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ON THE ANALYTIC REFLECTION OF A MINIMAL SURFACE

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For a long time it has been known that in a Euclidean space one can reflect a minimal surface across a part of its boundary if the boundary contains a line segment, or if the minimal surface meets a plane orthogonally along the boundary. The proof of this fact makes use of H. A. Schwarz's reflection principle for harmonic functions.

In this paper we show that a minimal surface, as a conformal and harmonic map from a Riemann surface into $\mathbf{R}^3$, can also be reflected analytically if it meets a plane at a constant angle.

THEOREM 1. *Let $\Sigma \subset \mathbf{R}^3$ be a minimal surface with nonempty boundary $\partial\Sigma$ and let Π be a plane. Suppose that $L \subset \Sigma \cap \Pi$ is a C^1 curve, Σ is C^1 along L, and at all points of L the tangent plane to Σ makes a fixed angle $0 < \theta < 90°$ with Π. Then Σ can be analytically extended across L to a minimal surface $\overline{\Sigma}$ satisfying the following properties*:

(i) $\overline{\Sigma} = \Sigma \cup \Sigma^*$, *where Σ^* is the set of all images p^* of $p \in \Sigma$ under an analytic reflection map $*$.*

(ii) *p and p^* are separated by Π in such a way that*

$$\operatorname{dist}(p, \Pi) = \operatorname{dist}(p^*, \Pi).$$

(iii) *The Gauss map $g: \overline{\Sigma} \to \mathbf{C}$ satisfies*

$$\overline{g(p)} \cdot g(p^*) = \left(\tan\frac{\theta}{2}\right)^{-2}.$$

(iv) *$p^* \in \Sigma^*$ is a branch point (geometric) if and only if $p \in \Sigma$ is.*

(v) *The map $*$ is a single-valued immersion if Σ is simply connected and L is connected, or Σ is doubly connected and L is closed.*

(vi) *If $*$ is single-valued, then Σ^* has finite total curvature if and only if Σ does.*

(vii) *If $\partial\Sigma = L$, then $\overline{\Sigma}$ is complete.*

Proof. Let x, y, z be coordinates of $\mathbf{R}^3$ such that $\Pi = \{(x, y, z): z = 0\}$. Since x, y, z are harmonic functions on the minimal surface Σ, one can find conjugate harmonic (possibly multiple-valued) functions $\overline{x}, \overline{y}, \overline{z}$ to x, y, z respectively on Σ. Then

$$u = x + i\overline{x}, \quad v = y + i\overline{y}, \quad w = z + i\overline{z}$$

are holomorphic (possibly multiple-valued) functions on Σ, and

$$du = dx + id\overline{x}, \quad dv = dy + id\overline{y}, \quad dw = dz + id\overline{z}$$

are holomorphic 1-forms on Σ. Introduce $z, \overline{z}$ as conformal parameters on Σ. Then Σ can be recaptured by setting

$$x = \mathrm{Re}\int^{w} du, \quad y = \mathrm{Re}\int^{w} dv, \quad z = \mathrm{Re}\int^{w} dw.$$

From the conjugacy of $\overline{x}, \overline{y}, \overline{z}$ to x, y, z, it follows that

$$du^2 + dv^2 + dw^2 = 0.$$

Define a holomorphic differential ω and a meromorphic function g on Σ by

$$\omega = du - idv, \quad g = \frac{dw}{du - idv}.$$

Then we have

$$\begin{aligned} x &= \mathrm{Re}\int^{w} \frac{1}{2}\left(-g + \frac{1}{g}\right) dw, \\ y &= \mathrm{Re}\int^{w} \frac{i}{2}\left(g + \frac{1}{g}\right) dw, \\ z &= \mathrm{Re}\int^{w} dw. \end{aligned} \tag{1}$$

It is well known that g is exactly the Gauss map of the surface Σ.

Put $-\Sigma = \{(x, y, -z) \colon (x, y, z) \in \Sigma\}$ and define a Riemann surface $\widetilde{\Sigma}$ by $\widetilde{\Sigma} = \Sigma \cup (-\Sigma)$. For any $p = (x, y, z) \in \Sigma$, let $-p = (x, y, -z) \in -\Sigma$. Since $z = 0$ on $\Sigma \cap (-\Sigma)$ $(\supset L)$, we can extend the conformal parameters $z, \overline{z}$ over to $\widetilde{\Sigma}$ (across L) by the usual reflection with respect to Π, that is,

$$z(-p) = -z(p) \quad \text{and} \quad \overline{z}(-p) = \overline{z}(p) \quad \text{for any } -p \in -\Sigma.$$

Hence we see that dw is a well-defined holomorphic 1-form on the Riemann surface $\widetilde{\Sigma}$.

Now note that the constant angle hypothesis implies

$$|g(p)| = \left(\tan\frac{\theta}{2}\right)^{-1} \quad \text{for all } p \in L.$$

In other words, g maps L into a circle in $\mathbf{C}$. Since Σ is C^1 along L and L plays the same role in the Riemann surface $\widetilde{\Sigma}$ as a line does in $\mathbf{C}$, we can extend the Gauss map g holomorphically over to $\widetilde{\Sigma}$ (across L) as follows. Define an extension of g, still called g, by

$$(2) \qquad g(-p) = \left(\tan^2 \frac{\theta}{2} \cdot \overline{g(p)}\right)^{-1}, \qquad -p \in -\Sigma.$$

Clearly g is holomorphic on $-\Sigma$ and continuous on $\widetilde{\Sigma}$. Let $h\colon \mathbf{C} \to \mathbf{C}$ be a linear transformation which maps the circle $|w| = (\tan\frac{\theta}{2})^{-1}$ onto the imaginary axis of $\mathbf{C}$. Then the real part of $h \circ g$ is continuous on $\widetilde{\Sigma}$ and harmonic on Σ and $-\Sigma$. Moreover we have

$$\begin{aligned} \operatorname{Re}[h \circ g(-p)] &= \operatorname{Re}[h \circ g(p)] = 0 \quad \text{for } p \in L, \\ \operatorname{Re}[h \circ g(-p)] &= -\operatorname{Re}[h \circ g(p)] \quad \text{for } -p \in -\Sigma. \end{aligned}$$

Hence by the reflection principle we conclude that $h \circ g$ is holomorphic on $\widetilde{\Sigma}$, and so is g.

Using this extended map g, the extended 1-form dw, and the Weierstrass representation formula (1), we can obtain the extended minimal surface $\overline{\Sigma}$. Here, for any $p \in \Sigma$, p^* is determined by integrating (1) over a contour on $\widetilde{\Sigma}$ from a fixed point to $-p$. In case Σ is multiply connected it may happen that the reflection map $*$ maps $p \in \Sigma$ to infinitely many points $p^* \in \Sigma^*$. Also we should discuss the case where $g(p) = 0$ or ∞. At such a point p, w cannot be a parameter of Σ. However dw and $\frac{1}{g} \pm g$ have a zero and a pole of the same order respectively at $-p$ as well as p. Consequently du and dv are holomorphic at $-p$ and thus Σ^* is well defined in a neighborhood of p^*. This proves conclusion (i).

Conclusion (ii) follows from the symmetry of $-\Sigma$ to Σ and the formula for z in (1).

(2) implies (iii).

Suppose p is a regular point. If the tangent plane to Σ at p is parallel to Π, then $dw = 0$ at p. For this reason, w is not a good conformal parameter near the point p. However, for any conformal parametrization in a neighborhood of p, the metric of the corresponding immersion is, by **[BC]**,

$$ds^2 = \frac{1}{2}(1 + |g|^2)^2 |\omega|^2 = \frac{1}{2}(|g| + |g|^{-1})^2 |dw|^2 .$$

Hence the ratio between the metrics at p and $-p$ is given by

$$\begin{aligned}\frac{ds^2(-p)}{ds^2(p)} &= \frac{\frac{1}{2}(\tan^{-2}\frac{\theta}{2}\cdot|g|^{-1}+\tan^2\frac{\theta}{2}\cdot|g|)^2|dw|^2}{\frac{1}{2}(|g|+|g|^{-1})^2|dw|^2} \\ &= \left(\frac{\tan^2\frac{\theta}{2}\cdot|g|+\tan^{-2}\frac{\theta}{2}\cdot|g|^{-1}}{|g|+|g|^{-1}}\right)^2.\end{aligned}$$

Note here that this ratio depends not on the parametrization of Σ but on the geometry of Σ. Furthermore one can easily show that

$$\begin{aligned}(3)\qquad & 0<\min\left(\tan^2\frac{\theta}{2},\tan^{-2}\frac{\theta}{2}\right) \\ &\leq \frac{ds(-p)}{ds(p)} \leq \max\left(\tan^2\frac{\theta}{2},\tan^{-2}\frac{\theta}{2}\right)<\infty.\end{aligned}$$

Therefore Σ^* is also regular at p^*. Since $\Sigma=(\Sigma^*)^*$ and $p=(p^*)^*$, we can obtain the converse similarly.

For (v), we note that in either case every contour in $\widetilde{\Sigma}$ is nullhomotopic or homotopic to a contour in Σ and that no forms in formula (1) have real periods on Σ. Hence $*$ is single-valued and so, by (3), an immersion.

To prove (vi), we use a formula for the Gauss curvature of Σ **[BC]**:

$$K=-\left[\frac{4|g'|}{|f|(1+|g|^2)^2}\right]^2.$$

The curvature ratio between p and $-p$ is given by

$$\frac{K(-p)}{K(p)}=\frac{\left[\frac{4|g'|}{\tan^6\frac{\theta}{2}\cdot|g|^3(1+\tan^{-4}\frac{\theta}{2}\cdot|g|^{-2})^2}\right]^2}{\left[\frac{4|g'|}{|g|^{-1}(1+|g|^2)^2}\right]^2}=\frac{\tan^4\frac{\theta}{2}\cdot(1+|g|^2)^4}{(1+\tan^4\frac{\theta}{2}\cdot|g|^2)^4}.$$

Therefore

$$\begin{aligned}& 0<\min\left(\tan^{12}\frac{\theta}{2},\tan^{-4}\frac{\theta}{2}\right) \\ &\leq \frac{K(-p)}{K(p)} \leq \max\left(\tan^{12}\frac{\theta}{2},\tan^{-4}\frac{\theta}{2}\right)<\infty,\end{aligned}$$

and the conclusion follows.

Finally it is not difficult to see that (vii) can be derived from (3). Thus the proof of the theorem is now complete.

COROLLARY. *Let Σ be a complete minimal surface of finite total curvature in $\mathbf{R}^3$. If an end E of Σ meets a plane along ∂E at a constant angle, then Σ is the catenoid.*

Proof. From Theorem 1 it follows that $\overline{E} = E \cup E^*$ is a complete minimal surface of finite total curvature with two ends. $\overline{E}$ must then be the catenoid [L]. Obviously, by the unique continuation property of a minimal surface, we have $\overline{E} = \Sigma$.

Let Σ be a minimal surface in $\mathbf{R}^3$ with Gauss map g. For any real number $0 < r < \infty$, let us denote by Σ_r the minimal immersion of Σ into $\mathbf{R}^3$ defined by the formula

$$\begin{aligned} x &= \operatorname{Re} \int^w \frac{1}{2}\left(-rg + \frac{1}{rg}\right) dw\,, \\ y &= \operatorname{Re} \int^w \frac{i}{2}\left(rg + \frac{1}{rg}\right) dw\,, \\ z &= \operatorname{Re} \int^w dw\,. \end{aligned}$$

Then we see that every minimal surface can be deformed into a 1-parameter family of minimal surfaces and that this deformation preserves the z-coordinate and multiplies the Gauss map by r.

THEOREM 2. *Assume $\Sigma \subset \mathbf{R}^3$ is a minimal surface with nonempty boundary $\partial\Sigma$ which makes a constant angle θ with a plane Π along $\partial\Sigma \cap \Pi$.*

(i) *For any real number $0 < r < \infty$, the minimal surface Σ_r makes a constant angle $\phi = 2\tan^{-1}(\frac{1}{r}\tan\frac{\theta}{2})$ with Π along $\partial\Sigma_r \cap \Pi$.*

(ii) *There exists a positive real number s such that the minimal surface Σ_s meets Π orthogonally along $\partial\Sigma_s \cap \Pi$, and the analytic extension $\overline{\Sigma}$ of Σ is the same as $(\Sigma_s \cup (\Sigma_s)^*)_{1/s}$, where $(\Sigma_s)^*$ is the usual reflection (mirror image) of Σ_s with respect to Π.*

Proof. (i) By hypothesis, $|g(p)| = (\tan\frac{\theta}{2})^{-1}$ for all $p \in \partial\Sigma \cap \Pi$. Then

$$|rg(p)| = r\left(\tan\frac{\theta}{2}\right)^{-1} = \left(\tan\frac{\phi}{2}\right)^{-1},$$

where $\phi = 2\tan^{-1}(\frac{1}{r}\tan\frac{\theta}{2})$. Since the deformation of Σ into Σ_r

preserves the z-coordinate and multiplies the Gauss map by r, Σ_r meets Π along $\partial\Sigma_r \cap \Pi$ at the constant angle ϕ.

(ii) Let s be the positive real number satisfying

$$2\tan^{-1}\left(\frac{1}{s}\tan\frac{\theta}{2}\right) = 90^\circ .$$

Then Σ_s meets Π orthogonally. Clearly we have

$$(\overline{\Sigma})_s = \overline{(\Sigma_s)} .$$

Since $\overline{(\Sigma_s)}$ is the union of Σ_s and its mirror image $(\Sigma_s)^*$ with respect to Π, we conclude that

$$\overline{\Sigma} = ((\overline{\Sigma})_s)_{1/s} = (\overline{(\Sigma_s)})_{1/s} = (\Sigma_s \cup (\Sigma_s)^*)_{1/s} .$$

REMARKS. 1. A nice example of the analytic reflection can be seen in the catenoid. Let Π_1, Π_2, and Π_3 be the parallel planes with $\mathrm{dist}(\Pi_1, \Pi_2) = \mathrm{dist}(\Pi_2, \Pi_3)$. Let Σ be the catenoid whose ends are parallel to the Π_i. Then Σ intersects the Π_i along circles at constant angles α_i. Assume $\alpha_2 \neq 90^\circ$ and define D_1, D_3 to be the two bounded components of $\Sigma \sim (\Pi_1 \cup \Pi_2 \cup \Pi_3)$. Then D_3 is the analytic reflection of D_1 with respect to Π_2 and D_1 is that of D_3. If we define D_+, D_- to be the components of $\Sigma \sim \Pi_2$, then $D_+ = (D_-)^*$ and $D_- = (D_+)^*$.

2. Embeddedness of Σ does not necessarily imply that of Σ^*.

3. If the tangent plane to Σ at p is parallel to Π, so is the tangent plane to Σ^* at p^*. This is clear in view of Theorem 1(iii).

4. Given an angle $0 < \theta < 90^\circ$, two points p_1, p_2 on Π, and a curve $\Gamma \subset \mathbf{R}^3$ from p_1 to p_2, one can construct an area minimizing surface Σ with the fixed boundary Γ and a free boundary $L \subset \Pi$ along which Σ meets Π at the angle θ as follows. Let Γ_1 be the line segment on Π from p_2 to p_1. We regard Γ, Γ_1 as 1-dimensional sets with orientation, i.e., 1-currents. Let S be a surface with $\partial S = \Gamma \cup \Gamma_S$, $\Gamma_S \subset \Pi$. Give S and Γ_S orientations, S is then called a 2-current, in such a way that $\partial S = \Gamma - \Gamma_S$. As sets, Γ_1 and Γ_S bound a planar domain $D \subset \Pi$ with $\partial D = \Gamma_1 \cup \Gamma_S$. Giving suitable orientations to each component of D, we can make D into a 2-current such that $\partial D = \Gamma_1 + \Gamma_S$. Let us fix an orientation of the plane Π. Then D, as a set, is divided into two disjoint domains D_1, D_2 such that D_1 and D_2 with the orientation inherited from Π can be thought of as 2-currents, and

$$D = D_1 - D_2 .$$

Now we define $\widetilde{A}(S)$, the *modified area* of S, by

$$\widetilde{A}(S) = \mathrm{Area}(S) + \cos\theta[\mathrm{Area}(D_1) - \mathrm{Area}(D_2)].$$

Let $\mathscr{F}$ be the family of all 2-currents S such that $\partial S - \Gamma$ is a 1-current on Π. Then it is not difficult to see that $-\infty < \inf\{\widetilde{A}(S) : S \in \mathscr{F}\}$ and therefore we can find a modified area minimizing current Σ. Σ, as a set, is a desired minimal surface, and by [T] it is Hölder continuously differentiable up to its free boundary. Thus we can analytically extend Σ across its free boundary $\partial\Sigma \sim \Gamma$ to obtain the *θ-reflection* Σ^* of Σ with respect to Π.

Open problems. 1. Is it possible to extend Theorem 1 to the case of a constant mean curvature surface in $\mathbf{R}^3$ or a minimal hypersurface in $\mathbf{R}^n$? It is well known that the answer is yes if a constant mean curvature surface (a minimal hypersurface respectively) meets a plane (a hyperplane respectively) orthogonally.

2. As a generalization of Corollary, is it true that if a complete constant mean curvature surface Σ of finite topological type intersects a plane at a constant angle $\neq 90°$, then Σ is a Delaunay's surface?

3. Given a compact convex body U in $\mathbf{R}^3$, one can construct a minimal disk D in U which makes a constant contact angle θ with the convex boundary ∂U? Grüter and Jost **[GJ]** solved the problem affirmatively when $\theta = 90°$.

4. Most complete minimal surfaces are known to have at least one plane of symmetry. However, some complete immersed minimal surfaces of genus zero constructed by H. Karcher do not have a plane of symmetry. Nevertheless, given a complete minimal surface in $\mathbf{R}^3$, can one find a plane which intersects the minimal surface at a constant angle?

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