m \tau_\mu(x)$ $(m = 0, 1, \dots, l-1)$ with respect to the boundary of B_μ vanish. From relations (4) and (6), using $\sup_{x\in S}|T(x)| \ll 1$, the inequality $\|H_l(x, \omega_N)\|_1 \gg N \cdot N^{-2l/(d-1)}$ follows.

Step 2. We begin with the case $1-d < \alpha < 3-d$. Consider the kernel $k_\alpha(\cos\theta_1) = |2\sin(\theta_1/2)|^\alpha$, which generates the distance function $|x-y|^\alpha$. We are looking for an inverse kernel $k_\alpha^{-1}(\cos\theta_1)$ such that the convolution equation

$$k_\alpha^{-1} * k_\alpha = h_1 \tag{7}$$

holds on S^{d-1}.

We have the expansion

$$k_\alpha(\cos\theta_1) \sim \sum_{n=0}^{\infty} a_n \cdot P_n^{(\lambda)}(\cos\theta_1), \tag{8}$$

where

$$a_n = c(\lambda, \alpha) \cdot \frac{(n+\lambda)\cdot\Gamma(n-\alpha/2)}{\Gamma(n+2\lambda+1+\alpha/2)} \quad \text{and}$$
$$c(\lambda, \alpha) = 2^{1+\alpha} \cdot \frac{\Gamma(2\lambda)\cdot\Gamma(\alpha/2+\lambda+1/2)}{\Gamma(\lambda+1/2)\cdot\Gamma(-\alpha/2)}.$$

Note that the expansion (8) holds for any value of α satisfying $1-d < \alpha$ and $\alpha \neq 0, 2, \ldots$. If we omit the factor $\Gamma(-\frac{\alpha}{2})$ in the denominator $c(\lambda, \alpha)$, we obtain a kernel of the type $|\sin(\theta_1/2)|^\alpha \log\sin(\theta_1/2)$ for these exceptional values of α. It is in this sense that we shall use the notation $k_\alpha(\cos\theta_1)$ for all $\alpha > 1-d$.

Proceeding quite formally, and using (5), we obtain a solution of (7) in the form

$$k_\alpha^{-1}(\cos\theta_1) \sim \sum_{n=1}^{\infty} b_n \cdot P_n^{(\lambda)}(\cos\theta_1), \tag{9}$$

where

$$b_n = c_1(\lambda, \alpha) \cdot \frac{(n+\lambda)^2}{n(n+2\lambda)} \cdot \frac{\Gamma(n+2\lambda+1+(\alpha/2))}{(n+\lambda)\cdot\Gamma(n-(\alpha/2))}.$$

Using Stirling's formula, and subtracting successively appropriate multiples of (8) (with α replaced by $4-2d-\alpha, 5-2d-\alpha, \ldots$) from (9), we obtain a representation

$$k_\alpha^{-1} = d_1 \cdot k_{4-2d-\alpha} + d_2 \cdot k_{5-2d-\alpha} + \cdots + d_s \cdot k_{s+3-2d-\alpha} + R_s \qquad (d-1 \neq 0), \tag{10}$$

where ΔR_s is bounded and continuous for $s = d+1$.

A rigorous proof of (10) is obtained in the following way. Let

$$\pi_r(\cos\theta_1) = \sum_{n=0}^{\infty}(n+\lambda)r^n \cdot P_n^{(\lambda)}(\cos\theta_1) \qquad (0 < r < 1)$$

be the Poisson kernel on S^{d-1}. Note that k_α^{-1} and h_1 are integrable over S^{d-1}, and hence that $\kappa = k_\alpha^{-1} * \pi_r$ solves the equation

$$\kappa * k_\alpha = h_1 * \pi_r.$$

Letting $r \to 1$, we obtain the desired result.

From (10), we further get the estimates

$$\begin{aligned} &|k_\alpha^{-1}(|x-y|)| \ll |x-y|^{4-2d-\alpha} \quad \text{and} \\ &|\Delta k_\alpha^{-1}(|x-y|)| \ll |x-y|^{2-2d-\alpha}. \end{aligned} \tag{11}$$

Step 3. We use $T * k_\alpha^{-1}$ as a test function for $U_\alpha(x, \omega_N)$, where T is the test function introduced in step 1. In view of the relation

$$\begin{aligned} &\int_S U_\alpha(x, \omega_N) \cdot (T * k_\alpha^{-1})(x)\, d\sigma(x) \\ &\quad = \int_S (U_\alpha * k_\alpha^{-1})(x) \cdot T(x)\, d\sigma(x) \\ &\quad = \int_S H_1(x, \omega_N) \cdot T(x)\, d\sigma(x), \end{aligned} \tag{12}$$

it is sufficient to estimate $\sup_{x\in S} |(T * k_\alpha^{-1})(x)|$.

For fixed $x \in S^{d-1}$, let $\Lambda' = \Lambda'_x$ be the set of subdomains $B_\mu \in \Lambda$ that contain some point y such that $|x-y| < N^{-1/(d-1)}$ holds. Let $\Lambda'' = \Lambda\backslash\Lambda'$ be the set of remaining B_μ's. We have

$$\begin{aligned} |(T * k_\alpha^{-1})(x)| &\ll \left|\sum_{\Lambda'} \int_{B_\mu} \Delta\tau_\mu(y) \cdot k_\alpha^{-1}(|x-y|)\, d\sigma(y)\right| \\ &\quad + \left|\sum_{\Lambda''} \int_{B_\mu} \tau_\mu(y) \cdot \Delta k_\alpha^{-1}(|x-y|)\, d\sigma(y)\right| \\ &\ll \sum_{\Lambda'} \int_{B_\mu} |x-y|^{4-2d-\alpha}\, d\sigma(y) \\ &\quad + \sum_{\Lambda''} \int_{B_\mu} |x-y|^{2-2d-\alpha}\, d\sigma(y) \\ &\ll N^{(d+\alpha-3)/(d-1)}. \end{aligned} \tag{13}$$

From (6), (12), and (13), the assertion follows.

In order to obtain the assertion of Theorem 1 in the case $2l-1-d < \alpha < 2l+1-d$ $(l = 2, 3, \ldots; \alpha \neq 2, 4, \ldots)$, we proceed in a similar way, solving the equation $k_\alpha^{-1} * k_\alpha = h_l$, and noting that

$$\sup_{x\in S} |(T * k_\alpha^{-1})(x)| \ll N \cdot N^{(\alpha-2l)/(d-1)}.$$

This argument also works for $\alpha = 2l-1-d$, $\alpha \neq 2, 4, \ldots$, whereas in the case $\alpha = 2, 4, \ldots$ the convolution equation which corresponds to (7) has no solution. However, if we define $U_\alpha(x, \omega_N)$ by

$$U_\alpha(x, \omega_N) = \sum_{j=1}^{N} |x - x_j|^\alpha \log|x - x_j| - N \cdot m'(\alpha, d)$$

for $\alpha = 2, 4, \ldots$, the assertion of Theorem 1 would also remain true in the exceptional cases.

This finishes our proof of Theorem 1.

3. Bounds for energy sums. In proving Theorem 2, we shall distinguish three cases.

The case $0 < \alpha < 2$. By formula (8), all the coefficients $a_n = a_n(\alpha)$ $(n \geq 1)$ in the expansion of $k_\alpha(\cos\theta_1)$ are negative. The addition formula for spherical harmonics (see [2], §11.4.) implies the following identity:

$$E_\alpha(\omega_N) = -c(\alpha, d) \int_S \left(\sum_{j=1}^{N} \delta_\alpha(|x - x_j|) \right)^2 d\sigma(x).$$

Here $c(\alpha, d)$ is a positive constant, and $\delta_\alpha(|x-x_j|)$ is a new distance function, generated by the kernel

$$\delta_\alpha(\cos\theta_1) \sim \sum_{n=1}^{\infty} (-(n+\lambda) \cdot a_n(\alpha))^{1/2} \cdot P_n^{(\lambda)}(\cos\theta_1).$$

In view of the expansion (8), the kernel $\delta_\alpha(\cos\theta_1)$ is of the type $\delta_\alpha(|x-y|) \sim |x-y|^{(1+\alpha-d)/2}$. Now choose the integer $l \geq 1$ such that $2l-1-d \leq (1+\alpha-d)/2 < 2l+1-d$. Consider again the convolution equation

$$\delta_\alpha^{-1} * \delta_\alpha = h_l.$$

Proceeding as in the proof of Theorem 1, we find that the inverse δ_α^{-1} has a representation of the following form:

$$\delta_\alpha^{-1} = \sum_{m=1}^{s} d_m \cdot k_{2l+m+1-2d-\beta} + R_s \qquad (d_1 \neq 0).$$

Here $\beta = (1+\alpha-d)/2$, and if we choose s large enough, $\Delta^l R_s$ will be bounded on S. From now on the proof is the same as in step 3 of the proof of Theorem 1, yielding

$$\left\| \sum_{j=1}^{N} \delta_\alpha(|x-x_j|) \right\|_1 \gg N^{(d-\alpha-1)/2(d-1)}.$$

From this and the Cauchy-Schwarz inequality, the inequality

$$E_\alpha(\omega_N) \le -c(\alpha, d) \cdot N^{1-\alpha/(d-1)}$$

follows immediately.

The case $1-d < \alpha < 3-d$. In the case of an unbounded kernel k_α, we have to proceed in a different way. Together with the kernel $k_\alpha(\theta) = (2-2\cos\theta_1)^{\alpha/2}$ consider the more general kernel

$$d_r(\cos\theta_1) = \left(r + \frac{1}{r} - 2\cos\theta_1\right)^{\alpha/2} \qquad (0 < r \le 1).$$

Let m_r be the mean value of $d_r(\cos\theta_1)$ over S^{d-1}, and let $d_r(|x-y|)$ be the distance function generated by $d_r(\cos\theta_1)$ on S^{d-1}. We have

$$\begin{aligned} E_\alpha(\omega_N) &= \sum_{j\ne k}(d_1(|x_j-x_k|) - m_1) \\ &= \sum_{j,k}(d_r(|x_j-x_k|) - m_r) - N\cdot d_r(0) \\ &\quad - N^2\cdot(m_1-m_r) + N\cdot m_1 \\ &\quad + \sum_{j\ne k}(d_1(|x_j-x_k|) - d_r(|x_j-x_k|)). \end{aligned}$$

First of all note that $d_r(|x-y|) \le d_1(|x-y|)$, and that

$$\sum_{j,k}(d_r(|x_j-x_k|) - m_r) \ge 0,$$

as all the coefficients of $d_r(\cos\theta_1) - m_r$ in the ultraspherical expansion are nonnegative. (This may be proved in the same way as Hilfssatz 6 in **[4]**, using the Rodrigues formula for ultraspherical polynomials.)

Hence

$$E_\alpha(\omega_N) \ge -N\cdot d_r(0) - N^2\cdot(m_1-m_r). \tag{14}$$

Now choose $r = 1 - N^{-1/(d-1)}$. We have $d_r(0) \ll N^{-\alpha/(d-1)}$ and $m_1 - m_r \ll N^{-1}\cdot N^{-\alpha/(d-1)}$. Inserting these estimates in (14) yields the desired result.

The case $3-d \leq \alpha \leq 0, d \geq 4$. Unfortunately, the preceding method does not seem to give the best result in the case $3-d < \alpha < 0$. Putting $r = 1-\varepsilon$, we obtain $d_r(0) \ll \varepsilon^\alpha$ and $m_1 - m_r \ll \varepsilon^2$ (instead of $\varepsilon^{d-1+\alpha}$ as above). Choosing $\varepsilon = N^{1/(2-\alpha)}$, assertion (c) of Theorem 2 follows.

In the logarithmic case, the same procedure yields

$$E_0(\omega_N) \leq \frac{N}{2}\log N + O(N),$$

which is best possible in dimension 3 (see **[9]**), but probably not in higher dimensions. This finishes our proof of Theorem 2.

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