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ISOMORPHISMS AMONG MONODROMY GROUPS AND APPLICATIONS TO LATTICES IN PU(1, 2)

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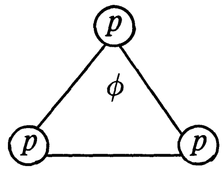
The discreteness of some monodromy groups in $\mathrm{PU}(1, 2)$ is proved. G. D. Mostow's conjecture on a necessary and sufficient condition for the discreteness of monodromy subgroups of $\mathrm{PU}(1, 2)$ is established. Some isomorphisms and inclusion relations among the monodromy groups are given.

1. Introduction. In **[DM]**, Deligne and Mostow define certain monodromy subgroups of $\mathrm{PU}(1, n)$ which are closely related to the groups Mostow studied in his earlier work **[M-1]**. The connection between these two is made clear in **[M-2]** and **[M-3]**. Each of the papers investigates the discreteness of the groups. Thereafter, in case $n > 3$, Mostow gives a necessary and sufficient condition for the groups to be discrete in $\mathrm{PU}(1, n)$ **[M-4]**. He conjectured that his condition would also hold in dimensions two and three (apart from stated exceptions). This paper considers the monodromy subgroups of $\mathrm{PU}(1, 2)$. The discreteness of some monodromy groups is proved in §3. Mostow's conjecture is verified in §4. The volumes of the fundamental domains for the groups are computed in §5 and are used to find the indices for the inclusion relations among the monodromy groups given in §6. The isomorphisms given throughout this paper were discovered using computer investigations of the fundamental domains as a guide. The proofs however are completely independent of the computer work. The following brief summary of **[DM]**, **[M-1]**, **[M-2]**, and **[M-3]** introduces notation and results needed in the remaining sections.

2. Preliminaries. Mostow's work on discrete groups generated by complex reflections. The following results are contained in **[M-1]** which arose out of Mostow's exploration of the limits of the validity in the case of $\mathbf{R}$-rank 1 groups of Margulis' Theorem, *Irreducible lattices in semisimple Lie groups of* $\mathbf{R}$*-rank greater than* 1 *are arithmetic.* Motivated by Makarov's (for $n = 3$) and Vinberg's (for $n \leq 5$) construction of nonarithmetic lattices in $\mathrm{SO}(n, 1)$ using reflections in faces of geodesic polyhedra in real hyperbolic n-space Rh^n, Mostow considered subgroups in the isometry group $\mathrm{PU}(n, 1)$ of complex hyperbolic space Ch^n generated by complex reflections. He defined a family of subgroups $\Gamma_{p,t}$ for $p = 3, 4, 5$ and $|t| < 3(\frac{1}{2} - \frac{1}{p})$ as follows.

Let V be a complex 3-dimensional vector space with basis e_1, e_2, e_3. An hermitian form H_ϕ on V corresponding to the Coxeter diagram:

(2.1)



where p is a positive integer and $\phi^3 = e^{\pi i t}$

is given by

$$(2.2)\qquad \langle e_1, e_2\rangle = \langle e_2, e_3\rangle = \langle e_3, e_1\rangle = -\alpha\phi \quad \text{where } \alpha = \frac{1}{2\sin(\frac{\pi}{p})}.$$

Set

$$(2.3)\qquad \eta = e^{\frac{\pi i}{p}}.$$

Then each R_i, $i = 1, 2, 3$ defined by

$$(2.4)\qquad R_i(x) = x + (\eta^2 - 1)\langle x, e_i\rangle e_i \quad \text{for } x \in V$$

is a **C**-reflection since it is a linear map of order p fixing each point of $e_i^\perp = \{x \in V;\ \langle x, e_i\rangle = 0\}$. We call $e_i^\perp$ the *mirror* of R_i and e_i the *mirror normal* of R_i. The group corresponding to the Coxeter diagram is $\Gamma_{p,t} = \langle\{R_i\}_{i=1}^3\rangle$, the group generated by the complex reflections. The group $\Gamma_{p,t}$ preserves the hermitian form H_ϕ. If we restrict our attention to $p > 2$ and $\arg(\phi^3) = t < 3(\frac{\pi}{2} - \frac{\pi}{p})$ it turns out that the signature of H_ϕ is (two $+$, one $-$) and hence $\Gamma_{p,t}$ is embedded in $\mathbf{U}(2, 1)$.

It is not at all clear which values of (p, t) result in a $\Gamma_{p,t}$ which is discrete in $\mathbf{U}(2, 1)$ however. This was the main problem Mostow faced. Computer exploration of the fundamental domains for these groups was essential in deciding which $\Gamma_{p,t}$ are discrete. Using the computer investigations to get a clearer picture of what was going on, he formulated and proved theorems with some technical details that can be found in [**M-1**, §6]. His strategy for proving discreteness of $\Gamma_{p,t}$ is based on: *if a smooth polyhedron F in a Riemannian manifold and a finite subset* Δ *of the isometry group together satisfy certain conditions on the codimension one and two faces of F and a related family of polyhedra, then the group* Γ *generated by* Δ *is a discrete* *subgroup of the isometry group and F is a fundamental domain for Γ modulo* $\mathrm{Aut}_\Gamma F$. Since Δ is only a finite subset it is possible to use the computer to figure out candidates for Δ and F. Mostow used these theorems to find a sufficient condition for $\Gamma_{p,t}$ to be discrete by solving for the set of (p, t) which give a polyhedron satisfying the codimension one and two conditions. He was able to prove that $\Gamma_{p,t}$ is a lattice for 17 values of (p, t); seven of these are nonarithmetic and are listed in §7. The codimension one and two conditions also give relations among the generators that result in a presentation for $\Gamma_{p,t}$, which was used later to show its relation with the monodromy groups defined by Deligne and Mostow.

The work of Deligne and Mostow. Define a function of $N-3$ variables $z_1, \dots, z_{N-3}$ by

$$f_{ij}(z_1, \dots, z_{N-3}) = \int_{z_i}^{z_j} \left(\prod_{k=1}^{N-3} (z - z_k)^{-\mu_k}\right) z^{-\mu_{N-2}}(z-1)^{-\mu_{N-1}}\, dz$$

where $\{z_1, \dots, z_{N-3}\}$ and $\{\mu_1, \dots, \mu_{N-1}\}$ are complex numbers and the path of integration is selected in $P - \{z_1, \dots, z_{N-3}, 0, 1, \infty\}$, $P = \mathbf{C} \cup \{\infty\}$, the complex projective line. Let μ_N be the order of the pole of the integrand at ∞. Then summing over all the μ's, one has $\sum_{k=1}^N \mu_k = 2$. For this reason we define a *disc N-tuple* to be an N-tuple of real numbers $\mu = \{\mu_1, \dots, \mu_N\}$ satisfying $0 < \mu_k < 1$ for $k = 1, \dots, N$ and $\sum_{k=1}^N \mu_k = 2$ and restrict our attention to such μ.

The f_{ij} are multivalued hypergeometric functions of $N-3$ variables studied by Schwarz in case $N = 4$ and Picard in case $N = 5$. Deligne and Mostow studied the monodromy of these hypergeometric functions via flat vector bundles and cohomology with local coefficients with the following results **[DM]**.

Let $S = \{1, \dots, N\}$ and P^S be the set of functions from S to P. Let M be the subset of injective maps from S to P, i.e. $M = \{(z_1, \dots, z_N) \in P^N;\ z_i \neq z_j \text{ for } i \neq j\}$. Then PGL_2 acts on P^S by Möbius transformations in each coordinate and we set $Q = \mathrm{PGL}_2 \backslash M$. Note that $Q = \{(z_1, \dots, z_{N-3});\ z_i \in P\ z_i \neq 0, 1, \infty \text{ and } z_i \neq z_j \text{ for } i \neq j\}$. *Remark*: For the sake of simplicity, we first defined the multivalued function f_{ij} as a function of the $N-3$ variables $z_1, \dots, z_{N-3}$. However in **[DM]** they are studied on the space Q, thereby permitting a symmetric role for each of $z_1, \dots, z_N$. Since PGL_2 sends any three distinct points of P to any other, we can choose $(z_{N-2}, z_{N-1}, z_N) = (0, 1, \infty)$.

There are $N-2$ linearly independent integrals among the f_{ij} and by taking them as projective coordinates of a point in the projective space P^{N-3}, one gets a multivalued map

$$\omega_\mu : Q \longrightarrow P^{N-3}.$$

From the map ω_μ we obtain a well-defined map from the simply connected cover of Q to P^{N-3} which is $\pi_1(Q)$-equivariant. The action of $\pi_1(Q)$ on P^{N-3} is called the *monodromy action.* We define Γ_μ as the image of $\pi_1(Q)$ in $\mathrm{PGL}(N-2)$. If μ is a disc N-tuple, Γ_μ preserves an hermitian form of signature $(1, N-3)$. A main result in **[DM]** is:

THEOREM (*Deligne-Mostow*). *If $\mu = (\mu_1, \dots, \mu_N)$ is a disc N-tuple which satisfies the condition*

(INT) *For all $1 \le i \ne j \le N$, such that $\mu_i + \mu_j < 1$,*
$$(1-\mu_i-\mu_j)^{-1} \in \mathbf{Z}.$$

Then Γ_μ is a lattice in the projective unitary group $\mathrm{PU}(1, N-3)$.

In their proof they consider the following partial compactification of Q. A point $y \in P^S$ is called *μ-stable* if and only if for all $z \in P$,

$$\sum_{y(s)=z} \mu_s < 1.$$

The set of all μ-stable points is denoted M_{st}. The partial compactification, Q_{st}, is the quotient space $\mathrm{PGL}_2 \backslash M_{st}$. Let $\widetilde{Q} \to Q$ be the cover corresponding to the kernel of the monodromy action and $\widetilde{Q}_{st}$ the Fox completion of $\widetilde{Q} \to Q$ over Q_{st}. Deligne and Mostow extend the map ω_μ to a map $\tilde{\omega}_\mu$ from $\widetilde{Q}_{st}$ to B^+, a complex ball in P^{N-3}. They prove that $\tilde{\omega}_\mu : \widetilde{Q}_{st} \longrightarrow B^+$ is a topological covering map and as the ball is simply connected, an isomorphism. The homeomorphism $\tilde{\omega}_\mu$ transforms the fibers of the projection $\widetilde{Q}_{st} \longrightarrow Q_{st}$ into the orbits of Γ_μ and so we have $B^+/\Gamma_\mu \simeq Q_{st}$. Hence the task of computing the volume of the fundamental domain for Γ_μ acting on the ball is equivalent to computing the volume of Q_{st}. We make use of this fact in §5.

Although the condition INT is sufficient to prove the discreteness of the monodromy groups Γ_μ, one would like a necessary condition for discreteness. Towards that end, Mostow **[M-2]** weakened the integrality condition to a condition Σ INT: there is a subset $S_1 \subset \{1, \dots, N\}$

such that $\mu_i = \mu_j$ for all $i, j \in S_1$ and for all $i \neq j$ such that $\mu_i + \mu_j < 1$,

$$(1 - \mu_i - \mu_j)^{-1} \in \begin{cases} \frac{1}{2}\mathbf{Z} & \text{if } i, j \in S_1, \\ \mathbf{Z} & \text{otherwise.} \end{cases}$$

He proved a theorem that he states in **[M-2**, §2] as follows. Let $S = S_1 \cup S_2$ with S_1 as above and $S_2 = S \setminus S_1$. Let Σ denote the permutation group of S_1. Then Σ operates on P^S by permutation of factors and hence on the subset M and on Q.

Let Q' denote the subset of Q on which Σ operates freely; Q' is an open dense submanifold of Q. Let 0 be a base point in Q', let $\overline{0}$ denote the orbit $\Sigma 0$. The monodromy homomorphism can be extended to $\pi_1(Q'/\Sigma, \overline{0})$ (the exact homotopy sequence of the fibration

$$\begin{array}{ccc} \Sigma & \longrightarrow & Q' \\ & & \downarrow \\ & & Q'/\Sigma \end{array}$$

gives the exact sequence

$$1 \longrightarrow \pi_1(Q') \longrightarrow \pi_1(Q'/\Sigma) \longrightarrow \Sigma \longrightarrow 1,$$

and we consider $\pi_1(Q')$ as a subgroup of $\pi_1(Q'/\Sigma)$) and for the image of this monodromy homomorphism we write $\Gamma_{\mu\Sigma}$.

THEOREM (*Mostow*). *Assume* $\mu = (\mu_s)_{s \in S}$ *satisfies condition* ΣINT. *Then* $\Gamma_{\mu\Sigma}$ *is a lattice in* $\mathrm{PU}(1, N-3)$.

In fact, Γ_μ is a lattice, since the exact sequence

$$1 \longrightarrow \Gamma_\mu \longrightarrow \Gamma_{\mu\Sigma} \longrightarrow \Sigma \longrightarrow 1$$

implies $\Gamma_{\mu\Sigma}$ is a lattice whenever Γ_μ is. The complete list of all μ satisfying the half integral condition ΣINT but not INT is given in §7. This list includes some μ not found in **[M-2]**.

Mostow was led to an investigation of $\Gamma_{\mu\Sigma}$ by the similarities between Γ_μ and $\Gamma_{p,t}$ in the case $N = 5$. Although these lattices are different, it turns out that the $\Gamma_{p,t}$ are conjugate in PU(1,2) to a subgroup of $\Gamma_{\mu\Sigma}$ of index at most three (the relation is made explicit in the next section). For this reason, we can consider the $\Gamma_{p,t}$ as included in the list of μ satisfying ΣINT.

Next Mostow gives a necessary condition for discreteness when he proves in **[M-4]** the converse to the previous theorem in case $N > 6$.

THEOREM (*Mostow*). *Assume* $N > 5$ *and* μ *is a disc* N*-tuple. If* Γ_μ *is discrete in* $\mathrm{PU}(1, N-3)$, *then* μ *satisfies condition* $\Sigma\mathrm{INT}$ *except for*

$$\mu = \left(\frac{1}{12}, \frac{3}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}\right).$$

In this paper we deal with the Γ_μ subgroups of $\mathrm{PU}(1, 2)$.

The relation between Γ_μ and $\Gamma_{p,t}$ via braid groups. In the case $N = 5$, Mostow shows in **[M-3]** how $\Gamma_{p,t}$ and Γ_μ are related via the braid group. We use this connection extensively and therefore reproduce part of that discussion here in the current notation. We begin with the definition of a braid group.

Let L_1 and L_2 be two parallel lines in the plane $y = 0$ of (x, y, z) space, L_1 at $z = r_1$ and L_2 at $z = r_2$. Let $P_i = (i, 0, r_1)$, $Q_i = (i, 0, r_2)$, $i = 1, \dots, n$.

A *braided N-path* is a set of N paths $c_i(t)$ in $\mathbf{R}^3$ $(i = 1, \dots, N)$ satisfying

(1) $c_i(t) = (x_i(t), y_i(t), t)$, $r_1 \le t \le r_2$, $c_i(r_1) = P_i$, $c_i(r_2) \in \{Q_1, \dots, Q_N\}$.

(2) The paths do not intersect.

Two braided N-paths are regarded as *equivalent* if and only if it is possible to deform the one configuration into the other respecting conditions (1) and (2) throughout the deformation; note that one does permit r_1, r_2 to vary so long as $r_1 < r_2$ is respected. We define a braid to be an equivalence class of braided N-paths. The fact that r_1 and r_2 can vary allows one to define an associative multiplication of braids. The braid in which no paths intertwine is the identity braid. It is easy to see that an inverse of a braid is defined by its mirror image. Thus the set of braids forms a group under multiplication. We call this the *braid group on N-strings* in $\mathbf{R}^3$ and denote it by $B_N(\mathbf{R}^2)$.

Each braid b in $B_N(\mathbf{R}^2)$ effects a permutation $\overline{b}$ of $\{1, \dots, N\}$. The map $\pi\colon b \to \overline{b}$ is a homomorphism of $B_N(\mathbf{R}^2)$ onto Σ_N, the permutation group on N letters. Let

$$C_N = \operatorname{Ker} \pi.$$

C_N is called the *colored braid group* or *pure braid group.*

A braided N-path can be regarded as a deformation of the N distinct points in $\mathbf{R}^2$ and it is a topological fact that this deformation can be extended to an isotopy of $\mathbf{R}^2$. In fact, the N points can be taken anywhere in $\mathbf{R}^2$. We can also consider N-string braids whose endpoints lie anywhere on the 2-sphere $S^2 = \mathbf{R}^2 \cup \infty$. In that case the deformations can take place in $S^2 \times \mathbf{R}$ rather than $\mathbf{R}^2 \times \mathbf{R}$. We distinguish this braid group from the previous one by denoting them $B_N(S^2)$ and $B_N(\mathbf{R}^2)$ respectively.

Recall the M was defined as the set of all injective maps from $S = \{1, \dots, N\}$ to P. Fix a base point of M as $0 = (1, 2, 3, \dots, N)$. Then $\pi_1(M, 0)$ consists of N paths $c_i(t)$ in P, $0 \le t \le 1$ with $c_i(0) = c_i(1) = i$, $1 \le i \le n+3$ and such that $(c_i(t), t)$ in $P \times \mathbf{R}$ do not intersect. That is, $\pi_1(M, 0)$ is precisely the colored braid group $C_N(P)$ on N strings in P.

In order to describe the relation between $\Gamma_{p,t}$ and $\Gamma_{\mu\Sigma}$ Mostow chooses a set of generators for the pure braid group on 5 strings in P that is stable under the permutation group of the subset S_1 of punctures $S = \{z_1, z_2, z_3, z_4, z_5\}$.

Assume $S_1 = \{z_1, z_2, z_3\}$ and assume $\mu_1 = \mu_2 = \mu_3$.

Identify the projective line P with S^2, the 2-sphere with its standard metric. Choose z_1, z_2, z_3 equally spaced on the equator of S^2 with z_4 and z_5 at the North and South poles respectively. Denote by $(i\,j)$ for any $i \ne j$ with $i, j \in \{1, 2, 3, 4, 5\}$ the pure braid that moves z_i along the shortest path to a point near z_j, then makes a small circuit in the positive sense around z_j, and then returns to its original position. For $i, j \in \{1, 2, 3\}$ let $i \diagup j$ denote the braid that interchanges i and j via a half-turn isotopy in the positive sense that leaves each point fixed outside of a small neighborhood of the shortest arc joining i to j.

Let J denote the cyclic permutation $1 \to 2 \to 3 \to 1$ of $\{1, 2, 3, 4, 5\}$. We denote also by J its realization as a rotation by angle $2\pi/3$ in the positive sense around the North pole of P, and its realization as a braid in $B_5(P)$. Let θ denote the monodromy homomorphism and set

$$
\begin{aligned}
(2.5)\quad & A_i = \theta((4\,i)), \qquad A_i' = \theta((5\,i)), \\
& B_i = \theta((i-1\;\; i+1)), \qquad \text{(cyclicly permuting } i = 1, 2, 3) \\
& R_i = \theta(i-1 \diagup i+1), \\
& B_i' = \theta((4\,5)_i),
\end{aligned}
$$

where the circuit $(4\,5)_i$ is chosen so as to cross the equator only on the short arc $(i-1, i+1)$. We shall use the following identities coming from the braid group:

$$
\begin{aligned}
(2.6)\qquad & J^{-1} R_i R_{i+1} = A_i^{-1}, \qquad J R_{i+1} R_i = A_{i+1}'^{-1}, \\
& J R_i = R_{i+1} J, \qquad \text{(cyclicly permuting } i = 1, 2, 3).
\end{aligned}
$$

The product of the pure braids (2 3)(4 3)(3 1)(4 2)(4 1)(1 2) is in the center of the colored braid group $C_4(\mathbf{R}^2)$ on 4-strings in $\mathbf{R}^2$, and therefore its image in Γ_μ is central in Γ_μ, and therefore central in PU(1,2) since Γ_μ is of finite covolume in $\mathrm{PU}(1, 2)$, by a well known result of A. Selberg. Inasmuch as $\mathrm{PU}(1, 2)$ has only the identity element in its center, we get

$$B_1 A_3 A_2 B_2 A_1 B_3 = 1.$$

The group Γ_μ is generated by any five of $\{A_1, A_2, A_3, B_1, B_2, B_3\}$. Additional identities coming from the braid group are:

$$\begin{aligned} A_i' &= A_{i-1} A_{i+1} B_i, \\ B_i' &= A_{i-1}^{-1} A_i^{-1} A_{i+1}^{-1}, \\ B_i &= R_i^2, \\ A_i B_i &= B_i A_i, \\ A_i A_j' &= A_j' A_i \quad \text{for } j \neq i, \\ B_i B_j' &= B_j' B_i \quad \text{for } j \neq i. \end{aligned} \tag{2.7}$$

For any i, j with $i \neq j$, set

$$k_{ij} = (1 - \mu_i - \mu_j)^{-1}. \tag{2.8}$$

We assume that μ *satisfies condition* $\Sigma\,\mathrm{INT}$ *for* S_1. Then k_{ij} is an integer except when $i, j \in \{1, 2, 3\}$. For any $i, j \in \{1, 2, 3\}$ we set

$$k = \begin{cases} k_{ij} & \text{if } k_{ij} \in \mathbf{Z}, \\ 2k_{ij} & \text{otherwise}, \end{cases}$$
$$k_4 = k_{4i}, \quad k_5 = k_{5i} \qquad (i, j = 1, 2, 3).$$

Then Γ_μ has the presentation

(2.9) Generators: $A_1, A_2, A_3, B_1, B_2, B_3$

Relations: $A_i B_i = B_i A_i, \qquad B_1 A_3 A_2 B_2 A_1 B_3 = 1,$

$$A_i^{k_4} = 1, \qquad B_i^k = 1,$$

$$(A_{i-1} A_{i+1} B_i)^{k_5} = A_i'^{k_5} = 1,$$

$$(A_1^{-1} A_2^{-1} A_3^{-1})^{k_{45}} = B_i'^{k_{45}} = 1.$$

The group $\Gamma_{\mu\Sigma}$ has the additional generators R_1, R_2, R_3. Set $\Gamma_\mu^* = \langle R_1, R_2, R_3 \rangle$, the subgroup of $\Gamma_{\mu\Sigma}$ generated by R_1, R_2, R_3. One can derive a presentation for Γ_μ^* which coincides with the presentation for $\Gamma_{p,t}$ given in **[M-1]** if one takes (p, t) and μ related by

$$\mu_1 = \mu_2 = \mu_3 = \frac{1}{2} - \frac{1}{p}, \tag{2.10}$$

$$\mu_4 = \frac{1}{4} + \frac{3}{2p} - \frac{t}{2}, \qquad \mu_5 = \frac{1}{4} + \frac{3}{2p} + \frac{t}{2},$$

that is,

$$p = \left(\frac{1}{2} - \mu_1\right)^{-1}, \qquad t = \mu_5 - \mu_4.$$

By the strong rigidity theorem for $\mathrm{PU}(1, n)$, $n > 1$ Mostow concludes:

THEOREM (*Mostow*). *The lattices* $\Gamma_{p,t}$ *are conjugate in* $\mathrm{PU}(1, 2)$ *to the subgroup* Γ_μ^* *of* $\Gamma_{\mu\Sigma}$ *with* μ *and* (p, t) *related as above, and*

$$\Gamma_{\mu\Sigma} \simeq \langle J, \Gamma_{p,t} \rangle.$$

The specific relations between Γ_μ, $\Gamma_{\mu\Sigma}$, and $\Gamma_{p,t}$ in all cases are given in §7.

3. The discreteness of some monodromy groups. Mostow proved that $\Sigma\,\mathrm{INT}$ is a necessary and sufficient condition for the discreteness of $\Gamma_\mu \subset \mathrm{PU}(1, n)$ for all $n \geq 3$ except for $\mu = (\frac{1}{12}, \frac{3}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12}, \frac{5}{12})$ in dimension 3. He discovered that in dimension 2 the situation is more complicated **[M-4]**. There are several disc 5-tuples μ such that Γ_μ could be proved discrete even though the μ do not satisfy $\Sigma\,\mathrm{INT}$. However, for three of the μ he could not determine if the Γ_μ were discrete. Here we give a list of the three μ with the corresponding (p, t):

$$\left(\frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{7}{30}, \frac{14}{30}\right) \longleftrightarrow \left(15, \frac{7}{30}\right)$$
$$\left(\frac{11}{24}, \frac{11}{24}, \frac{11}{24}, \frac{5}{24}, \frac{10}{24}\right) \longleftrightarrow \left(24, \frac{5}{24}\right)$$
$$\left(\frac{20}{42}, \frac{20}{42}, \frac{20}{42}, \frac{8}{42}, \frac{16}{42}\right) \longleftrightarrow \left(42, \frac{4}{21}\right).$$

All previous methods for proving the discreteness of Γ_μ or $\Gamma_{p,t}$ are insufficient. The theorem in this section settles the question of whether or not these groups are discrete.

We begin by computing the normal vectors to the mirrors of the reflections $\{A_i\}_{i=1,3}$, their inner products, and the eigenvectors of $\{B'_i\}_{i=1,3}$. Using (2.4) and the fact that $\alpha = \frac{i}{\eta-\bar{\eta}}$, the matrices of the R_i can be written in the e-basis as follows:

$$R_1 = \begin{pmatrix} \eta^2 & -\eta i\overline{\phi} & -\eta i\phi \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.1}$$

$$R_2 = \begin{pmatrix} 1 & 0 & 0 \\ -\eta i\phi & \eta^2 & -\eta i\overline{\phi} \\ 0 & 0 & 1 \end{pmatrix},$$

$$R_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\eta i\overline{\phi} & -\eta i\phi & \eta^2 \end{pmatrix}.$$

Using (2.6) we find that

$$A_1^{-1} = \begin{pmatrix} -\eta i\phi & \eta^2 & -\eta i\overline{\phi} \\ 0 & 0 & 1 \\ 0 & -\eta^3 i\overline{\phi} & -\eta^2\overline{\phi}^2 - \eta i\phi \end{pmatrix}, \tag{3.2}$$

$$A_2^{-1} = \begin{pmatrix} -\eta^2\overline{\phi}^2 - \eta i\phi & 0 & -\eta^3 i\overline{\phi} \\ -\eta i\overline{\phi} & -\eta i\phi & \eta^2 \\ 1 & 0 & 0 \end{pmatrix},$$

$$A_3^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ -\eta^3 i\overline{\phi} & -\eta^2\overline{\phi}^2 - \eta i\phi & 0 \\ \eta^2 & -\eta i\overline{\phi} & -\eta i\phi \end{pmatrix}.$$

The characteristic polynomial of A_1^{-1} is:

$$\det(A_1^{-1} - \lambda I) = -(\lambda + \eta i\phi)(\lambda + \eta i\phi)(\lambda + \eta^2\overline{\phi}^2).$$

So we take the third column of:

$$A_1^{-1} - (-\eta i\phi)I = \begin{pmatrix} 0 & \eta^2 & -\eta i\overline{\phi} \\ 0 & \eta i\phi & 1 \\ 0 & -\eta^3 i\overline{\phi} & -\eta^2\overline{\phi}^2 \end{pmatrix}$$

as the mirror normal for A_1. Similar computations show that the mirror normals for the $\{A_i\}_{i=1,3}$ are:

$$(3.3)\quad a_1 = \begin{pmatrix} -\eta i\overline{\phi} \\ 1 \\ -\eta^2\overline{\phi}^2 \end{pmatrix}, \qquad a_2 = \begin{pmatrix} -\eta^2\overline{\phi}^2 \\ -\eta i\overline{\phi} \\ 1 \end{pmatrix}, \qquad a_3 = \begin{pmatrix} 1 \\ -\eta^2\overline{\phi}^2 \\ -\eta i\overline{\phi} \end{pmatrix}.$$

From (2.1) and (2.2) the matrix of H_ϕ in the e_i base is:

$$(3.4)\qquad H_\phi = \begin{pmatrix} 1 & -\alpha\phi & -\alpha\overline{\phi} \\ -\alpha\overline{\phi} & 1 & -\alpha\phi \\ -\alpha\phi & -\alpha\overline{\phi} & 1 \end{pmatrix}.$$

Note that since the $\{a_i\}_{i=1,3}$ are related by a cyclic permutation of entries we have

$$\langle a_1, a_2\rangle = \langle a_2, a_3\rangle = \langle a_3, a_1\rangle, \text{ and}$$
$$\langle a_1, a_1\rangle = \langle a_2, a_2\rangle = \langle a_3, a_3\rangle.$$

We compute

$$\begin{aligned} &\langle a_1, a_2\rangle \\ &= a_1^t H_\phi \overline{a_2} = (-\eta i\overline{\phi}, 1, -\eta^2\overline{\phi}^2) \begin{pmatrix} 1 & -\alpha\phi & -\alpha\overline{\phi} \\ -\alpha\overline{\phi} & 1 & -\alpha\phi \\ -\alpha\phi & -\alpha\overline{\phi} & 1 \end{pmatrix} \begin{pmatrix} -\overline{\eta}^2\phi^2 \\ \overline{\eta} i\phi \\ 1 \end{pmatrix} \\ &= \alpha\phi(\overline{\eta}^2 - 3) + 2\alpha\eta i\overline{\phi}^2 + 2\overline{\eta} i\phi - \eta^2\overline{\phi}^2. \end{aligned}$$

and

$$\langle a_1, a_1\rangle = 1 + \frac{\eta^2 i\overline{\phi}^3 + \overline{\eta}^2 i\phi^3}{\eta - \overline{\eta}}.$$

Hence

$$(3.5)\qquad \begin{aligned} &\frac{\langle a_1, a_2\rangle}{(\langle a_1, a_1\rangle\langle a_2, a_2\rangle)^{\frac{1}{2}}} \\ &\qquad = \frac{\alpha\phi(\overline{\eta}^2 - 3) + 2\alpha\eta i\overline{\phi}^2 + 2\overline{\eta} i\phi - \eta^2\overline{\phi}^2}{\left|1 + \left(\frac{\eta^2 i\overline{\phi}^3 + \overline{\eta}^2 i\phi^3}{\eta - \overline{\eta}}\right)\right|}. \end{aligned}$$

From (2.7) we compute

$$(3.6)\qquad B_1' = A_3^{-1}A_1^{-1}A_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \Xi & -\eta^6 & 0 \\ \Upsilon & 0 & -\eta^6 \end{pmatrix},$$

$$B_2' = A_1^{-1}A_2^{-1}A_3^{-1} = \begin{pmatrix} -\eta^6 & \Upsilon & 0 \\ 0 & 1 & 0 \\ 0 & \Xi & -\eta^6 \end{pmatrix},$$

$$B_3' = A_2^{-1}A_3^{-1}A_1^{-1} = \begin{pmatrix} -\eta^6 & 0 & \Xi \\ 0 & -\eta^6 & \Upsilon \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$\Xi = \eta^5 i\phi - \eta^4\overline{\phi}^2 - \eta^2\overline{\phi}^2 - \eta i\phi,$$
$$\Upsilon = \eta^5 i\overline{\phi} - \eta^4\phi^2 - \eta^2\phi^2 - \eta i\overline{\phi}.$$

We shall see that for the cases we are interested in, the Ξ and Υ simplify, making the computations much cleaner. The characteristic polynomial of B_1' is:

$$\det(B_1' - \lambda I) = (1-\lambda)(-\eta^6-\lambda)(-\eta^6-\lambda).$$

The eigenvectors corresponding to the eigenvalues $-\eta^6$, $-\eta^6$, 1 are

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1+\eta^6 \\ \Xi \\ \Upsilon \end{pmatrix}$$

respectively. We raise B_1' to a power n by letting P be the matrix of eigenvectors

$$P = \begin{pmatrix} 1+\eta^6 & 0 & 0 \\ \Xi & 1 & 0 \\ \Upsilon & 0 & 1 \end{pmatrix},$$

and therefore

$$B_1'^n = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-\eta^6)^n & 0 \\ 0 & 0 & (-\eta^6)^n \end{pmatrix} P^{-1}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ \Xi\left(\frac{1-(-\eta^6)^n}{1+\eta^6}\right) & (-\eta^6)^n & 0 \\ \Upsilon\left(\frac{1-(-\eta^6)^n}{1+\eta^6}\right) & 0 & (-\eta^6)^n \end{pmatrix}.$$

Similarly for B_2' and B_3' we get

$$B_2'^n = \begin{pmatrix} (-\eta^6)^n & \Upsilon\left(\frac{1-(-\eta^6)^n}{1+\eta^6}\right) & 0 \\ 0 & 1 & 0 \\ 0 & \Xi\left(\frac{1-(-\eta^6)^n}{1+\eta^6}\right) & (-\eta^6)^n \end{pmatrix},$$

$$B_3'^n = \begin{pmatrix} (-\eta^6)^n & 0 & \Xi\left(\frac{1-(-\eta^6)^n}{1+\eta^6}\right) \\ 0 & (-\eta^6)^n & \Upsilon\left(\frac{1-(-\eta^6)^n}{1+\eta^6}\right) \\ 0 & 0 & 1 \end{pmatrix}.$$

THEOREM 3.1. *The groups* $\Gamma_{15,\frac{7}{30}}$, $\Gamma_{24,\frac{5}{24}}$, *and* $\Gamma_{42,\frac{4}{21}}$ *are lattices in* $\mathrm{PU}(1,2)$. *More precisely,*

$$\Gamma_{15,\frac{7}{30}} \simeq \Gamma_{3,\frac{1}{30}},$$
$$\Gamma_{24,\frac{5}{24}} \simeq \Gamma_{3,\frac{1}{12}},$$
$$\Gamma_{42,\frac{4}{21}} \simeq \Gamma_{3,\frac{5}{42}}.$$

Proof. The above isomorphisms that prove the discreteness of $\Gamma_{15,\frac{7}{30}}$, $\Gamma_{24,\frac{5}{24}}$, and $\Gamma_{42,\frac{4}{21}}$ are three in a more general class of isomorphisms. Let $\Gamma_{3,t}$ with $t \in \{\frac{1}{30}, \frac{1}{12}, \frac{5}{42}, \frac{7}{30}, \frac{1}{3}\}$ denote the lattices from **[M-1]**. Then we will prove

$$\Gamma_{3,t} \simeq \Gamma_{\frac{12}{1-6t}, \frac{1}{4}-\frac{t}{2}}.$$

We consider the action of the groups on the image of $V^- = \{v \in V;\ \langle v, v\rangle < 0\}$ in the complex projective space. Since the signature of the hermitian form is (1 negative, n positive), the image of V^- is a complex 2-dimensional ball. We find reflections $\{C_i\}_{i=1,3}$ of order 3 in $\Gamma_{p,\frac{1}{4}-\frac{t}{2}}$ whose mirror normals $\{c_i\}_{i=1,3}$ satisfy

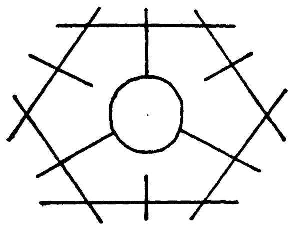
$$\langle c_1, c_2\rangle = \langle c_2, c_3\rangle = \langle c_3, c_1\rangle = -\alpha\phi,$$

where $-\alpha\phi$ is the inner product of the mirror normals, $\{e_i\}$, corresponding to the generators of $\Gamma_{3,t}$ and hence

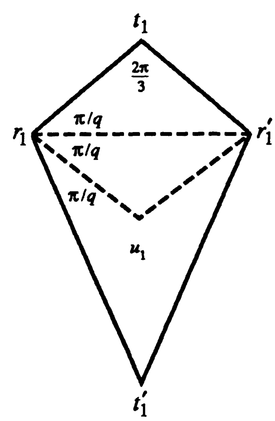
$$-\alpha\phi = \frac{-e^{\frac{\pi i t}{3}}}{2\sin\frac{\pi}{3}} = -\frac{\sqrt{3}}{3}e^{\frac{\pi i t}{3}}.$$

Since the action of $\mathrm{PU}(1,2)$ on the ball is transitive, the isometry of the ball taking the system of mirror normals $\{e_i\}_{i=1,3}$ for the generators $\{R_i\}_{i=1,3}$ of $\Gamma_{3,t}$ to the system of mirror normals $\{c_i\}_{i=1,3}$ induces a monomorphism of $\Gamma_{3,t}$ to $\langle\{C_i\}_{i=1,3}\rangle$, the subgroup generated by the $\{C_i\}_{i=1,3}$. Then we show that the $\{C_i\}_{i=1,3}$ generate $\Gamma_{p,\frac{1}{4}-\frac{t}{2}}$.

We motivate the choice of the $\{C_i\}_{i=1,3}$. Each reflection on V has a fixed point set in the projective space consisting of a point (the image of the mirror normal) and a line (the image of the mirror). Note that the line and not the point lies in the negative ball since the mirror normals lie in the positive cone. Let $e_i^\perp$, $a_i^\perp$, $a_i'^\perp$, and $b_i'^\perp$ denote the lines in the ball fixed by the B_i, A_i, A_i', and B_i' respectively. The following diagram gives the configuration of these lines in the ball; all intersections of the complex lines are orthogonal.



These lines play an important role in the fundamental domain Ω defined in [M-1]. Since $\Gamma_{p,\frac{1}{4}-\frac{t}{2}}$ does *not* satisfy the necessary conditions for it to be a discrete group, the Ω is *not* a fundamental domain for $\Gamma_{p,\frac{1}{4}-\frac{t}{2}}$. However, studying Ω does give a clue to the choice of the $\{C_i\}_{i=1,3}$. Label the points defined by the diagram as follows: $t_1 = a_1^\perp \cap e_1^\perp$, $t_1' = a_1'^\perp \cap e_1^\perp$, $r_1 = b_3'^\perp \cap e_1^\perp$, and $r_1' = b_2'^\perp \cap e_1^\perp$. In the complex geodesic line $e_1^\perp$, the points r_1, r_1', t_1, and t_1' form a geodesic quadrilateral.



The reflection B_3' stabilizes $e_1^\perp$ and affects a rotation in $e_1^\perp$ around r_1 through an angle $2\pi\frac{3}{q} = \frac{6\pi}{q}$, where q is defined by

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{6}.$$

From (2.10) the $\mu = (\mu_1, \ldots, \mu_5)$ corresponding to $(p, \frac{1}{4} - \frac{t}{2})$ is given by

$$\mu = \left(\frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{1}{6} + \frac{1}{p}, 2\left(\frac{1}{6} + \frac{1}{p}\right)\right). \tag{3.7}$$

So k_4 and k_{45}, the orders of the A_i and B'_i respectively, are (refer to (2.8) and (2.9))

$$k_4 = (1 - \mu_1 - \mu_4)^{-1} = 3 \qquad \text{and} \qquad k_{45} = (1 - \mu_4 - \mu_5)^{-1} = \frac{q}{3}.$$

Now, k_{45} is the reason μ does *not* satisfy Σ INT. The Ω is too large to be a fundamental domain for $\Gamma_{p, \frac{1}{4} - \frac{t}{2}}$ and $\frac{6\pi}{q}$, the angle of rotation of B'_3, is too large. If one can choose $n \in \mathbf{Z}$ such that $3n \equiv 1 \bmod q$, then $B_3'^n$ is a rotation through an angle $\frac{2\pi}{q}$. Hence $B_3'^n t_1 = u_1$ and $B_3'^{-n} u_1 = t_1$. Since A_1 is a rotation through an angle $\frac{2\pi}{3}$ at t_1, $C_1 \equiv B_3'^n A_1 B_3'^{-n}$ is a rotation through $\frac{2\pi}{3}$ at u_1. Let $C_i = B_{i-1}'^n A_i B_{i-1}'^{-n}$ for $i = 1, 2, 3$. Adding $\{C_i\}_{i=1,3}$ to the set Δ that gave rise to Ω (see §2) should cut down the size of Ω and it turns out that these are exactly the reflections needed.

Recall that we need order 3 reflections $\{C_i\}_{i=1,3}$ whose mirror normals $\{c_i\}_{i=1,3}$ satisfy

$$\langle c_1, c_2 \rangle = \langle c_2, c_3 \rangle = \langle c_3, c_1 \rangle = -\frac{\sqrt{3}}{3} e^{\frac{\pi i}{3}}.$$

Since we are choosing $C_i = B_{i-1}'^n A_i B_{i-1}'^{-n}$ for $i = 1, 2, 3$, the mirror normals are precisely $c_i = B_{i-1}'^n a_i$ for $i = 1, 2, 3$. We begin with a lemma.

LEMMA. *Let $\{B'_i\}_{i=1,3}$ be the reflections in $\Gamma_{p, \frac{1}{4} - \frac{t}{2}}$ defined previously. Define q by $\frac{1}{p} + \frac{1}{q} = \frac{1}{6}$. For $t \in \{\frac{1}{30}, \frac{1}{12}, \frac{5}{42}, \frac{7}{30}, \frac{1}{3}\}$ we can choose $n \in \mathbf{Z}$ such that $3n \equiv 1 \bmod q$. Define $C_i = B_{i-1}'^n A_i B_{i-1}'^{-n}$ for $i = 1, 2, 3$. Then the mirror normals for the $\{C_i\}_{i=1,3}$ are*

$$c_1 = B_3'^n a_1 = \begin{pmatrix} 0 \\ \eta^2 \\ -\eta^2 \overline{\phi}^2 \end{pmatrix}, \quad c_2 = B_1'^n a_2 = \begin{pmatrix} -\eta^2 \overline{\phi}^2 \\ 0 \\ \eta^2 \end{pmatrix},$$

$$c_3 = B_2'^n a_3 = \begin{pmatrix} \eta^2 \\ -\eta^2 \overline{\phi}^2 \\ 0 \end{pmatrix},$$

where $\eta = e^{\frac{\pi i}{p}}$ and $\phi^3 = e^{\pi i(\mu_5 - \mu_4)}$.

Proof of Lemma. We have computed $B_i^{\prime n}$ previously in this section. For the specific n and q chosen above, the $B_i^{\prime n}$ can be simplified as follows.

Since $3n \equiv 1 \bmod q$,

$$\frac{3n}{q} \equiv \frac{1}{q} \bmod \mathbf{Z} \quad \text{and} \quad \frac{-3n}{q} \equiv \frac{-1}{q} \bmod \mathbf{Z}.$$

But $\frac{-1}{q} = \frac{1}{p} - \frac{1}{6}$ so we have

$$n\left(\frac{3}{p} - \frac{1}{2}\right) \equiv \frac{1}{p} - \frac{1}{6} \bmod \mathbf{Z},$$

which implies

$$e^{2\pi i(\frac{1}{p}-\frac{1}{6})} = [e^{2\pi i(\frac{3}{p}-\frac{1}{2})}]^n.$$

We write this equation as

$$-\eta^2 e^{\frac{2\pi i}{3}} = (-\eta^6)^n.$$

From this we get

$$\left[\frac{1-(-\eta^6)^n}{1+\eta^6}\right] = \frac{1+e^{\frac{2\pi i}{3}}\eta^2}{1+\left(e^{\frac{2\pi i}{3}}\eta^2\right)^3} = \frac{1}{e^{-\frac{2\pi i}{3}}\eta^4 + e^{-\frac{\pi i}{3}}\eta^2 + 1}.$$

Since (3.7) gives $\mu_5 - \mu_4 = \frac{1}{6} + \frac{1}{p}$, we have

$$\phi^3 = e^{\pi i(\mu_5 - \mu_4)} = e^{\frac{\pi i}{6}}\eta.$$

Substituting for $\overline{\phi}^3$ in Ξ gives

$$\begin{aligned} \Xi &= \eta^5 i\phi - \eta^4\overline{\phi}^2 - \eta^2\overline{\phi}^2 - \eta i\phi \\ &= \eta i\phi e^{\frac{2\pi i}{3}}(e^{-\frac{2\pi i}{3}}\eta^4 + e^{-\frac{\pi i}{3}}\eta^2 + e^{-\frac{\pi i}{3}} + e^{\frac{\pi i}{3}}). \end{aligned}$$

Combined with the above we get

$$\Xi\left(\frac{1-(-\eta^6)^n}{1+\eta^6}\right) = e^{\frac{2\pi i}{3}}\eta i\phi = -e^{-\frac{\pi i}{3}}\eta i\phi.$$

Similarly,

$$\Upsilon = -\eta i\overline{\phi}(e^{-\frac{2\pi i}{3}}\eta^4 + e^{-\frac{\pi i}{3}}\eta^2 + 1),$$

so

$$\Upsilon\left(\frac{1-(-\eta^6)^n}{1+\eta^6}\right) = -\eta i\overline{\phi}.$$

Note also that

$$\eta^3 i\overline{\phi}^3 + e^{-\frac{\pi i}{3}}\eta^2 = \eta^2.$$

Together, these equations imply

$$(3.8) \qquad B_1'^n = \begin{pmatrix} 1 & 0 & 0 \\ -e^{-\frac{\pi i}{3}}\eta i\phi & e^{-\frac{\pi i}{3}}\eta^2 & 0 \\ -\eta i\overline{\phi} & 0 & e^{-\frac{\pi i}{3}}\eta^2 \end{pmatrix}$$

and hence

$$B_1'^n a_2 = \begin{pmatrix} -\eta^2\overline{\phi}^2 \\ 0 \\ \eta^2 \end{pmatrix} = c_2.$$

From the symmetry in the $\{B_i'\}_{i=1,3}$ and the $\{a_i\}_{i=1,3}$,

$$c_1 = \begin{pmatrix} 0 \\ \eta^2 \\ -\eta^2\overline{\phi}^2 \end{pmatrix}, \qquad c_2 = \begin{pmatrix} -\eta^2\overline{\phi}^2 \\ 0 \\ \eta^2 \end{pmatrix}, \qquad c_3 = \begin{pmatrix} \eta^2 \\ -\eta^2\overline{\phi}^2 \\ 0 \end{pmatrix}.$$

We now show that

$$\frac{\langle c_1, c_2\rangle}{(\langle c_1, c_1\rangle\langle c_2, c_2\rangle)^{\frac{1}{2}}} = -\frac{\sqrt{3}}{3}e^{\frac{\pi i t}{3}}.$$

First

$$\langle c_1, c_2\rangle = -\alpha\phi - \overline{\phi}^2.$$

Next

$$\langle c_1, c_1\rangle = 2 + \alpha(\phi^3 + \overline{\phi}^3).$$

Hence

$$\begin{aligned} \frac{\langle c_1, c_2\rangle}{(\langle c_1, c_1\rangle\langle c_2, c_2\rangle)^{\frac{1}{2}}} &= \frac{-\alpha\phi - \overline{\phi}^2}{2 + \alpha(\phi^3 + \overline{\phi}^3)} \\ &= -\overline{\eta}\phi\frac{(\eta i + \eta^2\overline{\phi}^3 - \overline{\phi}^3)}{2(\eta - \overline{\eta}) + i(\phi^3 + \overline{\phi}^3)}. \end{aligned}$$

Now, since $p = \frac{12}{1-6t}$, notice that

$$-\overline{\eta}\phi = -e^{\frac{\pi i}{p}}e^{\frac{\pi i}{3}(\frac{1}{4}-\frac{t}{2})} = -e^{\frac{\pi i t}{3}}.$$

Using simple facts about 6th roots of unity and the previous equations involving $\overline{\phi}^3$, one shows that

$$\frac{\eta i + \eta^2\overline{\phi}^3 - \overline{\phi}^3}{2(\eta - \overline{\eta}) + i(\phi^3 + \overline{\phi}^3)} = \frac{\sqrt{3}}{3}.$$

The final step is to show that the $\{C_i\}_{i=1,3}$ generate the whole group. Since $\Gamma_{\frac{12}{1-6t},\frac{1}{4}-\frac{t}{2}}$ is generated by the $\{R_i\}_{i=1,3}$, we show $R_i^{-1} = JC_iC_{i-1}$ for $i = 1, 2, 3$. By symmetry we need only exhibit the case $i = 1$, and since we only need projective equality, we show $R_1JC_1C_3 = e^{\frac{2\pi i}{3}}\overline{\eta}^2 I$. From (3.2) and (3.8) we have

$$\begin{aligned} C_1 &= B_3'^n A_1 B_3'^{-n} \\ &= \begin{pmatrix} \overline{\eta}i\overline{\phi} & 0 & 0 \\ 0 & e^{-\frac{\pi i}{3}}\overline{\eta}i\overline{\phi} - \overline{\eta}^2\phi^2 & e^{\frac{2\pi i}{3}}\overline{\phi}^2 - e^{\frac{\pi i}{3}}\overline{\eta}i\phi \\ 0 & e^{\frac{\pi i}{3}}\overline{\eta}^2 & e^{\frac{\pi i}{3}}\overline{\eta}i\overline{\phi} \end{pmatrix} \end{aligned}$$

and hence

$$C_3 = \begin{pmatrix} e^{-\frac{\pi i}{3}}\overline{\eta}i\overline{\phi} - \overline{\eta}^2\phi^2 & e^{\frac{2\pi i}{3}}\overline{\phi}^2 - e^{\frac{\pi i}{3}}\overline{\eta}i\phi & 0 \\ e^{\frac{\pi i}{3}}\overline{\eta}^2 & e^{\frac{\pi i}{3}}\overline{\eta}i\overline{\phi} & 0 \\ 0 & 0 & \overline{\eta}i\overline{\phi} \end{pmatrix}.$$

Finally from this and (3.1) we get

$$R_1JC_1C_3 = \begin{pmatrix} e^{\frac{2\pi i}{3}}\overline{\eta}^2 & 0 & 0 \\ 0 & e^{\frac{2\pi i}{3}}\overline{\eta}^2 & 0 \\ 0 & 0 & e^{\frac{2\pi i}{3}}\overline{\eta}^2 \end{pmatrix}.$$

This completes the proof of Theorem 3.1. Notice that in addition to the three isomorphisms stated in the theorem we have also included in the proof

$$\Gamma_{-12,\frac{1}{12}} \simeq \Gamma_{3,\frac{1}{3}} \quad \text{and} \quad \Gamma_{-30,\frac{4}{30}} \simeq \Gamma_{3,\frac{7}{30}}.$$

All five of these isomorphisms play an important role in the next section.

4. Mostow's conjecture on the discreteness of monodromy groups in PU(1, 2).

Mostow's Conjecture. *Let μ be a disc 5-tuple. Then Γ_μ is discrete in* PU(1, 2) *if and only if μ satisfies* ΣINT *or Γ_μ is commensurable with Γ_ν where ν is a disc 5-tuple satisfying* ΣINT.

Mostow found that any μ not satisfying ΣINT with Γ_μ discrete **[M-4]** is on the following list of nine (the corresponding (p, t) is given whenever $\mu_1 = \mu_2 = \mu_3$):

$$\left(\frac{1}{12}, \frac{3}{12}, \frac{5}{12}, \frac{5}{12}, \frac{10}{12}\right)$$
$$\left(\frac{1}{10}, \frac{1}{10}, \frac{4}{10}, \frac{7}{10}, \frac{7}{10}\right)$$
$$\left(\frac{3}{14}, \frac{3}{14}, \frac{4}{14}, \frac{9}{14}, \frac{9}{14}\right)$$
$$\left(\frac{4}{18}, \frac{5}{18}, \frac{5}{18}, \frac{11}{18}, \frac{11}{18}\right)$$
$$\left(\frac{7}{12}, \frac{7}{12}, \frac{7}{12}, \frac{1}{12}, \frac{2}{12}\right) \longleftrightarrow \left(-12, \frac{1}{12}\right)$$
$$\left(\frac{8}{15}, \frac{8}{15}, \frac{8}{15}, \frac{2}{15}, \frac{4}{15}\right) \longleftrightarrow \left(-30, \frac{4}{30}\right)$$
$$\left(\frac{13}{30}, \frac{13}{30}, \frac{13}{30}, \frac{7}{30}, \frac{14}{30}\right) \longleftrightarrow \left(15, \frac{7}{30}\right)$$
$$\left(\frac{11}{24}, \frac{11}{24}, \frac{11}{24}, \frac{5}{24}, \frac{10}{24}\right) \longleftrightarrow \left(24, \frac{5}{24}\right)$$
$$\left(\frac{20}{42}, \frac{20}{42}, \frac{20}{42}, \frac{8}{42}, \frac{16}{42}\right) \longleftrightarrow \left(42, \frac{4}{21}\right).$$

Theorem 3.1 proves that five of the nine have Γ_μ commensurable with Γ_ν, ν satisfying ΣINT. For the four remaining μ in the list we prove the following theorem.

THEOREM 4.1. *There exist monomorphisms*

$$\Gamma_{3,0} \hookrightarrow \Gamma_{(\frac{5}{12}, \frac{5}{12}, \frac{3}{12}, \frac{1}{12}, \frac{10}{12})}$$
$$\Gamma_{5, \frac{7}{10}} \hookrightarrow \Gamma_{(\frac{1}{10}, \frac{1}{10}, \frac{4}{10}, \frac{7}{10}, \frac{7}{10})},$$

and the following isomorphisms

$$\Gamma_{7, \frac{3}{14}} \longleftrightarrow \Gamma_{(\frac{3}{14}, \frac{3}{14}, \frac{4}{14}, \frac{9}{14}, \frac{9}{14})}$$
$$\Gamma_{9, \frac{1}{18}} \longleftrightarrow \Gamma_{(\frac{5}{18}, \frac{5}{18}, \frac{4}{18}, \frac{11}{18}, \frac{11}{18})}.$$

Proof. The monomorphisms are explicitly constructed as before. In this case we map the generators $\{R_i\}_{i=1,3}$ of $\Gamma_{3,0}$ and $\Gamma_{5,\frac{7}{10}}$ to reflections in the corresponding Γ_μ. Hence we must find mirror normals $\{c_i\}_{i=1,3}$ that satisfy

$$\langle c_1, c_2\rangle = \langle c_2, c_3\rangle = \langle c_3, c_1\rangle = -\alpha\phi = \frac{-1}{2\sin\frac{\pi}{3}} = \frac{-1}{\sqrt{3}}$$

in the case of $\Gamma_{3,0}$, and

$$\langle c_1, c_2\rangle = \langle c_2, c_3\rangle = \langle c_3, c_1\rangle = -\alpha\phi = \frac{-e^{\frac{7\pi i}{30}}}{2\sin\frac{\pi}{5}}$$

in the case of $\Gamma_{5,\frac{7}{30}}$. The computations are complicated by the fact that no three of the μ_i are equal in either case. We begin by giving generalized matrices and mirror normals in terms of the μ parameters.

Given $\mu = (\mu_1, \dots, \mu_5)$, associate to each μ_i a complex number

$$M_i = e^{2\pi\sqrt{-1}(1-\mu_i)}.$$

Recalling how the B_i, A_i, A_i' and B_i' were defined in §2 we see that the multipliers for them are $M_{i-1}M_{i+1}$, M_iM_4, M_iM_5, and M_4M_5 respectively. By multiplier we mean, for example

$$B_i(x) = x + (M_{i-1}M_{i+1} - 1)\langle x, e_i\rangle e_i.$$

Let

$$\alpha_{i+1\ i-1} = \alpha_{i-1\ i+1} = \left(\frac{\sin\pi\mu_{i-1}\sin\pi\mu_{i+1}}{\sin\pi(\mu_i+\mu_{i-1})\sin\pi(\mu_i+\mu_{i+1})}\right)^{\frac{1}{2}}.$$

Then the matrix of H_ϕ with respect to the e-base (the $\{e_i\}_{i=1,3}$ are the unit normals to the mirrors of $\{B_i\}_{i=1,3}$) is:

$$H = \begin{pmatrix} 1 & \alpha_{12}M_3^{\frac{1}{2}}M_4^{\frac{1}{3}} & \alpha_{13}M_2^{-\frac{1}{2}}M_4^{-\frac{1}{3}} \\ \alpha_{21}M_3^{-\frac{1}{2}}M_4^{-\frac{1}{3}} & 1 & \alpha_{23}M_1^{\frac{1}{2}}M_4^{\frac{1}{3}} \\ \alpha_{31}M_2^{\frac{1}{2}}M_4^{\frac{1}{3}} & \alpha_{32}M_1^{-\frac{1}{2}}M_4^{-\frac{1}{3}} & 1 \end{pmatrix}.$$

Now one can write down the matrices of the B_i and the A_i in the e-base. In this general setting the e-base comes from normalizing an e'-base that satisfies

$$\langle e_i', e_i'\rangle = \frac{\sin\pi(\mu_{i-1}+\mu_{i+1})\sin\pi\mu_3}{\sin\pi\mu_{i-1}\sin\pi\mu_{i+1}}.$$

Set $\beta_i = \langle e_i', e_i'\rangle^{\frac{1}{2}}$ and $\beta_{ij} = \frac{\beta_i}{\beta_j}$. A computation like the one in §3 with the more general matrices gives the mirror normals to the A_i as

$$a_1 = \begin{pmatrix} -M_4^{\frac{1}{3}}\beta_{13} \\ -M_4^{\frac{2}{3}}\beta_{23} \\ -1 \end{pmatrix}, \quad a_2 = \begin{pmatrix} -1 \\ -M_4^{\frac{1}{3}}\beta_{21} \\ -M_4^{\frac{2}{3}}\beta_{31} \end{pmatrix}, \quad a_3 = \begin{pmatrix} -M_4^{\frac{2}{3}}\beta_{12} \\ -1 \\ -M_4^{\frac{1}{3}}\beta_{32} \end{pmatrix}.$$

We now show that $\Gamma_{3,0} \hookrightarrow \Gamma_{\frac{5}{12}, \frac{3}{12}, \frac{5}{12}, \frac{1}{12}, \frac{10}{12}}$. Let

$$c_1 = a_2, \quad c_2 = e^{\frac{\pi i}{6}} e_1, \quad c_3 = e^{-\frac{5\pi i}{18}} A_2 \cdot e_3.$$

The $\{c_i\}_{i=1,3}$ are given by

$$c_1 = \begin{pmatrix} -1 \\ -M_4^{\frac{1}{3}} \beta_{21} \\ -M_4^{\frac{2}{3}} \beta_{31}, \end{pmatrix}, \qquad c_2 = \begin{pmatrix} e^{\frac{\pi i}{6}} \\ 0 \\ 0 \end{pmatrix}, \qquad c_3 = e^{-\frac{5\pi i}{18}} \begin{pmatrix} M_2 M_4^{\frac{1}{3}} \beta_{13} \\ M_2 M_4^{\frac{2}{3}} \beta_{23} \\ M_2 M_4 + 1 \end{pmatrix}.$$

Since $\langle e_i, e_i \rangle = 1$, c_2 and c_3 are clearly unit vectors and it is not hard to see that c_1 is also. One computes that

$$\langle c_1, c_2 \rangle = -e^{-\frac{\pi i}{6}} \left[1 + \frac{M_3^{-\frac{1}{2}} \sin \pi \mu_2}{\sin \pi(\mu_2 + \mu_3)} + \frac{M_3^{\frac{1}{2}} \sin \pi \mu_3}{\sin \pi(\mu_2 + \mu_3)} \right],$$

$$\langle c_2, c_3 \rangle = e^{\frac{4\pi i}{9}} M_4^{-\frac{1}{3}} \left[M_2^{-1} \frac{\beta_1}{\beta_3} + \frac{M_2^{-1} M_3^{\frac{1}{2}} \sin \pi \mu_4}{\beta_1 \beta_3 \sin \pi \mu_3} \right.$$
$$\left. + \frac{M_2^{-\frac{1}{2}} (M_2^{-1} M_4^{-1} + 1) \sin \pi \mu_4}{\beta_1 \beta_3 \sin \mu_2} \right],$$

$$\langle c_3, c_1 \rangle = \frac{-e^{-\frac{5\pi i}{18}}}{\beta_1 \beta_3} \left[M_2 M_4^{\frac{1}{3}} \left(\beta_1^2 + \beta_2^2 + \beta_3^2 + 4 \sin \pi \mu_4 \right) \right.$$
$$+ M_2^{\frac{1}{2}} \frac{\sin \pi \mu_4}{\sin \pi \mu_2} \left(M_2 M_4^{\frac{4}{3}} + M_4^{\frac{1}{3}} + M_4^{-\frac{2}{3}} \right)$$
$$\left. + \frac{M_4^{-\frac{2}{3}}}{\sin \pi \mu_1} \left(M_1^{-\frac{1}{2}} \sin \pi \mu_4 + \beta_3^2 \sin \pi \mu_1 \right) \right].$$

After substituting $\mu = \left(\frac{5}{12}, \frac{3}{12}, \frac{5}{12}, \frac{1}{12}, \frac{10}{12} \right)$ for the μ_i, one verifies that indeed

$$\langle c_1, c_2 \rangle = \langle c_2, c_3 \rangle = \langle c_3, c_1 \rangle = -\frac{1}{\sqrt{3}}.$$

Now for the $\Gamma_{\frac{1}{10}, \frac{7}{10}, \frac{1}{10}, \frac{7}{10}, \frac{4}{10}}$ case we choose mirror normals $\{c_i\}_{i=1,3}$ such that

$$\langle c_1, c_2 \rangle = \langle c_2, c_3 \rangle = \langle c_3, c_1 \rangle = -\frac{e^{\frac{7\pi i}{30}}}{2 \sin \frac{\pi}{5}}.$$

Let

$$c_1 = e^{\frac{\pi i}{3}} a_1, \quad c_2 = (A_1 A_3 A_1)^{-1} a_1, \quad c_3 = e^{\frac{13\pi i}{5}} B_2 e_1.$$

In this case

$$(A_1A_3A_1)^{-1} = \begin{pmatrix} e^{\frac{3\pi i}{5}}+e^{-\frac{3\pi i}{5}} & 2\cos(\frac{2\pi}{5})e^{\frac{\pi i}{5}} & e^{\frac{2\pi i}{5}}+e^{-\frac{2\pi i}{5}}+e^{-\frac{4\pi i}{5}}+1 \\ \frac{-e^{-\frac{\pi i}{5}}}{2\cos\frac{2\pi}{5}} & 0 & \frac{e^{-\frac{\pi i}{5}}+1}{2\cos\frac{2\pi}{5}} \\ e^{\frac{4\pi i}{5}}+e^{-\frac{4\pi i}{5}}+e^{\frac{2\pi i}{5}} & 2\cos\frac{2\pi}{5} & e^{\frac{\pi i}{5}}+e^{-\frac{\pi i}{5}}+e^{-\frac{3\pi i}{5}} \end{pmatrix}.$$

Also

$$B_2 = \begin{pmatrix} 1 & 0 & 0 \\ (M_1M_3-1)H_{12} & M_1M_3 & (M_1M_3-1)H_{32} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ e^{\frac{2\pi i}{5}} & e^{-\frac{2\pi i}{5}} & e^{\frac{\pi i}{5}} \\ 0 & 0 & 1 \end{pmatrix}$$

since

$$H = \begin{pmatrix} 1 & \frac{e^{\frac{11\pi i}{10}}}{2\sin\frac{\pi}{5}} & -\left(\frac{\sin\frac{\pi}{10}}{\sin\frac{4\pi}{10}}\right)i \\ \frac{e^{-\frac{11\pi i}{10}}}{2\sin\frac{\pi}{5}} & 1 & \frac{e^{\frac{11\pi i}{10}}}{2\sin\frac{\pi}{5}} \\ \left(\frac{\sin\frac{\pi}{10}}{\sin\frac{4\pi}{10}}\right)i & \frac{e^{-\frac{11\pi i}{10}}}{2\sin\frac{\pi}{5}} & 1 \end{pmatrix}.$$

Thus we get

$$c_1 = e^{\frac{\pi i}{3}}a_1 = e^{\frac{\pi i}{3}} \begin{pmatrix} -e^{\frac{\pi i}{5}} \\ \frac{-e^{\frac{2\pi i}{5}}}{2\cos\frac{2\pi}{5}} \\ -1 \end{pmatrix} = \begin{pmatrix} -e^{\frac{8\pi i}{15}} \\ \frac{-e^{\frac{11\pi i}{15}}}{2\cos\frac{2\pi}{5}} \\ -e^{\frac{\pi i}{3}} \end{pmatrix},$$

$$c_2 = (A_1A_3A_1)^{-1}a_1 = \begin{pmatrix} 0 \\ \frac{-e^{-\frac{\pi i}{5}}}{2\cos\frac{2\pi}{5}} \\ 0 \end{pmatrix},$$

and

$$c_3 = e^{\frac{13\pi i}{5}} B_2 e_1 = e^{\frac{13\pi i}{5}} \begin{pmatrix} 1 \\ e^{\frac{2\pi i}{5}} \\ 0 \end{pmatrix} = \begin{pmatrix} e^{\frac{13\pi i}{5}} \\ -e^{\frac{4\pi i}{15}} \\ 0 \end{pmatrix},$$

which gives

$$\langle c_1, c_2 \rangle = \frac{e^{-\frac{\pi i}{6}}}{4\cos\frac{2\pi}{5}\sin\frac{\pi}{5}} + \frac{e^{\frac{14\pi i}{15}}}{(2\cos\frac{2\pi}{5})^2} + \frac{e^{-\frac{17\pi i}{30}}}{4\cos\frac{2\pi}{5}\sin\frac{\pi}{5}}$$

$$= \frac{-e^{\frac{7\pi i}{30}}}{2\sin\frac{\pi}{5}} \cdot \frac{1}{(2\cos\frac{2\pi}{5})^2},$$

$$\langle c_2, c_3 \rangle = \frac{-e^{-\frac{\pi i}{6}}}{4\cos\frac{2\pi}{5}\sin\frac{\pi}{5}} + \frac{e^{-\frac{7\pi i}{15}}}{2\cos\frac{2\pi}{5}} = \frac{-e^{\frac{7\pi i}{30}}}{2\sin\frac{\pi}{5}} \cdot \frac{1}{2\cos\frac{2\pi}{5}},$$

and

$$\langle c_3, c_1 \rangle = -e^{\frac{\pi i}{3}} + \frac{e^{\frac{7\pi i}{30}}}{4\cos\frac{2\pi}{5}\sin\frac{\pi}{5}} - \frac{e^{\frac{\pi i}{30}}\cos\frac{2\pi}{5}}{\sin\frac{\pi}{5}}$$

$$- \frac{e^{-\frac{11\pi i}{30}}}{2\sin\frac{\pi}{5}} - \frac{e^{\frac{8\pi i}{15}}}{2\cos\frac{2\pi}{5}} - \frac{e^{\frac{\pi i}{30}}}{2\sin\frac{\pi}{5}}$$

$$= \frac{-e^{\frac{7\pi i}{30}}}{2\sin\frac{\pi}{5}} \cdot \frac{1}{2\cos\frac{2\pi}{5}}.$$

Now since we find that

$$\langle c_1, c_1 \rangle = \frac{1}{(2\cos\frac{2\pi}{5})^2},$$

$$\langle c_2, c_2 \rangle = \frac{1}{(2\cos\frac{2\pi}{5})^2}, \quad \text{and} \quad \langle c_3, c_3 \rangle = 1;$$

after normalizing the c_i we are left with

$$\langle c_1, c_2 \rangle = \langle c_2, c_3 \rangle = \langle c_3, c_1 \rangle = \frac{-e^{\frac{7\pi i}{30}}}{2\sin\frac{\pi}{5}}$$

as required.

Notice that since all four groups are arithmetic lattices, the $\Gamma_{p,t}$ inject as subgroups of finite index. Hence the Γ_μ are commensurable with groups satisfying ΣINT.

The isomorphisms are proved in a similar way and are in fact part of a more general statement given by Deligne and Mostow in a paper to appear. From the viewpoint of Theorem 6.2 we can make the following statement that includes these isomorphisms. For each $p \in \{5, 6, 7, 8, 9, 10, 12, 18\}$ the following groups are isomorphic:

$$\Gamma_{\left(\frac{1}{2}-\frac{1}{p}, \frac{1}{2}-\frac{1}{p}, \frac{1}{2}-\frac{1}{p}, \frac{1}{2}-\frac{1}{p}, \frac{4}{p}\right)} \simeq \Gamma_{\left(\frac{1}{2}-\frac{2}{p}, \frac{1}{2}-\frac{2}{p}, \frac{1}{2}+\frac{1}{p}, \frac{1}{2}+\frac{1}{p}, \frac{2}{p}\right)}$$
$$\simeq \Gamma_{\left(\frac{1}{2}-\frac{1}{p}, \frac{1}{2}-\frac{1}{p}, \frac{1}{2}-\frac{1}{p}, \frac{1}{2}+\frac{2}{p}, \frac{1}{p}\right)}.$$

Together Theorem 3.1 and Theorem 4.1 verify Mostow's conjecture.

5. The volumes of fundamental domains for the Γ_μ. In general it is difficult to compute the index of one infinite group inside another. In §6 we determine indices using ratios of the volumes of fundamental domains computed here. Let Ω be the region defined in **[M-1]**. Ω is a fundamental domain for $\Gamma_{p,t}$ modulo $\langle J \rangle$, the subgroup generated by J, the cyclic automorphism of order 3 permuting the generators of $\Gamma_{p,t}$. Some of the $\Gamma_{p,t}$ do not contain J in which case Ω is a fundamental domain. Carrying out for general (p, t) the computation done in **[MS]** for $(5, \frac{1}{20})$ gives the following theorem of Mostow and Siu.

THEOREM 5.1. *Let $\Gamma_{p,t}$ be a lattice with $p = 3, 4, 5$ and $|t| < \frac{1}{2} - \frac{1}{p}$. Then*

$$\operatorname{vol}(\Omega) = 2\pi^2 \left[3\left(\frac{1}{2} - \frac{1}{p}\right)^2 - t^2 \right].$$

In case $|t| > \frac{1}{2} - \frac{1}{p}$ we have the following.

THEOREM 5.2. *Let $\Gamma_{p,t}$ be a lattice with $p = 3, 4, 5$ and $\frac{1}{2} - \frac{1}{p} < |t| < 3(\frac{1}{2} - \frac{1}{p})$. Then*

$$\operatorname{vol}(\Omega) = \pi^2 \left[3\left(\frac{1}{2} - \frac{1}{p}\right) - t \right]^2.$$

Proof. The computation is the same as in **[MS]** except when $\frac{1}{2} - \frac{1}{p} < |t| < 3(\frac{1}{2} - \frac{1}{p})$ the combinatorial type of Ω changes; that is, the $\Delta_{321}, \Delta_{213}, \Delta_{132}$ collapse to a point and hence drop out of the computation. Also, the quadrilateral $t_{23}p_{31}t_{32}p_{21}$ in $\tilde{R}_1 \cap \tilde{R}_1^{-1}$ has angles

$$\begin{aligned}
\angle p_{21} t_{32} p_{31} &= \pi \left(t + \left(\frac{1}{2} - \frac{1}{p} \right) \right), \\
\angle p_{21} t_{23} p_{31} &= \pi \left(t - \left(\frac{1}{2} - \frac{1}{p} \right) \right), \\
\angle t_{23} p_{21} t_{32} &= \angle t_{23} p_{31} t_{32} = \pi \left(\frac{6-p}{2p} \right).
\end{aligned}$$

Therefore the area of $\tilde{R}_1 \cap \tilde{R}_1^{-1}$ is

$$2\pi - \left[2\pi t + 2\pi \left(\frac{6-p}{2p} \right) \right] = 2\pi \left[1 - t - \left(\frac{6-p}{2p} \right) \right].$$

The angles in Δ_{123} are

$$\begin{aligned}
\angle t_{13} s_{23} \tilde{s}_{21} &= \angle t_{13} \tilde{s}_{21} s_{23} = \frac{\pi}{2} \left(t - \left(\frac{1}{2} - \frac{1}{p} \right) \right), \\
\angle s_{23} t_{13} \tilde{s}_{21} &= \angle s_{12} t_{31} \tilde{s}_{31} = \frac{2\pi}{p},
\end{aligned}$$

and so the area of Δ_{123} is

$$\pi - \left[\frac{2\pi}{p} + \pi \left(t - \left(\frac{1}{2} - \frac{1}{p} \right) \right) \right] = \pi \left[\frac{3}{2} - \frac{3}{p} - t \right].$$

Carrying out the computation with these values and without the Δ_{321} term yields the result.

Next, when $p > 5$, we have $\mu_4 + \mu_5 < 1$ and the fixed point set of the $\{B_i'\}_{i=1,3}$ are lines not points, resulting in an increase in the number of 2-faces in the computation. There is still a great deal of cancellation, but not quite enough. Integrals of the logarithm of the Jacobian of the element J over surfaces that are not geodesics remain. Rather than trying to evaluate these integrals, we use an alternate method to compute the volumes.

We remarked in §2 that we could use the fact that $B^+/\Gamma_\mu \simeq Q_{st}$, in case μ satisfies INT (i.e. $\Gamma_{p,t}$ with p even), to compute the volume.

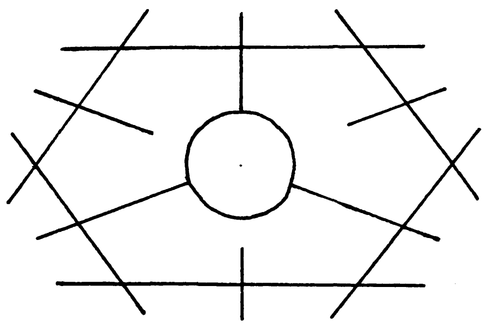
We begin by choosing a torsion free subgroup $\Gamma_0 \triangleleft \Gamma_\mu$ of index m in Γ_μ. If we define $Y \equiv B^+/\Gamma_0$ then the projection

$$\begin{array}{c} Y = B^+/\Gamma_0 \\ \pi \big\downarrow \\ Q_{st} \simeq B^+/\Gamma_\mu \end{array}$$

is an m to 1 covering map off the branch locus. This implies that with respect to the volume induced from the ball, $\mathrm{vol}(Y) = m \cdot \mathrm{vol}(Q_{st})$. But $\mathrm{vol}(Y) = \frac{8\pi^2}{3} \cdot \chi(Y)$. Hence we need only compute $\chi(Y)$. But $\chi(Y) = m \cdot \chi(Q_{st})$ – correction for ramification. We proceed with this calculation.

We work under the assumption that μ satisfies INT. Let $L_{ij} = \{z | z_i = z_j\}$ whenever $\mu_i + \mu_j < 1$. The L_{ij} are all exceptional lines in Q_{st} if $\mu_i + \mu_j < 1$ for all i, j **[DM]** and π ramifies only over the $L_{ij} \simeq P^1$. If four of the lines are blown down, Q_{st} may be $\mathbf{P}^2$ and then the lines L_{ij} are *not* exceptional. However, under the assumption that $\mu_i + \mu_j < 1$ for all $i, j \in S$, there are ten exceptional lines with the following configuration

(5.3)



Notice that the line L_{ij} is the line fixed by $(i\,j)$. That is, $L_{i-1,i+1}$ comes from B_i, $i = 1, 2, 3$, L_{i4} and L_{i5} come from A_i and A'_i, $i = 1, 2, 3$ respectively, and L_{45} comes from the B'_i. The ramification over the lines comes from the orders of the corresponding element as follows.

We want to determine the ramification of $\pi : Y \longrightarrow Q_{st}$ over a point $y \in Q_{st}$. Let V be a suitably small neighborhood of y in Q_{st} (for precise details see **[DM**, §8.2 **]**). Define the *decomposition group* D_y to be the image of $\pi_1(V \cap Q, 0)$ in $\pi_1(Q, 0)$. Then

$$\pi^{-1}(y) \simeq \pi_0(\pi^{-1}(V \cap Q)) \simeq D_y \backslash \pi_1(Q, 0)/\theta^{-1}(\Gamma_0)$$

so

$$|\pi^{-1}(y)| = |\theta(D_y) \backslash \Gamma_\mu/\Gamma_0| = \frac{|\Gamma_\mu/\Gamma_0|}{|\theta(D_y)|} = \frac{m}{|\theta(D_y)|}.$$

If $y \in Q$ then $\pi_1(V \cap Q, 0)$ is trivial and $|\pi^{-1}(y)| = m$ which we know since π is an m to 1 covering map except over the L_{ij}. Hence we need only determine for each $y \in L_{ij}$ the order of the image of D_y under the monodromy homomorphism θ.

If y is on L_{ij} but not on any of the other lines, then $V \cap Q$ is just $\mathbf{C}^2$ with a complex line removed and hence the decompostion group is generated by a loop around the line. The image under θ of this loop has order equal to the order of the element associated to L_{ij}. Hence

$$|\theta(D_y)| = k_{ij} \qquad \text{(recall that } k_{ij} = (1-\mu_i-\mu_j)^{-1}\text{)}.$$

Next consider $y \in L_{ij} \cap L_{lq}$. Then $V \cap Q$ is $\mathbf{C}^2$ with two lines removed and the image of the decomposition group is the sum of two cyclic groups, hence

$$|\theta(D_y)| = k_{ij}k_{lq}.$$

We can now proceed with the following theorem.

THEOREM 5.3. *Let μ be a disc 5-tuple with $\mu_1 = \mu_2 = \mu_3$ that satisfies* INT *and such that $\mu_i + \mu_j < 1$ for all $i \neq j$. Then*

$$\operatorname{vol}(B^+/\Gamma_\mu) = \pi^2 \left[\frac{p^2 + 12p - 60}{p^2} - 4t^2 \right].$$

Proof. From the above discussion we need only compute $\chi(Y)$. Choose a triangulation on Q_{st} that includes the triangulation of each $L_{ij} \simeq P^1$ which consists of vertices at $0, 1, \infty$ on the equator and at $i, -i$, the North and South poles and includes the edges connecting each of $0, 1, \infty$ to the other four points. Also choose this triangulation such that if two of the L_{ij} intersect, then the intersection point is a vertex of the triangulation of both lines. Take π^{-1} of this triangulation of Q_{st} as a triangulation of Y. Let $\nu_l =$ the number of l-dimensional cells in the triangulation. Now we compute $\nu_l(Y)$ as $m \cdot \nu_l(Q_{st})$ with corrections for l-dimensional cells in the L_{ij}.

$$\nu_0(Y) = m \cdot \nu_0(Q_{st}) - \sum_{\substack{L_{ij}\cap L_{lq} \\ i,j,l,q \text{ distinct} \\ \text{in } \{1,\dots,5\}}} \left(m - \frac{m}{k_{ij}k_{lq}}\right) - 2 \sum_{\substack{L_{ij} \\ 1\le i<j\le 5}} \left(m - \frac{m}{k_{ij}}\right).$$

Here the first sum is a correction term for the vertices that are intersection points in figure (5.3). The second sum is the correction term for the remaining two vertices in each line L_{ij}, $1 \le i < j \le 5$. Next note that there are 9 edges in the triangulation of each L_{ij}. Hence

$$\nu_1(Y) = m \cdot \nu_1(Q_{st}) - 9 \sum_{\substack{L_{ij} \\ 1\le i<j\le 5}} \left(m - \frac{m}{k_{ij}}\right).$$

Similarly there are six 2-faces in each L_{ij} so

$$\nu_2(Y) = m \cdot \nu_2(Q_{st}) - 6 \sum_{\substack{L_{ij} \\ 1\le i<j\le 5}} \left(m - \frac{m}{k_{ij}}\right).$$

Off the lines L_{ij} we have

$$\nu_3(Y) = m \cdot \nu_3(Q_{st}), \qquad \nu_4(Y) = m \cdot \nu_4(Q_{st}).$$

Taking the alternating sum we get

$$\chi(Y) = m \cdot \chi(Q_{st}) - \sum_{\substack{i,j,l,q \\ \text{distinct}}} \left(m - \frac{m}{k_{ij}k_{lq}}\right) + \sum_{1\le i<j\le 5} \left(m - \frac{m}{k_{ij}}\right).$$

Since $\mu_1 = \mu_2 = \mu_3$ we have

$$\mu = \left(\frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{1}{4} + \frac{3}{2p} - \frac{t}{2}, \frac{1}{4} + \frac{3}{2p} + \frac{t}{2}\right).$$

Under our assumption that μ satisfies INT, the k_{ij} are integers and we compute for $i = 1, 2, 3$ that

$$k_{i-1,i+1} = \left(\frac{2}{p}\right)^{-1}, \qquad k_{i4} = \left(\frac{1}{4} - \frac{1}{2p} + \frac{t}{2}\right)^{-1},$$

$$k_{i5} = \left(\frac{1}{4} - \frac{1}{2p} - \frac{t}{2}\right)^{-1}, \qquad k_{45} = \left(\frac{1}{2} - \frac{3}{p}\right)^{-1}.$$

In this case Q_{st} is complex projective 2-space P^2 with four points blown up, so $\chi(Q_{st}) = 3 + 4 = 7$. From figure (5.3) note that there are 6 points where A_i meets A'_j, 3 points where B_i meets A_i, 3 points where B_i meets A'_i, and 3 points where B_i meets B'_j, hence

$$\begin{aligned}\chi(Y) = m \cdot \Bigg[7 &- 6\left(1 - \left[\left(\frac{1}{4} - \frac{1}{2p}\right)^2 - \frac{t^2}{4}\right]\right) \\ &- 3\left(1 - \frac{2}{p}\left(\frac{1}{4} - \frac{1}{2p} + \frac{t}{2}\right)\right) \\ &- 3\left(1 - \frac{2}{p}\left(\frac{1}{4} - \frac{1}{2p} - \frac{t}{2}\right)\right) - 3\left(1 - \frac{2}{p}\left(\frac{1}{2} - \frac{3}{p}\right)\right)\Bigg] \\ = m \cdot &\left[\frac{3}{8p^2}(p^2 + 12p - 60) - \frac{3}{2}t^2\right]\end{aligned}$$

From this we compute

$$\begin{aligned}\operatorname{vol}(Q_{st}) &= \frac{1}{m}\operatorname{vol}(Y) = \frac{1}{m}\cdot\frac{8\pi^2}{3}\chi(Y)\\ &= \frac{1}{m}\cdot\frac{8\pi^2}{3}\left(m\left[\frac{3}{8p^2}(p^2+12p-60)-\frac{3}{2}t^2\right]\right)\\ &= \pi^2\left[\frac{p^2+12p-60}{p^2}-4t^2\right].\end{aligned}$$

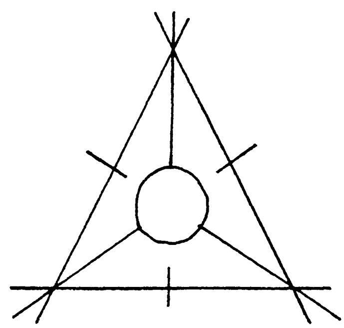
This completes the proof. Next we consider a set of μ which satisfy INT but now with $\mu_i+\mu_5>1$ for $i=1,2,3$.

THEOREM 5.4. *Set* $\mu=(\frac{1}{2}-\frac{1}{p},\frac{1}{2}-\frac{1}{p},\frac{1}{2}-\frac{1}{p},\frac{1}{p},\frac{1}{2}+\frac{2}{p})$. *For* $p\in\{8,10,12,18\}$, μ *is a disc 5-tuple that satisfies* INT *and*

$$\operatorname{vol}(B^+/\Gamma_\mu)=\pi^2\left[\frac{8(p-5)}{p^2}\right].$$

Proof. Under these assumptions, the lines in Q_{st} coming from the A_i', i.e. L_{i5}, $i=1,2,3$ are blown down and the configuration of lines is

(5.4)



The computation is the same as in Theorem 5.3 except we omit the A_i' lines and need to determine the ramification over the points where three lines meet. From the proof of Lemma 10.3 in [**DM**] we have that the order of the decomposition group is

$$\left[2\left(\frac{1}{k_1}+\frac{1}{k_2}+\frac{1}{k_3}-1\right)^{-1}\right]^2$$

where the k_i correspond to the three intersecting lines. Notice that this point of intersection is $A_{i-1}\cap A_{i+1}\cap B_i$ for $i=1,2,3$. Since in this case $k_{i-1,i+1}=\frac{p}{2}$ and $k_{i4}=2$, $i=1,2,3$, we have that the order of the decomposition group at $A_{i-1} \cap A_{i+1} \cap B_i$ is

$$\left[2\left(\frac{1}{2}+\frac{1}{2}+\frac{2}{p}-1\right)^{-1}\right]^2 = p^2.$$

Using this information we can complete the calculation.

$$\begin{aligned}
\nu_0(Y) &= m \cdot \nu_0(Q_{st}) - \sum_{\substack{L_{i-1,i+1} \cap L_{45} \\ i=1,2,3}} \left(m - \frac{m}{k_{i-1,i+1}k_{45}}\right) \\
&\quad - \sum_{\substack{L_{i-1,i+1} \cap L_{i4} \\ i=1,2,3}} \left(m - \frac{m}{k_{i-1,i+1}k_{i4}}\right) \\
&\quad - \sum_{i=1,2,3} \left(m - \frac{m}{p^2}\right) - 2 \sum_{\substack{L_{ij} \\ 1 \le i < j \le 4}} \left(m - \frac{m}{k_{ij}}\right) \\
\nu_1(Y) &= m \cdot \nu_1(Q_{st}) - 9 \sum_{\substack{L_{ij} \\ 1 \le i < j \le 4}} \left(m - \frac{m}{k_{ij}}\right) \\
\nu_2(Y) &= m \cdot \nu_2(Q_{st}) - 6 \sum_{\substack{L_{ij} \\ 1 \le i < J \le 4}} \left(m - \frac{m}{k_{ij}}\right) \\
\nu_3(Y) &= m \cdot \nu_3(Q_{st}) \\
\nu_4(Y) &= m \cdot \nu_4(Q_{st})
\end{aligned}$$

Taking the alternating sum and noting that since three lines have been blown down, $\chi(Q_{st}) = 4$, we have

$$\begin{aligned}
\chi(Y) = m\Bigg[4 &- 3\left(1 - \frac{2}{p}\left(\frac{1}{2} - \frac{3}{p}\right)\right) - 3\left(1 - \frac{2}{p}\left(\frac{1}{2}\right)\right) - 3\left(1 - \frac{1}{p^2}\right) \\
&+ 3\left(1 - \frac{1}{2}\right) + 3\left(1 - \frac{2}{p}\right) + 3\left(1 - \left(\frac{1}{2} - \frac{3}{p}\right)\right)\Bigg] \\
= m &\cdot \left[\frac{3(p-5)}{p^2}\right].
\end{aligned}$$

This gives

$$\operatorname{vol}(B^+/\Gamma_\mu) = \frac{8\pi^2}{3}\left[\frac{3(p-5)}{p^2}\right] = \pi^2\left[\frac{8(p-5)}{p^2}\right]$$

to complete the proof.

We have only dealt with μ satisfying INT. In the case of μ satisfying ΣINT we study the $\Gamma_{\mu\Sigma}$ of [M-2] mentioned in §1. The proofs are much the same, with some modification that we give in the proof of Theorem 5.1$'$. We use the notation 5.1$'$ since we are considering exactly the same class of groups as in Theorem 5.1. The formula for the volume is divided by 3 since the formula in Theorem 5.1 is for a fundamental domain of $\Gamma_{p,t}$ modulo $\langle J\rangle$. If $J \in \Gamma_{p,t}$ then $\Gamma_{p,t} \simeq \Gamma_{\mu\Sigma}$ and the formula in Theorem 5.1 is too large by a factor of 3. If $J \notin \Gamma_{p,t}$, then $\langle \Gamma_{p,t}, J\rangle \simeq \Gamma_{\mu\Sigma}$ and Theorem 5.1$'$ gives the volume of a fundamental domain of $\Gamma_{\mu\Sigma}$.

THEOREM 5.1$'$. *Let μ be a disc 5-tuple that satisfies* ΣINT *and such that $\mu_i+\mu_j < 1$ for all i, j except $\mu_4+\mu_5 \geq 1$ (this is equivalent to $p \leq 5$). Then the volume of the fundamental domain for $\Gamma_{\mu\Sigma}$ is*

$$\frac{2\pi^2}{3}\left[3\left(\frac{1}{2}-\frac{1}{p}\right)^2 - t^2\right].$$

Proof. The role of Q_{st} in the previous theorems is played by Q_{st}/Σ as follows. As in §2 let $S = S_1 \cup S_2$ be a decomposition of the set S into disjoint subsets and assume that $\mu_s = \mu_t$ for all $s, t \in S_1$. Let Σ denote the permutation group of S_1. Then Σ operates on P^S by permutation of factors and hence on the subset M. Let Q' denote the subset of Q on which Σ acts freely, 0 a base point in Q', and $\overline{0}$ denote the orbit $\Sigma 0$, and let

$$\theta_\Sigma : \pi_1(Q'/\Sigma, \overline{0}) \longrightarrow \operatorname{Aut} B^+$$

denote the monodromy homomorphism. Then we have

$$Q_{st}/\Sigma \simeq B^+/\Gamma_{\mu\Sigma}$$

where

$$\Gamma_{\mu\Sigma} = \pi_1(Q'/\Sigma, \overline{0})/\operatorname{Ker}\theta_\Sigma.$$

Now we choose $\Gamma_0 \lhd \Gamma_{\mu\Sigma}$ torsion free with $|\Gamma_{\mu\Sigma}/\Gamma_0| = m$ as before. Let $Y = B^+/\Gamma_0$ and $\pi\colon Y \longrightarrow Q_{st}/\Sigma$ be the ramified cover. Let $y \in Q_{st}$ and V be a suitably small neighborhood of y in Q_{st} so that the image of $\pi_1(V \cap Q', o)$ in $\pi_1(Q', 0)$ is the decomposition group D_y as before. Then the image of

$$\pi_1(\tau(V \cap Q'), \overline{0}) \longrightarrow \pi_1(Q'/\Sigma, \overline{0})$$

is the decomposition group, $D_{\tau(y)}$ (τ is the orbit map $Q' \to Q'/\Sigma$). We need to compute the order of $\theta_\Sigma(D_{\tau(y)})$ for all $y \in Q_{st}$ such that $|\pi^{-1}(\tau(y))| \neq m$. In addition to the points in the L_{ij} we must consider points where the action of Σ is *not* free. Next we determine all such points, where in this case we have $S_1 = \{1, 2, 3\}$.

Let σ denote the transposition (12). Then σ acts on M_{st} by

$$(z_1, z_2, z_3, z_4, z_5) \xrightarrow{\sigma} (z_2, z_1, z_3, z_4, z_5).$$

The line $L_{12} = \{(z, z, z_3, z_4, z_5)\}$ is fixed by σ. To find points in Q_{st} fixed by σ we consider the PGL_2 action and solve for points that satisfy

$$\begin{aligned}&(z_1, z_2, z_3, z_4, z_5)\\ &\qquad = (gz_2, gz_1, gz_3, gz_4, gz_5) \quad \text{for some } g \in \mathrm{PGL}_2.\end{aligned}$$

If g fixes three points then $g =$ identity, so assume $z_3 = z_4$. Then by changing g we can assume $z_3 = z_4 = \infty$ and that the other fixed point is $z_5 = 0$. This implies $g(z) = az$ for some $a \in \mathbf{C}$, hence $z_1 = a^2 z_1$ and we take $a = -1$ to get the point $(z, -z, \infty, \infty, 0)$ in L_{34}. Similarly we get $(z, -z, \infty, 0, \infty)$ in L_{35}. Although in cases later on we have $(z, -z, 0, \infty, \infty)$ in L_{45}, this point does not appear here since $\mu_4 + \mu_5 > 1$.

The permutations (13) and (23) fix the lines L_{13} and L_{23} respectively, and contribute points in L_{24}, L_{25} and L_{14}, L_{15} exactly as above. In the quotient Q_{st}/Σ, the lines coming from the B_i, $i = 1, 2, 3$ (i.e. $L_{i-1\,i+1}$) are identified so there is only one resulting line in the quotient denoted by b. Likewise the L_{i4}, $i = 1, 2, 3$ are identified, as are the L_{i5}, $i = 1, 2, 3$ and we label the resulting lines a and a' according to the associated elements. The points $(z, -z, \infty, \infty, 0)$, $(z, \infty, -z, \infty, 0)$, and $(\infty, z, -z, \infty, 0)$ are identified in Q_{st}/Σ and we call the resulting point $a_\sigma \in a$. Similarly the image of $(z, -z, \infty, 0, \infty)$ in Q_{st}/Σ is called $a'_\sigma \in a'$.

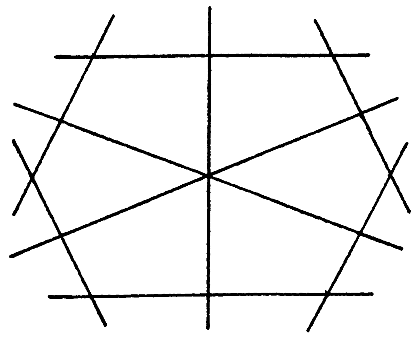
Finally, we must check the 3-cycles. Let J denote the permutation (123). Then

$$(z_1, z_2, z_3, z_4, z_5) \xrightarrow{J} (z_3, z_1, z_2, z_4, z_5)$$

clearly fixes the point $r = (z, z, z, z_4, z_5)$ where the $L_{i-1\,i+1}$, $i = 1, 2, 3$ intersect in Q_{st}. Next we take the PGL_2 action into account and solve

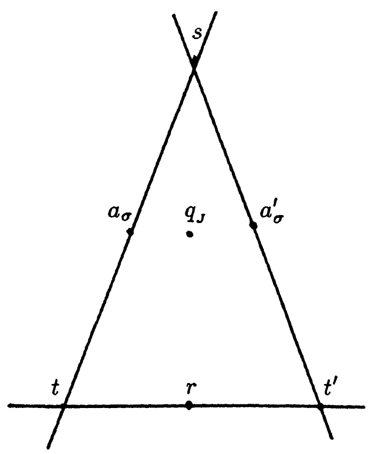
$$\begin{aligned}&(z_1, z_2, z_3, z_4, z_5)\\ &\qquad = (gz_3, gz_3, gz_2, gz_4, gz_5) \quad \text{for some } g \in \mathrm{PGL}_2.\end{aligned}$$

Since $z_4 \neq z_5$ in this case, we assume $z_4 = 0$ and $z_5 = \infty$. Then from $z_1 = a^3 z_1$ we take $a = \omega = e^{\frac{2\pi i}{3}}$ and denote the image of $(1, \omega, \omega^2, 0, \infty)$ in Q_{st}/Σ by q_J. Hence in this case the configuration of lines in Q_{st},



becomes

(5.1)



in Q_{st}/Σ.

Recall that when μ satisfies ΣINT, $S_1 = \{1, 2, 3\}$ is the set of indices where $k = k_{ij} = (1 - \mu_i - \mu_j)^{-1}$, $i, j \in S_1$ is a half integer. The permutation group Σ on S_1 was introduced so that the order of the image under θ_Σ of a loop around the image in Q_{st}/Σ of the **C**-line coming from when two coordinates z_i, z_j coincide is $2k_{ij}$. For the precise details see §3 in [**M-2**], specifically Lemma 3.9. Thus the decomposition group has order $2k_{ij}$ at those points in L_{ij} fixed only by the transposition (ij).

There are two points r and q_J remaining. The point q_J is isolated from the lines L_{ij} and since the group fixing q_J is $\langle (123) \rangle$, the decompostion group has order 3. The point $r = (z, z, z, z_4, z_5)$ is the intersection point of the lines $L_{i-1\,i+1}$, $i = 1, 2, 3$ coming from the B_i, $i = 1, 2, 3$. The decomposition group has as generators, the images of loops passing around each of the lines and, as a subgroup of $\Gamma_{\mu\Sigma}$ acting on the ball and fixing a point in the ball, it is generated by conjugate reflections of order p. By the classification of complex reflection groups in $\mathbf{C}^2$, the decomposition group has diagram

$$p \overset{3}{\text{———}} p$$

The order of this group is $24(\frac{p}{p-6})^2$. We complete the proof as before.

Choose a triangulation of Q_{st}/Σ that includes vertices in each of the lines a, a', b exactly as before. In each line choose the points labeled in (5.1) as three of the vertices, i.e. in a choose the points s, t, and a_σ, in a' choose s, t' and a'_σ, and in b choose t, t', and r. From the previous discussions we list the order of the decomposition group at each point.

$$\begin{array}{rl}
s & k_{i4}k_{i5} \\
t & 2kk_{i4} \\
t' & 2kk_{i5} \\
r & 24\left(\dfrac{p}{p-6}\right)^2 \\
a_\sigma & 2k_{i4} \\
a'_\sigma & 2k_{i5}.
\end{array}$$

The order for all other points in a, a', and b is k_{i4}, k_{i5}, and $2k$ respectively. We also include a correction term in $\nu_0(Y)$ for the point q_j.

$$\begin{aligned}
\nu_0(Y) &= m \cdot \nu_0(Q_{st}/\Sigma) - (\text{correction terms for } s, t, t', r, a_\sigma, a'_\sigma, q_J) \\
&\qquad - 2\,(\text{correction term for remaining vertex in } a,\ a',\ \text{and } b) \\
\nu_1(Y) &= m \cdot \nu_1(Q_{st}/\Sigma) \\
&\qquad - 9\,(\text{correction term for an edge in each of } a,\ a',\ \text{and } b) \\
\nu_2(Y) &= m \cdot \nu_2(Q_{st}/\Sigma) \\
&\qquad - 6\,(\text{correction term for a 2-face in each of } a,\ a',\ \text{and } b) \\
\nu_3(Y) &= m \cdot \nu_3(Q_{st}/\Sigma) \\
\nu_4(Y) &= m \cdot \nu_4(Q_{st}/\Sigma).
\end{aligned}$$

Now taking the alternating sum, writing out each term, and taking the value of $\chi(Q_{st}/\Sigma)$ from **[KLW]** we have

$$\begin{aligned}
\chi(Y) = m \cdot \Bigg[4 &- \left(1 - \left[\left(\frac{1}{4} - \frac{1}{2p}\right)^2 - \frac{t^2}{4}\right]\right) - \left(1 - \frac{1}{p}\left(\frac{1}{4} - \frac{1}{2p} - \frac{t}{2}\right)\right) \\
&- \left(1 - \frac{1}{p}\left(\frac{1}{4} - \frac{1}{2p} + \frac{t}{2}\right)\right) - \left(1 - \frac{(6-p)^2}{24p^2}\right) \\
&- \left(1 - \frac{1}{2}\left(\frac{1}{4} - \frac{1}{2p} - \frac{t}{2}\right)\right) \\
&- \left(1 - \frac{1}{2}\left(\frac{1}{4} - \frac{1}{2p} + \frac{t}{2}\right)\right) - \left(1 - \frac{1}{3}\right) + \left(1 - \frac{1}{p}\right) \\
&+ \left(1 - \left(\frac{1}{4} - \frac{1}{2p} - \frac{t}{2}\right)\right) + \left(1 - \left(\frac{1}{4} - \frac{1}{2p} + \frac{t}{2}\right)\right) \Bigg] \\
= m \cdot \Bigg[&\frac{3(p-2)^2}{16p^2} - \frac{t^2}{4} \Bigg].
\end{aligned}$$

Therefore

$$\operatorname{vol}(B^+/\Gamma_{\mu\Sigma}) = \frac{8\pi^2}{3}\left[\frac{3(p-2)^2}{16p^2} - \frac{t^2}{4}\right] = \frac{2\pi^2}{3}\left[3\left(\frac{1}{2} - \frac{1}{p}\right)^2 - t^2\right].$$

We use this method to compute the volumes in the rest of the cases where μ satisfies ΣINT. The only differences are in the configuration of lines and whether $\Sigma = \Sigma_3$ or Σ_4. We begin with the μ satisfying ΣINT and such that $\mu_i + \mu_j < 1$ for all $i \neq j$.

THEOREM 5.5. *Set* $\mu = (\frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{1}{6} + \frac{1}{p}, \frac{1}{3} + \frac{2}{p})$. *When* $p = 7$ *or* $p = 9$, μ *is a disc* 5*-tuple that satisfies* ΣINT *and*

$$\operatorname{vol}(B^+/\Gamma_{\mu\Sigma}) = \frac{\pi^2}{6}\left[\frac{p^2 + 12p - 60}{p^2} - 4t^2\right].$$

Proof. In this case S_1 is still $\{1, 2, 3\}$ so $\Sigma = \Sigma_3$. Since $\mu_4 + \mu_5 < 1$, there is a line L_{45} in Q_{st} and the configuration of lines in Q_{st} is as shown in (5.3). Hence in addition to the points that are fixed by elements of Σ_3 found in the proof of 5.1$'$, we must look for points where $z_4 = z_5$.

In the case of transpositions σ, we mentioned before that there is a point $b'_\sigma \in Q_{st}/\Sigma_3$ which is the image of $(z, -z, 0, \infty, \infty) \in M_{st}$.

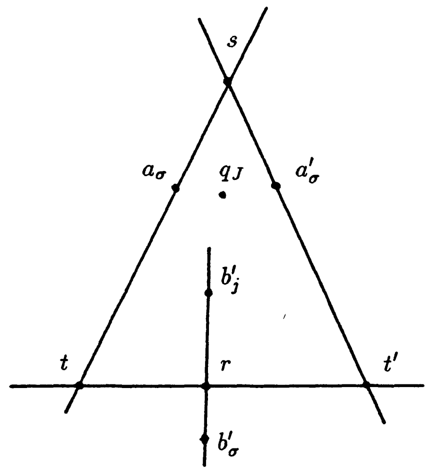
Now for the three cycle (123) which we denote by J, we solve

$$\begin{aligned}
&(z_1, z_2, z_3, z_4, z_5) \\
&\quad = (gz_3, gz_1, gz_2, gz, gz) \quad \text{for some } g \in \mathrm{PGL}_2.
\end{aligned}$$

Set $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$ and note that $g(z) = 1/1 - z$ maps $0 \to 1 \to \infty \to 0$ and has fixed points $z = -\omega, -\omega^2$. Note that $(0, 1, \infty, -\omega, -\omega)$ and $(0, 1, \infty, -\omega^2, -\omega^2)$ are identified in Q_{st}/Σ_3 since $(12) \cdot (0, 1, \infty, -\omega^2, -\omega^2) = (1, 0, \infty, -\omega^2, -\omega^2) = (g0, g1, g\infty, g(-\omega), g(-\omega))$ for $g(z) = 1 - z$. We denote the image in Q_{st}/Σ_3 of $(0, 1, \infty, -\omega, -\omega)$ by b'_J.

Hence the configuration of lines and points where Σ_3 does not act freely is

(5.5)



Choosing a triangulation as in the previous theorem and writing out $\nu_i(Y)$ with correction terms for the points a_σ, t, s, a'_σ, t', r, b'_σ, b'_J and q_J as before we get (using $\chi(Q_{st}/\Sigma_3) = 5$ from **[KLW]**)

$$\begin{aligned}
\chi(Y) = m \cdot \Bigg[5 &- \left[1 - \frac{1}{2}\left(\frac{1}{4} - \frac{1}{2p} + \frac{t}{2}\right)\right] - \left[1 - \frac{1}{p}\left(\frac{1}{4} - \frac{1}{2p} + \frac{t}{2}\right)\right] \\
&- \left[1 - \left(\left(\frac{1}{4} - \frac{1}{2p}\right)^2 - \frac{t^2}{4}\right)\right] - \left[1 - \frac{1}{2}\left(\frac{1}{4} - \frac{1}{2p} - \frac{t}{2}\right)\right] \\
&- \left[1 - \frac{1}{p}\left(\frac{1}{4} - \frac{1}{2p} - \frac{t}{2}\right)\right] - \left[1 - \frac{1}{p}\left(\frac{1}{2} - \frac{3}{p}\right)\right] \\
&- \left[1 - \frac{1}{2}\left(\frac{1}{2} - \frac{3}{p}\right)\right] - \left[1 - \frac{1}{3}\left(\frac{1}{2} - \frac{3}{p}\right)\right] - \left[1 - \frac{1}{3}\right] \\
&+ \left[1 - \frac{1}{p}\right] + \left[1 - \left(\frac{1}{4} - \frac{1}{2p} + \frac{t}{2}\right)\right] \\
&\qquad + \left[1 - \left(\frac{1}{4} - \frac{1}{2p} - \frac{t}{2}\right)\right] + \left[1 - \left(\frac{1}{2} - \frac{3}{p}\right)\right]\Bigg] \\
= m \cdot &\left[\frac{p^2 + 12p - 60}{16p^2} - \frac{t^2}{4}\right].
\end{aligned}$$

Hence we compute the volume as before and get

$$\begin{aligned}\operatorname{vol}(Q_{st}/\Sigma_3) &= \frac{8\pi^2}{3}\left[\frac{p^2+12p-60}{16p^2} - \frac{t^2}{4}\right] \\ &= \frac{\pi^2}{6}\left[\frac{p^2+12p-60}{p^2} - 4t^2\right].\end{aligned}$$

Next we consider a case where $\Sigma = \Sigma_4$ and $\mu_i + \mu_j < 1$ for all i, j.

THEOREM 5.6. *Set* $\mu = (\frac{1}{2}-\frac{1}{p}, \frac{1}{2}-\frac{1}{p}, \frac{1}{2}-\frac{1}{p}, \frac{1}{2}-\frac{1}{p}, \frac{4}{p})$. *When* $p = 7$ *or* $p = 9$, μ *is a disc* 5*-tuple that satisfies* ΣINT *and*

$$\operatorname{vol}(B^+/\Gamma_{\mu\Sigma}) = \frac{\pi^2}{6}\left[\frac{8(p-5)}{p^2}\right].$$

Proof. Here we must find the points where Σ_4 does *not* act freely. We begin with the transpositions. Clearly the line L_{ij} where $z_i = z_j$, $1 \le i < j \le 4$, is fixed by the transposition (ij). Note that these lines are identified in the quotient Q_{st}/Σ_4 (the lines L_{i5}, $i = 1, 2, 3$ and L_{45} are also identified, but they are not fixed by any element of Σ_4). There are points of the form $(z, -z, 0, \infty, \infty)$ that include any permutation of the first four coordinates. These points are identified in the quotient and, as in previous theorems, we denote the point in the quotient by b'_σ. In addition, there are points of the form $(z, -z, \infty, \infty, 0)$ that are not only fixed by (12), interchanging z and $-z$, but also (34) and $(12)(34)$. The image of these points in the quotient is a_σ as before. Next we consider the rest of the points fixed by $(12)(34)$.

For the product $(12)(34)$ we solve

$$\begin{aligned}&(gz_1, gz_2, gz_3, gz_4, gz_5) \\ &\qquad = (z_2, z_1, z_4, z_3, z_5) \quad \text{for some } g \in \mathrm{PGL}_2.\end{aligned}$$

Notice that g^2 fixes the five points z_i, $i = 1, \ldots, 5$ and hence must be the identity. We assume $z_5 = \infty$ and so $g(z)$ is the involution $g(z) = c - z$ for some $c \in \mathbf{C}$. This gives a line of fixed points $(y, c-y, w, c-w, \infty)$ which, after applying

$$g(z) = \frac{z-y}{c-2y}$$

and changing coordinates, can be written $(0, 1, x, 1-x, \infty)$. This line contains the point $(0, 1, \frac{1}{2}, \frac{1}{2}, \infty) \in a_3$ which gets identified with a preimage of a_σ in Q_{st} by

$$g(x, -x, \infty, \infty, 0) = \left(0, 1, \frac{1}{2}, \frac{1}{2}, \infty\right) \qquad \text{where } g(z) = \frac{z-x}{2z}.$$

The line also passes through $a_1 \cap b_1 = t_1 = (0, 1, 1, 0, \infty)$ and $a_2 \cap b_2 = t_2 = (0, 1, 0, 1, \infty)$ when $x = 1$ and 0 respectively. If we denote this line by l_3 there are similar lines l_i, $i = 1, 2$, each intersecting a_i and passing through two points t_{i-1} and t_{i+1}. The l_i, $i = 1, 2, 3$ are identified by Σ_4 in the quotient.

Although the lines are fixed by a subgroup of order 2 there are points in the l_i that are fixed by a cyclic group of order 4 (e.g. $\langle(1423)\rangle$). These are points fixed by $(12)(34)$ where $z_5 \neq \infty$ and come from the involution $g(z) = \frac{-1}{z}$, hence

$$(0, \infty, 1, -1, z_5) = g(\infty, 0, -1, 1, z_5),$$

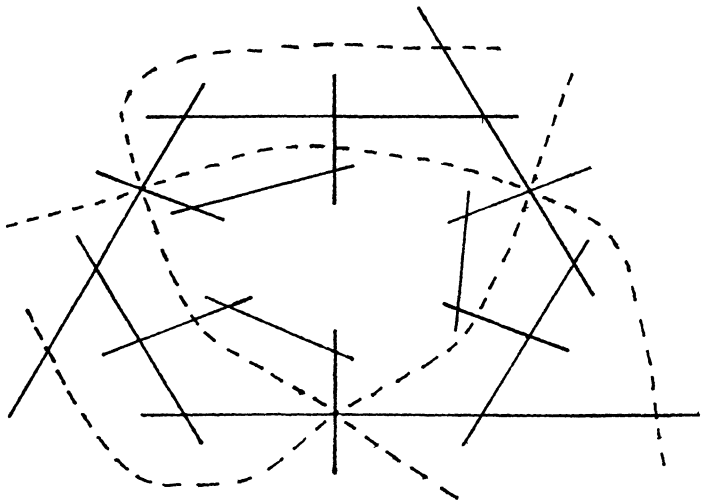
gives the points $(0, \infty, 1, -1, i)$ and $(0, \infty, 1, -1, -i)$. We can see that these points are also fixed by (1423) using $g(z) = \frac{1+z}{1-z}$,

$$g(0, \infty, 1, -1, i) = (1, -1, \infty, 0, i).$$

We denote the image of these points in Q_{st}/Σ_4 by $q_{\sigma\hat{\sigma}} \in l$.

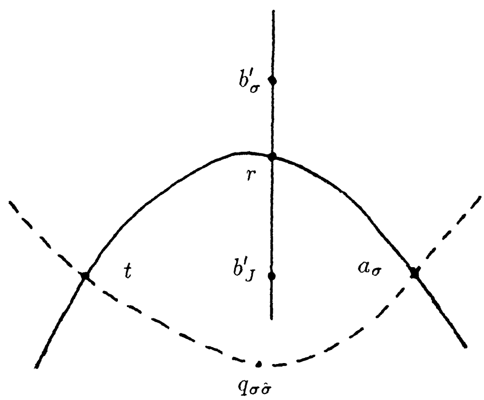
The 3-cycles in this case where $S_1 = \{1, 2, 3, 4\}$ have fixed points $(1, \omega, \omega^2, 0, \infty)$ and $(0, 1, \infty, -\omega, -\omega)$ as before except that permutations in the first *four* coordinates are allowed. The additional points don't add any new points in the quotient Q_{st}/Σ_4 as they all get identified, and we continue to label the points the quotient as $b'_J \in b' = a'$ and the isolated point q_J.

The configuration of lines in Q_{st} (where the l_i are shown as dotted lines) is



which becomes

(5.6)

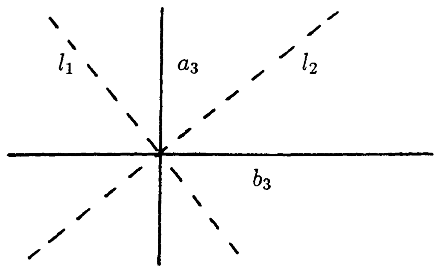


in the quotient Q_{st}/Σ_4.

The image of the decomposition group under θ_Σ at a_σ is generated by two commuting reflections of order 2 and p, and hence being the sum of two cyclic groups has order $2p$.

The other decomposition groups are identical to previous cases, except at the point t. Let U be a small ball around a preimage of t in Q_{st}. Let $U' = U \cap Q'$. A preimage of t in Q_{st} is of the form

(x, x, y, y, z) and locally the configuration of lines is



The point (x, x, y, y, z) has a dihedral group, D_4 (the Sylow 2-subgroup of Σ_4) as isotropy group. Pick a base point $0 \in U'$ and let $\overline{0} = \tau(0)$. We need to determine the image under θ_Σ of $\pi_1(U'/D_4, \overline{0}) \hookrightarrow \pi_1(Q'/\Sigma_4, \overline{0})$.

Consider the exact sequence

$$1 \longrightarrow \pi_1(U', 0) \longrightarrow \pi_1(U'/D_4, \overline{0}) \longrightarrow D_4 \longrightarrow 1.$$

Now U' is homeomorphic to $\mathbf{C}^2$ minus the four lines, l_1, l_2, a_3, b_3, so $\pi_1(U', 0)$ is generated by $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ (γ_i, $i = 1, 2, 3, 4$ conjugate to a small positive loop around l_1, l_2, a_3, b_3 respectively) with relations those expressing that $\gamma_1\gamma_2\gamma_3\gamma_4$ (conjugate to a small loop around the origin on a general line through the origin in $\mathbf{C}^2$) is central. Next write $D_4 = V \ltimes \mathbf{Z}_2$, the semidirect product of the 4-group and $\mathbf{Z}_2$, where we take the 4-group $V = \langle \overline{a}, \overline{b} \rangle$ generated by the permutations fixing the a_3 and b_3 lines, and $\mathbf{Z}_2 = \langle \overline{l} \rangle$, generated by a permutation fixing the line l_1.

If we think of $\pi_1(U', 0)$ as a subgroup of $\pi_1(U'/D_4, \overline{0})$ and write $\overline{\gamma}_i$, $i = 1, 2, 3, 4$ for the image of the γ_i, then $\pi_1(U'/D_4, \overline{0})$ is generated by $\overline{\gamma_i}$, $i = 1, 3, 4$ with the property:

$$\overline{\gamma_i}^2 = \gamma_i \qquad i = 1, 2, 3, 4,$$

and where the $\overline{\gamma_i}$, $i = 1, 2, 3, 4$ map to $\overline{l_1}, \overline{l_2}, \overline{a}, \overline{b}$ in D_4 respectively. Since the map ω_μ of $\widetilde{Q}$ to the ball is etale, it follows that $\theta_\Sigma(\overline{\gamma_1})$ has order 2. As noted before, $\theta_\Sigma(\overline{\gamma_3})$ and $\theta_\Sigma(\overline{\gamma_4})$ have order p so the image of θ_Σ is $(\mathbf{Z}_p \oplus \mathbf{Z}_p) \times \mathbf{Z}_2$. The order of the group is $2p^2$ and we can proceed with the calculation as before.

Choosing a triangulation as before and, after writing out the $\nu_i(Y)$ with the usual correction terms and taking $\chi(Q_{st}/\Sigma_4)$ from **[KLW]**, we have

$$\begin{aligned}\chi(Y) = m \cdot \Bigg[4 &- \left(1 - \frac{1}{p}\left(\frac{1}{2} - \frac{3}{p}\right)\right) - \left(1 - \frac{1}{2p^2}\right) - \left(1 - \frac{1}{2p}\right) \\ &- \left(1 - \frac{1}{2}\left(\frac{1}{2} - \frac{3}{p}\right)\right) - \left(1 - \frac{1}{3}\left(\frac{1}{2} - \frac{3}{p}\right)\right) - \left(1 - \frac{1}{4}\right) \\ &- \left(1 - \frac{1}{3}\right) + \left(1 - \frac{1}{p}\right) + \left(1 - \left(\frac{1}{2} - \frac{3}{p}\right)\right) + \left(1 - \frac{1}{2}\right)\Bigg] \\ = m \cdot \left[\frac{p-5}{2p^2}\right].\end{aligned}$$

Hence

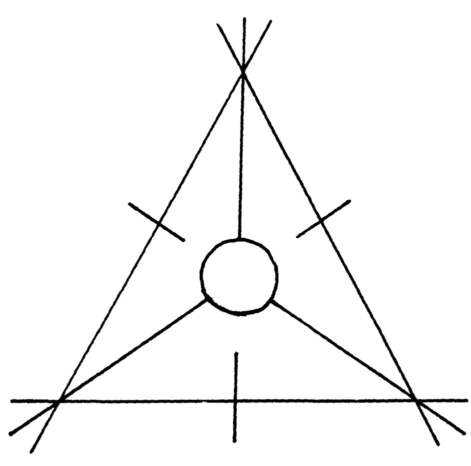
$$\operatorname{vol}(B^+/\Gamma_{\mu\Sigma}) = \frac{8\pi^2}{3}\left[\frac{p-5}{2p^2}\right] = \frac{\pi^2}{6}\left[\frac{8(p-5)}{p^2}\right].$$

The following theorem is used with Theorem 5.6 in §6 to prove that an inclusion of one class of groups in another is actually an isomorphism.

THEOREM 5.7. *Set* $\mu = (\frac{1}{2}-\frac{1}{p}, \frac{1}{2}-\frac{1}{p}, \frac{1}{2}-\frac{1}{p}, \frac{1}{p}, \frac{1}{2}+\frac{2}{p})$. *When* $p = 7$ *or* $p = 9$, μ *is a disc* 5*-tuple that satisfies* ΣINT *and*

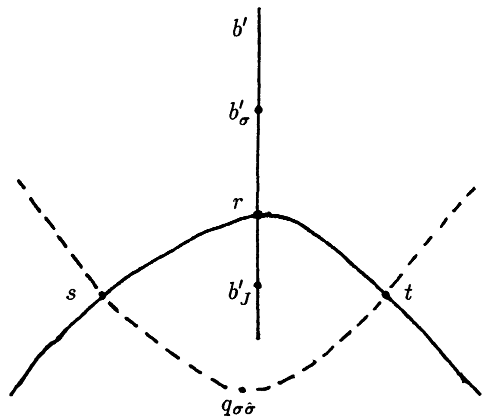
$$\operatorname{vol}(B^+/\Gamma_{\mu\Sigma}) = \frac{\pi^2}{6}\left[\frac{8(p-5)}{p^2}\right].$$

Proof. $S_1 = \{1, 2, 3\}$ and the configuration of lines in Q_{st}



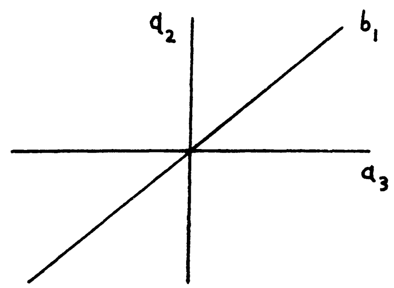
becomes

(5.7)



in Q_{st}/Σ_3. Here the ramification about the b-line is p and about the a-line is 2, hence the a-line plays the role of the l-line of the previous theorem. For example, the decomposition group at the point t is precisely the same as the decomposition group at a_σ in the previous case since they are both the intersection point of lines of ramification 2 and p.

The only point that doesn't correspond exactly to a point in the previous theorem is s. In this case, the isotropy group in Σ_3 of (x, y, y, y, z) (a preimage of s) is just $\mathbf{Z}_2 = \langle(23)\rangle$. Let U be a neighborhood of (x, y, y, y, z) in Q_{st} and $U' = U \cap Q'$. Then the configuration of lines is

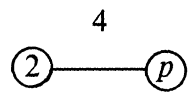


We must find the image under θ_Σ of $\pi_1(U'/\mathbf{Z}_2, \bar{0}) \hookrightarrow \pi_1(Q'/\Sigma_3, \bar{0})$. Consider now the exact sequence

$$1 \longrightarrow \pi_1(U', 0) \longrightarrow \pi_1(U'/\mathbf{Z}_2, \bar{0}) \longrightarrow \mathbf{Z}_2 \longrightarrow 1.$$

Let $\pi_1(U', 0)$ be generated by $\gamma_1, \gamma_2, \gamma_3$ (γ_i, $i = 1, 2, 3$ conjugate to a small positive loop around b_1, a_2, a_3 respectively) with relations those expressing that $\gamma_1\gamma_2\gamma_3$ (conjugate to a small loop around the origin on a general line through the origin in $\mathbf{C}^2$) is central. Write $\mathbf{Z}_2 = \langle \overline{b} \rangle$, $\overline{b}$ the permutation fixing b_1.

$\pi_1(U'/\mathbf{Z}_2, \overline{0})$ is generated by $\overline{\gamma_1}$, γ_2 (where $\overline{\gamma_1}^2 = \gamma_1$ and the image of $\overline{\gamma_1}$ in $\mathbf{Z}_2$ is $\overline{b}$). Since p is odd and $\theta_\Sigma(\gamma_1)$ has order p, $\theta_\Sigma(\langle \overline{\gamma_1}, \gamma_2 \rangle) = \theta_\Sigma(\langle \overline{\gamma_1}^2, \gamma_2 \rangle)$. Also, $\theta_\Sigma(\gamma_2)$ has order 2. Thus the image of θ_Σ is generated by a reflection of order 2 and one of order p. The resulting group is not the sum of cyclic groups nor the dihedral group because the image has a central subgroup of order p coming from $\gamma_1\gamma_2\gamma_3$. Hence by the classification of subgroups generated by complex reflections of order 2 and of order p, the group is



which is of order $2p^2$.

We thus arrive at the remarkable fact (5.7.1) *the configuration of lines for Q^μ_{st}/Σ_4 and Q^ν_{st}/Σ_3 match, i.e. even the orders of the decomposition groups at each point match up,* where μ and ν are

$$\mu = \left(\frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{4}{p}\right) \quad \text{and}$$
$$\nu = \left(\frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{1}{p}, \frac{1}{2} + \frac{2}{p}\right).$$

Hence the computation in this case is exactly as in Theorem 5.6 which gives the stated result. In fact, Deligne and Mostow prove in a paper to appear that

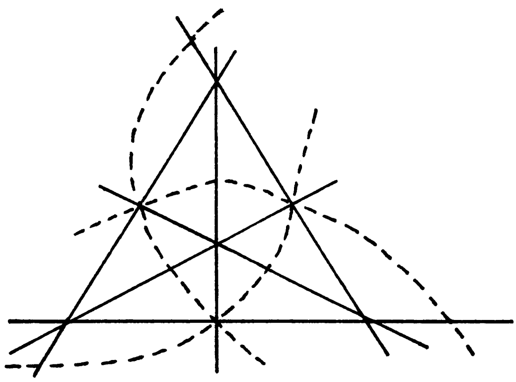
$$Q^\mu_{st}/\Sigma_4 \simeq Q^\nu_{st}/\Sigma_3.$$

The final computation is for the group $\Gamma_{5,\frac{1}{2}}$, a μ of the above type except $\mu_i + \mu_5 > 1$ for all i.

THEOREM 5.8. *Set $\mu = (\frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{1}{2} - \frac{1}{p}, \frac{4}{p})$. The only case where μ is a disc 5-tuple that satisfies ΣINT with $\frac{1}{2} + \frac{3}{p} > 1$ is $p = 5$, in which case*

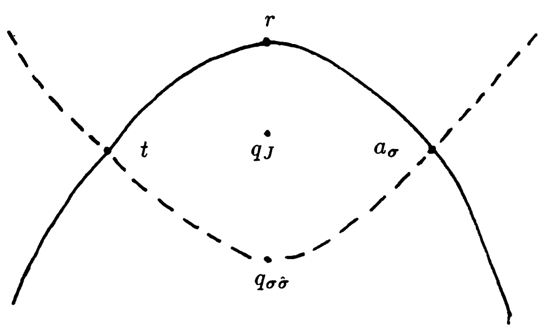
$$\operatorname{vol}(B^+/\Gamma_{\mu\Sigma}) = \pi^2 \left[\frac{(p-4)^2}{3p^2}\right].$$

Proof. Here we have the same configuration of lines in Q_{st} as in Theorem 5.6 except that the lines L_{i5}, $i = 1, 2, 3, 4$ are blown down. That is, the configuration in Q_{st} (where the l_i are again shown as dotted lines)



becomes in the quotient Q_{st}/Σ_4

(5.8)



The proof follows exactly as the others. The orders of the decomposition groups are listed below since the points are exactly like ones previously discussed.

r	$\dfrac{24p^2}{(6-p)^2}$
t	$2p^2$
a_σ	$2p$
q_J	3
$q_{\sigma\hat{\sigma}}$	4

Now we have

$$\begin{aligned}\chi(Y) = m \cdot \Bigg[5 - \left(1 - \frac{(6-p)^2}{24p^2}\right) - \left(1 - \frac{1}{2p^2}\right) - \left(1 - \frac{1}{2p}\right) \\ - \left(1 - \frac{1}{3}\right) - \left(1 - \frac{1}{4}\right) + \left(1 - \frac{1}{p}\right) + \left(1 - \frac{1}{2}\right)\Bigg] \\ = m \cdot \left[\frac{(p-4)^2}{8p^2}\right].\end{aligned}$$

Therefore we have

$$\mathrm{vol}(B^+/\Gamma_{\mu\Sigma}) = \frac{8\pi^2}{3} \cdot \left[\frac{(p-4)^2}{8p^2}\right] = \pi^2 \left[\frac{(p-4)^2}{3p^2}\right].$$

This completes the proof and §5.

6. Isomorphisms among monodromy groups in PU(1,2). These theorems were discovered during work on Mostow's conjecture. The similarities between the orders of reflections in the groups suggested various isomorphisms. The computer investigation revealed that in many instances isomorphisms could indeed be constructed. The first is a more general statement of Theorem 3.1.

THEOREM 6.1. *For each* $t \in \{0, \pm\frac{1}{30}, \pm\frac{1}{18}, \pm\frac{1}{12}, \pm\frac{5}{42}, \pm\frac{1}{6}, \pm\frac{7}{30}, \pm\frac{1}{3}\}$ *there is a monomorphism*:

$$\Gamma_{\frac{12}{1+6t}, \frac{1}{4}+\frac{t}{2}} \hookrightarrow \Gamma_{3,t}$$

which is an isomorphism whenever 3 *does not divide* $\frac{12}{1-6t}$.

Proof. This theorem can also be stated in terms of the parameter p as follows. For each $p \in \{4, 5, 6, 7, 8, 9, 10, 12, 15, 18, 24, 42, \infty, -30, -12\}$ there is a monomorphism:

$$\Gamma_{p, \frac{1}{p}+\frac{1}{6}} \longrightarrow \Gamma_{3, \frac{2}{p}-\frac{1}{6}}$$

$$\{R_i\}_{i=1,2,3} \longrightarrow \{A_j\}_{j=2,1,3}$$

which is an isomorphism only when 3 does **not** divide $\frac{6p}{p-6}$.

Observe that in $\Gamma_{\frac{12}{1+6t}, \frac{1}{4}+\frac{t}{2}}$ we have

$$\langle e_i, e_{i+1}\rangle = -\alpha\phi = \frac{-e^{\frac{\pi i}{3}\left(\frac{1}{4}+\frac{t}{2}\right)}}{2\sin\frac{\pi(1+6t)}{12}}.$$

We now show that we can map R_1, R_2, R_3 of $\Gamma_{\frac{12}{1+6t}, \frac{1}{4}+\frac{t}{2}}$ to A_2, A_1, A_3, respectively in $\Gamma_{3,t}$. Using (3.5) and noting that in $\Gamma_{3,t}$

$$\eta^2 = -\overline{\eta} \quad \text{and} \quad 1 + \eta^2 = \eta,$$

we find

$$\frac{\langle a_2\,,\,a_1\rangle}{(\langle a_2\,,\,a_2\rangle\langle a_1\,,\,a_1\rangle)^{\frac{1}{2}}} = \frac{-3\alpha\overline{\phi} - 2\eta i\overline{\phi} - 2\alpha\overline{\eta} i\phi^2 + \alpha\eta^2\overline{\phi} - \overline{\eta}^2\phi^2}{1 + \frac{i}{\eta-\overline{\eta}}\left(\eta^2\overline{\phi}^3 + \overline{\eta}^2\phi^3\right)}$$

$$= \frac{-i(\overline{\phi} + \eta^2\overline{\phi} + \overline{\eta} i\phi^2 - i\phi^2)}{\eta - \overline{\eta} - \overline{\eta} i\overline{\phi}^3 - \eta i\phi^3} = \frac{-i(\eta\overline{\phi} - \eta i\phi^2)}{\eta(1 - i\phi^3) - \overline{\eta}(1 + i\overline{\phi}^3)}$$

$$= \frac{-i\eta\overline{i}^{\frac{1}{2}}\phi^{\frac{1}{2}}(\overline{\phi}^{\frac{3}{2}} i^{\frac{1}{2}} + \phi^{\frac{3}{2}}\overline{i}^{\frac{1}{2}})}{\eta\overline{i}^{\frac{1}{2}}\phi^{\frac{3}{2}}(\overline{\phi}^{\frac{3}{2}} i^{\frac{1}{2}} + \phi^{\frac{3}{2}}\overline{i}^{\frac{1}{2}}) - \overline{\eta} i^{\frac{1}{2}}\overline{\phi}^{\frac{3}{2}}(\phi^{\frac{3}{2}}\overline{i}^{\frac{1}{2}} + \overline{\phi}^{\frac{3}{2}} i^{\frac{1}{2}})}$$

$$= \frac{-i\eta\overline{i}^{\frac{1}{2}}\phi^{\frac{1}{2}}}{\eta\overline{i}^{\frac{1}{2}}\phi^{\frac{3}{2}} - \overline{\eta} i^{\frac{1}{2}}\overline{\phi}^{\frac{3}{2}}} = \frac{-e^{\frac{\pi\imath}{3}\left(\frac{1}{4}+\frac{t}{2}\right)}}{2\sin\pi\left(\frac{1+6t}{12}\right)}$$

as required. Notice that for $t \in \{-\frac{1}{30}\,,\,-\frac{1}{12}\,,\,-\frac{5}{42}\,,\,-\frac{7}{30}\,,\,-\frac{1}{3}\}$ we have that 3 does not divide $\frac{12}{1-6t}$. This is precisely the condition that allows us to solve for the n in the Lemma of §3, and hence Theorem 3.1 proves that

$$\Gamma_{3\,,\,t} \simeq \Gamma_{\frac{12}{1+6t}\,,\,\frac{1}{4}+\frac{t}{2}}$$

for the above values of t. Using the volumes of the fundamental domains computed in §5 we find that the index of $\Gamma_{\frac{12}{1+6t}\,,\,\frac{1}{4}+\frac{t}{2}}$ in $\Gamma_{3\,,\,t}$ in the other cases is either 4 or 12 depending on whether or not $J \in \Gamma_{3\,,\,t}$. A more detailed discussion is given in §7.

Now we turn to a theorem that generalizes the fact proved in **[M-1]** that

$$\Gamma_{5\,,\,\frac{7}{10}} \simeq \Gamma_{5\,,\,\frac{1}{2}}.$$

It gives an isomorphism between a class of groups where $\Sigma = S_3$, the permutation group on three letters and the $\{A_i\}_{i=1\,,\,3}$ are reflections of order 2 and a class of groups where $\Sigma = S_4$ and which has no obvious reflections of order 2. Given integers $\pi\,,\,\rho\,,\,\sigma$ set

$$\mu(\pi\,,\,\rho\,,\,\sigma) = \left(\frac{1}{2} - \frac{1}{\pi},\ \frac{1}{2} - \frac{1}{\pi},\ \frac{1}{2} - \frac{1}{\pi},\ \frac{1}{2} + \frac{1}{\pi} - \frac{1}{\rho},\ \frac{1}{2} + \frac{1}{\pi} - \frac{1}{\sigma}\right).$$

Let $\Gamma_{\mu(\pi\,,\,\rho\,,\,\sigma)}$ be the corresponding group and $\Gamma_{\mu\Sigma(\pi\,,\,\rho\,,\,\sigma)}$ the extension defined in §2 coming from the maximal subset where the μ_i agree.

THEOREM 6.2. *For each* $p \in \{5\,,\,6\,,\,7\,,\,8\,,\,9\,,\,10\,,\,12\,,\,18\}$ *there is an isomorphism*

$$\Gamma_{\mu\Sigma(p\,,\,2\,,\,-p)} \simeq \Gamma_{\mu\Sigma(p\,,\,\frac{p}{2}\,,\,\frac{2p}{p-6})}.$$

Proof. Writing out the five μ_i in each case we get

$$\mu(p\,,\,2\,,\,-p)=\left(\frac{1}{2}-\frac{1}{p}\,,\;\frac{1}{2}-\frac{1}{p}\,,\;\frac{1}{2}-\frac{1}{p}\,,\;\frac{1}{p}\,,\;\frac{1}{2}+\frac{2}{p}\right)$$

and

$$\mu(p\,,\,\frac{p}{2}\,,\,\frac{2p}{p-6})=\left(\frac{1}{2}-\frac{1}{p}\,,\;\frac{1}{2}-\frac{1}{p}\,,\;\frac{1}{2}-\frac{1}{p}\,,\;\frac{1}{2}-\frac{1}{p}\,,\;\frac{4}{p}\right).$$

Notice that in going from the first to the second we've taken enough off μ_5 to make μ_4 equal the first three $\mu_1=\mu_2=\mu_3$. This is significant because the isomorphism is *not* among the Γ_μ *nor* the $\Gamma_{p,t}$. However, note that in the case of $\mu(p\,,\,2\,,\,-p)$ we have the corresponding $\Gamma_{p,t}\simeq\Gamma_{\mu\Sigma}$ which is generated by the R_i. Recalling the discussion of the braid group in §2, we want to map the $R_i\in\Gamma_{\mu(p,2,-p)}$, coming from turning $i-1$ around $i+1$, $i=1,2,3$ to the *square roots* of the $A_j\in\Gamma_{\mu(p,\frac{p}{2},\frac{2p}{p-6})}$, which lie in $\Gamma_{\mu\Sigma(p,\frac{p}{2},\frac{2p}{p-6})}$ and come from turning j around 4, $j=2,1,3$, respectively. For $\mu(p\,,\,2\,,\,-p)$ we have that (refer to (2.1), (2.2), and (2.10))

$$\langle e_1\,,\,e_2\rangle=\langle e_2\,,\,e_3\rangle=\langle e_3\,,\,e_1\rangle=-\alpha\phi=\frac{-e^{\frac{\pi i}{3}\left(\frac{1}{2}+\frac{1}{p}\right)}}{2\sin\frac{\pi}{p}}=\frac{-e^{\frac{\pi i}{6}}e^{\frac{\pi i}{3p}}}{2\sin\frac{\pi}{p}}.$$

Next notice that for $\mu(p\,,\,\frac{p}{2}\,,\,\frac{2p}{p-6})$, $t=\frac{5}{p}-\frac{1}{2}$ and so

$$\phi^3=e^{\pi it}=e^{\pi i\left(\frac{5}{p}-\frac{1}{2}\right)}=-i\eta^5.$$

Replacing each ϕ by expressions in η yields

$$\begin{aligned}\frac{\langle a_2\,,\,a_1\rangle}{(\langle a_2\,,\,a_2\rangle\langle a_1\,,\,a_1\rangle)^{\frac{1}{2}}}&=\frac{\alpha\eta^2\overline{\phi}-3\alpha\overline{\phi}-2\eta i\overline{\phi}-2\alpha\overline{\eta}i\phi^2-\overline{\eta}^2\phi^2}{1+\frac{i}{\eta-\overline{\eta}}(\eta^2\overline{\phi}^3+\overline{\eta}^2\phi^3)}\\&=\frac{-\alpha e^{\frac{\pi i}{6}}[\eta^{\frac{7}{3}}+2\eta^{\frac{1}{3}}+\overline{\eta}^{\frac{5}{3}}]}{1+\eta^2+1+\overline{\eta}^2}=-\alpha e^{\frac{\pi i}{6}}\eta^{\frac{1}{3}}=\frac{-e^{\frac{\pi i}{6}}e^{\frac{\pi i}{3p}}}{2\sin\frac{\pi}{p}}\end{aligned}$$

as required. This proves that $\Gamma_{\mu\Sigma(p,2,-p)}$ injects into $\Gamma_{\mu\Sigma(p,\frac{p}{2},\frac{2p}{p-6})}$. Consideration of the volumes of the fundamental domains computed in §5 (and listed in §7) indicates that this is an isomorphism. It is an isomorphism at the Γ_μ level only when $\Gamma_\mu\simeq\Gamma_{\mu\Sigma}$ (i.e. when the corresponding p is odd). Notice that for both $5,\frac{1}{2}$ and $5,\frac{7}{10}$ it is the case that $\Gamma_\mu\simeq\Gamma_{\mu\Sigma}\simeq\Gamma_{p,t}$.

7. Summary of specific information about Γ_μ and $\Gamma_{p,t}$. Here we give the specific information in dimension 2 mentioned in previous sections. This includes lists of lattices in the μ and p, t parameters, and the volumes of the fundamental domains for the lattices in PU(1,2). The following is the list of lattices given in **[M-1]**. For each p, t, the corresponding μ is given, where d is the denominator of the μ_i. The orders of the elements A_i, A_i', and B_i' are ρ, σ, and τ respectively. $\operatorname{Aut}\Omega$ indicates whether or not J is in $\Gamma_{p,t}$.

RCP	d	$d\mu_1$	$d\mu_4$	$d\mu_5$	ρ	σ	τ	p	t	$\operatorname{Aut}_\Gamma\Omega$	DM
1	12	2	9	9	12	12	-2	3	0	1	
2	30	5	22	23	10	15	-2	3	$\frac{1}{30}$	3	
3	18	3	13	14	9	18	-2	3	$\frac{1}{18}$	1	
4	24	4	17	19	8	24	-2	3	$\frac{1}{12}$	3	
5	42	7	29	34	7	42	-2	3	$\frac{5}{42}$	3	
6	6	1	4	5	6	∞	-2	3	$\frac{1}{6}$	1	
7	30	5	19	26	5	-30	-2	3	$\frac{7}{30}$	3	
8	12	2	7	11	4	-12	-2	3	$\frac{1}{3}$	3	
9	10	3	5	6	5	10	-10	5	$\frac{1}{10}$	3	
10	20	6	9	13	4	20	-10	5	$\frac{1}{5}$	3	
11	30	9	11	22	3	-30	-10	5	$\frac{11}{30}$	1	
12	10	3	2	9	2	-5	-10	5	$\frac{7}{10}$	3	
13	8	2	5	5	8	8	-4	4	0	3	10
14	12	3	7	8	6	12	-4	4	$\frac{1}{12}$	1	22
15	20	5	11	14	5	20	-4	4	$\frac{3}{20}$	3	26
16	4	1	2	3	4	∞	-4	4	$\frac{1}{4}$	3	3
17	12	3	5	10	3	-12	-4	4	$\frac{5}{12}$	1	23

Now we give the list of lattices satisfying INT in dimension 2 from **[DM]**.

DM	d	$d\mu_1$	$d\mu_2$	$d\mu_3$	$d\mu_4$	$d\mu_5$	ρ	σ	τ	p	t	$\mathrm{Aut}\,\Omega$	RCP
1	3	1	1	1	1	2	3	∞	∞	6	$\frac{1}{3}$	1	
2	4	2	2	2	1	1	4	4	2	∞	0	3	
3	4	1	1	1	2	3	4	∞	-4	4	$\frac{1}{4}$	3	16
4	5	2	2	2	2	2	5	5	5	10	0	3	
5	6	2	2	2	3	3	6	6	∞	6	0	1	
6	6	3	3	3	1	2	3	6	2	∞	$\frac{1}{6}$	1	
7	6	4	3	2	2	1							
8	6	2	2	2	1	5	2	-6	∞	6	$\frac{2}{3}$	3	
9	8	3	3	3	3	4	4	8	8	8	$\frac{1}{8}$	3	
10	8	2	2	2	5	5	8	8	-4	4	0	3	13
11	8	3	3	3	1	6	2	-8	8	8	$\frac{5}{8}$	3	
12	9	4	4	4	2	4	3	9	3	18	$\frac{4}{18}$	1	
13	10	4	4	4	1	7	2	-10	5	10	$\frac{6}{10}$	3	
14	12	5	5	5	4	5	4	6	4	12	$\frac{1}{12}$	3	
15	12	6	5	5	4	4							
16	12	5	5	5	3	6	3	12	4	12	$\frac{3}{12}$	1	
17	12	4	4	4	5	7	4	12	∞	6	$\frac{2}{12}$	3	
18	12	7	6	5	3	3							
19	12	7	7	4	4	2							
20	12	8	5	5	3	3							
21	12	5	5	5	1	8	2	-12	4	12	$\frac{7}{12}$	3	
22	12	3	3	3	7	8	6	12	-4	4	$\frac{1}{12}$	1	14
23	12	3	3	3	5	10	3	-12	-4	4	$\frac{5}{12}$	1	17
24	15	6	6	6	4	8	3	15	5	10	$\frac{4}{15}$	1	
25	18	8	8	8	1	11	2	-18	3	18	$\frac{10}{18}$	3	
26	20	5	5	5	11	14	5	20	-4	4	$\frac{3}{20}$	3	15
27	24	9	9	9	7	14	3	24	8	8	$\frac{7}{24}$	1	

The following is an updated version of the list in [**M-2**] of lattices satisfying Σ INT. All μ not satisfying Σ INT with Γ_μ discrete are added to the end of the list.

d	$d\mu_1$	$d\mu_2$	$d\mu_3$	$d\mu_4$	$d\mu_5$	ρ	σ	τ	p	t	AUT Ω
10	3	3	3	3	8	$\frac{5}{2}$	-10	-10	5	$\frac{1}{2}$	3
20	6	6	9	9	10						
14	5	5	5	5	8	$\frac{7}{2}$	14	14	7	$\frac{3}{14}$	3
18	7	7	7	7	8	$\frac{9}{2}$	6	6	9	$\frac{1}{18}$	1
18	7	7	7	5	10	3	18	6	9	$\frac{5}{18}$	1
6	1	1	2	3	5						
6	1	1	2	4	4						
6	1	1	3	3	4						
10	2	3	3	6	6						
12	2	2	4	7	9						
12	2	2	6	7	7						
18	2	7	7	10	10						
14	5	5	5	2	11	2	-7	14	7	$\frac{9}{14}$	3
18	7	7	7	2	13	2	-9	6	9	$\frac{11}{18}$	3
42	15	15	15	13	26	$\frac{7}{2}$	14	14	7	$\frac{13}{42}$	3
30	13	13	13	7	14	3	10	$\frac{10}{3}$	15	$\frac{7}{30}$	
24	11	11	11	5	10	3	8	$\frac{8}{3}$	24	$\frac{5}{24}$	
42	20	20	20	8	16	3	7	$\frac{7}{3}$	42	$\frac{4}{21}$	
12	7	7	7	1	2	3	4	$\frac{4}{3}$	-12	$\frac{1}{12}$	
30	16	16	16	4	8	3	5	$\frac{5}{3}$	-30	$\frac{4}{30}$	
10	1	1	4	7	7						
12	1	3	5	5	10						
14	3	3	4	9	9						
18	4	5	5	11	11						

The specific relations between Γ_μ, $\Gamma_{\mu\Sigma}$ and $\Gamma_{p,t}$. Mostow has shown that $\Gamma_{p,t}$ is conjugate to a subgroup of $\Gamma_{\mu\Sigma}$. The precise relation among Γ_μ, $\Gamma_{\mu\Sigma}$ and $\Gamma_{p,t}$ is summarized below.

Case 1. If μ satisfies INT (and hence p is even) and $J \in \Gamma_{p,t}$, then

$$\Gamma_\mu \xrightarrow{\text{index } n!} \Gamma_{\mu\Sigma} \simeq \Gamma_{p,t}$$

Case 2. If μ satisfies INT (p even) and $J \notin \Gamma_{p,t}$, then

$$\begin{tikzcd} \Gamma_\mu \arrow[r, "\text{index } n!"] & \Gamma_{\mu\Sigma} \simeq \langle J, \Gamma_{p,t}\rangle \\ & \Gamma_{p,t} \arrow[u, "\text{index } 3"'] \end{tikzcd}$$

Case 3. If μ satisfies ΣINT but not INT (hence p is odd) and $J \in \Gamma_{p,t}$, then

$$\Gamma_\mu \simeq \Gamma_{\mu\Sigma} \simeq \Gamma_{p,t}$$

Case 4. If μ satisfies ΣINT but not INT (p odd) and $J \notin \Gamma_{p,t}$, then

$$\begin{tikzcd} \Gamma_\mu \simeq \Gamma_{\mu\Sigma} \simeq \langle J, \Gamma_{p,t}\rangle \\ \Gamma_{p,t} \arrow[u, "\text{index } 3"'] \end{tikzcd}$$

The following lists give the volumes of the fundamental domains for Γ_μ, $\Gamma_{\mu\Sigma}$, and $\Gamma_{p,t}$ in Cases 1 thru 4. The configuration of lines column, headed "Config. No.", indicates which formula in §5 was used to compute the volume. Consideration of the volumes is used in §6 to compute indices and prove isomorphisms.

Case 1

Config. No.	p, t	Γ_μ	$\Gamma_{\mu\Sigma} \simeq \Gamma_{p,t}$	$\Gamma_{\mu\Sigma_4}$
5.1	$4, 0$	$6 \cdot \frac{\pi^2}{8}$	$\frac{\pi^2}{8}$	
5.1	$4, \frac{3}{20}$	$6 \cdot \frac{11\pi^2}{100}$	$\frac{11\pi^2}{100}$	
5.4	$6, \frac{2}{3}$	$2 \cdot \frac{\pi^2}{9}$	$\frac{1}{3} \cdot \frac{\pi^2}{9}$	
5.3	$8, \frac{1}{8}$	$12 \cdot \frac{\pi^2}{8}$		$\frac{1}{2} \cdot \frac{\pi^2}{8}$
5.4	$8, \frac{5}{8}$	$3 \cdot \frac{\pi^2}{8}$	$\frac{1}{2} \cdot \frac{\pi^2}{8}$	
5.3	$10, 0$	$8 \cdot \frac{\pi^2}{5}$		$\frac{1}{3} \cdot \frac{\pi^2}{5}$
5.4	$10, \frac{6}{10}$	$2 \cdot \frac{\pi^2}{5}$	$\frac{1}{3} \cdot \frac{\pi^2}{5}$	
5.3	$12, \frac{1}{12}$	$4 \cdot \frac{7\pi^2}{18}$		$\frac{1}{6} \cdot \frac{7\pi^2}{18}$
5.4	$12, \frac{7}{12}$	$\frac{7\pi^2}{18}$	$\frac{1}{6} \cdot \frac{7\pi^2}{18}$	
5.4	$18, \frac{10}{18}$	$2 \cdot \frac{13\pi^2}{81}$	$\frac{1}{3} \cdot \frac{13\pi^2}{81}$	

Case 2

Config. No.	p, t	Γ_μ	$\Gamma_{p,t}$	$\Gamma_{\mu\Sigma}$	$\Gamma_{\mu\Sigma_4}$
5.1	$4, \frac{1}{12}$	$2 \cdot \frac{13\pi^2}{36}$	$\frac{13\pi^2}{36}$	$\frac{1}{3} \cdot \frac{13\pi^2}{36}$	
5.2	$4, \frac{5}{12}$	$2 \cdot \frac{\pi^2}{9}$	$\frac{\pi^2}{9}$	$\frac{1}{3} \cdot \frac{\pi^2}{9}$	
5.3	$6, \frac{1}{3}$	$8 \cdot \frac{\pi^2}{9}$	$\frac{\pi^2}{9}$		$\frac{1}{3} \cdot \frac{\pi^2}{9}$
5.3	$8, \frac{7}{24}$	$8 \cdot \frac{11\pi^2}{72}$	$4 \cdot \frac{11\pi^2}{72}$	$\frac{4}{3} \cdot \frac{11\pi^2}{72}$	
5.3	$10, \frac{4}{15}$	$8 \cdot \frac{37\pi^2}{225}$	$4 \cdot \frac{37\pi^2}{225}$	$\frac{4}{3} \cdot \frac{37\pi^2}{225}$	
5.3	$12, \frac{3}{12}$	$8 \cdot \frac{\pi^2}{6}$	$4 \cdot \frac{\pi^2}{6}$	$\frac{4}{3} \cdot \frac{\pi^2}{6}$	
5.3	$18, \frac{4}{18}$	$8 \cdot \frac{13\pi^2}{81}$	$\frac{13\pi^2}{81}$		$\frac{1}{3} \cdot \frac{13\pi^2}{81}$

Case 3

Config. No.	$p\,,\,t$	$\Gamma_\mu \simeq \Gamma_{\mu\Sigma} \simeq \Gamma_{p\,,\,t}$
5.1	$3\,,\,\frac{1}{30}$	$\frac{1}{3}\cdot\frac{37\pi^2}{225}$
5.1	$3\,,\,\frac{1}{12}$	$\frac{1}{3}\cdot\frac{11\pi^2}{72}$
5.1	$3\,,\,\frac{5}{42}$	$\frac{1}{3}\cdot\frac{61\pi^2}{441}$
5.2	$3\,,\,\frac{7}{30}$	$\frac{1}{3}\cdot\frac{16\pi^2}{225}$
5.2	$3\,,\,\frac{1}{3}$	$\frac{1}{3}\cdot\frac{\pi^2}{36}$
5.1	$5\,,\,\frac{1}{10}$	$\frac{1}{3}\cdot\frac{13\pi^2}{25}$
5.1	$5\,,\,\frac{1}{5}$	$\frac{1}{3}\cdot\frac{23\pi^2}{50}$
5.2	$5\,,\,\frac{7}{10}$	$\frac{1}{3}\cdot\frac{\pi^2}{25}$
5.8	$5\,,\,\frac{1}{2}$	$\frac{1}{3}\cdot\frac{\pi^2}{25}$
5.6	$7\,,\,\frac{3}{14}$	$\frac{8}{3}\cdot\frac{\pi^2}{49}$
5.7	$7\,,\,\frac{9}{14}$	$\frac{8}{3}\cdot\frac{\pi^2}{49}$
5.5	$7\,,\,\frac{13}{42}$	$\frac{4}{3}\cdot\frac{61\pi^2}{441}$
5.7	$9\,,\,\frac{11}{18}$	$\frac{16}{3}\cdot\frac{\pi^2}{81}$

Case 4

Config. No.	$p\,,\,t$	$\Gamma_{p\,,\,t}$	$\Gamma_\mu \simeq \Gamma_{\mu\Sigma}$
5.1	$3\,,\,0$	$\frac{\pi^2}{6}$	$\frac{1}{3}\cdot\frac{\pi^2}{6}$
5.1	$3\,,\,\frac{1}{18}$	$\frac{13\pi^2}{81}$	$\frac{1}{3}\cdot\frac{13\pi^2}{81}$
5.2	$5\,,\,\frac{11}{30}$	$4\cdot\frac{16\pi^2}{225}$	$\frac{4}{3}\cdot\frac{16\pi^2}{225}$
5.6	$9\,,\,\frac{1}{18}$	$16\cdot\frac{\pi^2}{81}$	$\frac{16}{3}\cdot\frac{\pi^2}{81}$
5.5	$9\,,\,\frac{5}{18}$	$4\cdot\frac{13\pi^2}{81}$	$\frac{4}{3}\cdot\frac{13\pi^2}{81}$

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