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SKEINS AND HANDLEBODIES

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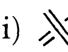
The Temperley-Lieb algebra is used to find a base for the vector space that is associated to a closed surface by the Topological Quantum Field Theory corresponding to the original Jones polynomial invariant.

1. Introduction. The Kauffman linear skein $\mathbb{S}M$ of an oriented 3-manifold M, that has a (possibly empty) finite collection of (framed) points specified in its boundary, is defined as follows. Throughout, A will be a fixed complex number later to be chosen to be a primitive $4r$th root of unity (though it is equally possible to work with the ring of Laurent polynomials in A, quotiented by the ideal generated by a cyclotomic polynomial). $\mathbb{S}M$ is the vector space of formal linear sums of isotopy classes of framed links in M of disjoint simple closed curves and arcs that agree with the specification in ∂M, quotiented by the following relations.

(i)  $= A$  $+ A^{-1}$ 

(ii) $L \cup U = (-A^{-2} - A^{2})L$.

In (i) a framing on a curve is depicted by a parallel to the curve, and in (ii) U is a zero-framed unknotted component in a ball disjoint from the link L. It is often convenient to project M to some surface F (for example, S^3 less two points projects to S^2, a handlebody projects to a disc-with-holes) and then $\mathbb{S}M$ is interpreted as the linear skein $\mathbb{S}F$ of link diagrams in F as in [5], [7], [10], the framings being determined by parallel curves in F. In particular the nth Temperley-Lieb algebra is the Kauffman skein of the ball with two sets of n points specified on its boundary. It is convenient to consider that via link diagrams in a rectangle with n specified points on the left edge, n points on the right edge, the product in the algebra coming from juxtaposition of the rectangles. Now, it is clear that $\mathbb{S}S^3 = \mathbb{C}$; in fact a zero-framed link corresponds in $\mathbb{S}S^3$ to its Jones polynomial evaluated at $t = A^{-4}$. Suppose that M is embedded in S^3, that M' is the closure of $S^3 - M$ and that M and M' have the same specified framed points in their common boundary. There is then a natural bilinear map

$$\mathbb{S}M \times \mathbb{S}M' \to \mathbb{C}$$

induced by the operation of taking framed links L and L' in M and M' and regarding $L \cup L'$ as an element of $\mathbb{S}S^3$. Such bilinear forms will be used in the usual way to associate to any element X of $\mathbb{S}M$ an element X^* of the dual of $\mathbb{S}M'$. This idea was considered, for the Temperley-Lieb algebra, in some detail in **[6]** where it was noted that the bilinear form could become degenerate when A was a root of unity.

The $\mathrm{Sl}_q(2, \mathbb{C})$ invariant introduced by Witten **[12]** for a closed 3-manifold "at a certain level" (here to be interpreted as "at a $4r$th primitive root of unity A"), was in **[7]** and **[8]** shown to be essentially an element of $\mathbb{S}S^3$ (*indirectly*) associated to any framed link in S^3 that produces the manifold by means of surgery. Atiyah and Witten **[1]** laid down the axioms for a "Topological Quantum Field Theory" within the framework of which it might be desirable to view quantum invariants. That involves the association of a vector space with every closed oriented surface and an element of that space to every oriented 3-manifold bounded by the surface. Vogel **[11]**, paying due regard to framings of manifolds, explained how the relevant Topological Quantum Field Theory for the $\mathrm{Sl}_q(2, \mathbb{C})$ invariant could be viewed in the context of the combinatorial methods of **[7]** and **[8]**. If a surface is regarded as the common boundary of a 3-manifold M in S^3, and the closure of its complement M' as already considered, then the vector space to be associated to that surface is the quotient of $\mathbb{S}M$ by the kernel of the map $X \mapsto X^*$. The purpose of this paper is briefly to give a base for that quotient space, thus in principle determining its dimension, and thus giving, from this viewpoint (as explained in **[1]**), an approach to the $\mathrm{Sl}_q(2, \mathbb{C})$-invariant of the product of a closed surface and a circle.

2. Using the Temperley-Lieb algebra. In what follows diagrams represent elements of the (Kauffman) linear skein of link diagrams in a rectangle with n specified points on the left edge and n' on the right edge. An integer i beside an arc will denote the intended presence of i parallel copies of that arc. When $n = n'$ this skein is the Temperley-Lieb algebra TL_n. Recall (for example from **[7]** or **[8]**) that this is generated, as an algebra, by elements 1, $e_1, e_2, \ldots, e_{n-1}$ where 1 is n arcs going straight from the left side of the rectangle to the right, and that e_i is the same except that the ith arc doubles back to the

$(i+1)$th point on the left edge, the ith and $(i+1)$th points on the right edge being connected similarly. For generic values of A, TL_n contains a central idempotent $f^{(n)}$ with the properties that $f^{(n)}e_i = 0$ for all i, $f^{(n)}f^{(n)} = f^{(n)}$ and $1 - f^{(n)}$ belongs to, and is indeed the identity of, the ideal generated by $\{e_1, e_2, \dots, e_{n-1}\}$. In the diagrams that follow a small square will denote the presence of an $f^{(n)}$ the relevant value of n being deducible from the labels on the strings entering the square; this follows the convention of [2]. The number Δ_n will be defined by

$$\Delta_n = (-1)^n (A^{2(n+1)} - A^{-2(n+1)})/(A^2 - A^{-2}),$$

this being characterised by the Chebyschev recurrence relation

$$\Delta_{n+1} = \Delta_1 \Delta_n - \Delta_{n-1}$$

where $\Delta_1 = -A^{-2} - A^2$ and $\Delta_0 = 1$. The inductive defining formula for $f^{(n)}$ is shown in Figure 1, $f^{(1)}$ being the identity in TL_1.

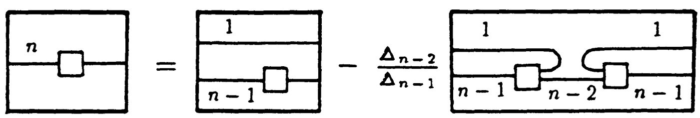


FIGURE 1

The formulae depicted in Figure 2 are immediately deducible (the third by induction on i).

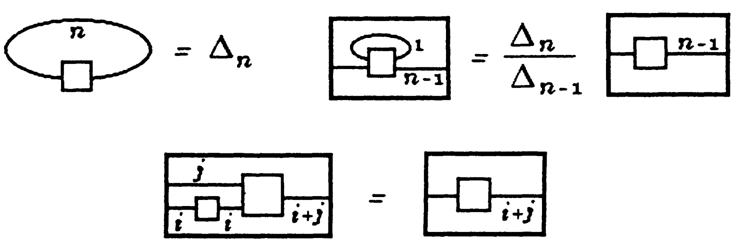


FIGURE 2

Note that when A is chosen to be a primitive $4r$th root of unity, $\Delta_n \neq 0$ when $n \leq r-2$, $\Delta_n = 0$ when $n = r-1$ and $f^{(n)}$ is not defined in $n \geq r$.

Consider the diagram shown in Figure 3.

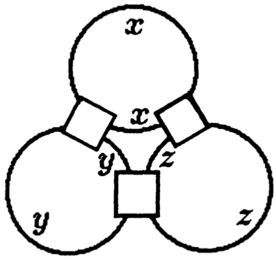


FIGURE 3

It consists of x parallel copies of a circle, y of another circle and z of a third with $f^{(x+y)}$, $f^{(y+z)}$ and $f^{(z+x)}$ inserted as shown. Let $\Gamma(x, y, z)$ be the element of $\mathbb{S}S^2$ that this diagram represents. It will be important to know when $\Gamma(x, y, z)$ is and is not zero. This element is thus calculated in the following which serves also as an exercise in the use of the Temperley-Lieb algebra (the result of the lemma is inherent in [4] and implied in [2]; a simultaneously derived version of this proof appears in [9]). Here $\Delta_n!$ denotes $\Delta_n \Delta_{n-1} \Delta_{n-2} \cdots \Delta_1$, this being interpreted as 1 if n is -1 or zero.

LEMMA 1.

$$\Gamma(x, y, z) = (\Delta_{x+y+z}!\Delta_{x-1}!\Delta_{y-1}!\Delta_{z-1}!)/(\Delta_{y+z-1}!\Delta_{z+x-1}!\Delta_{x+y-1}!).$$

Proof. Consider the equations depicted in Figure 4. The first line follows from the defining relation in Figure 1 (together with $f^{(z)}e_{z-1} = 0$), the second line follows by iterating the first line (and using the third equality of Figure 2).

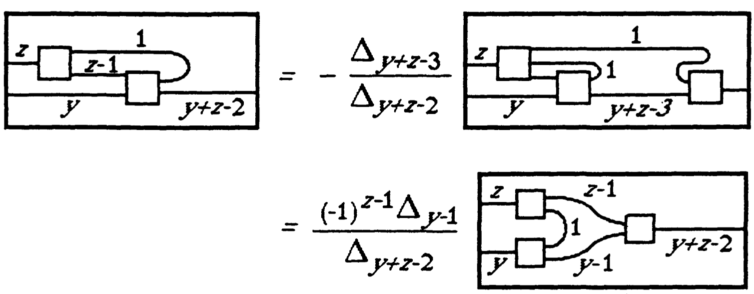


FIGURE 4

Next, the defining relation in Figure 1 followed by a double application of Figure 4 produces the identity of Figure 5.

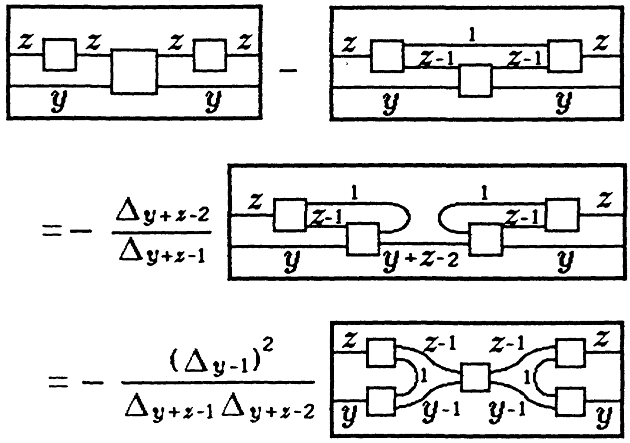


FIGURE 5

Now apply the identity of Figure 5 to the required object in Figure 3 using the second and third formula of Figure 2. The following recurrence relation results:

$$\Gamma(x, y, z) = \Gamma(x, y, z-1)\Delta_{x+z}/\Delta_{x+z-1}$$
$$- \Gamma(x+1, y-1, z-1)(\Delta_{y-1})^2/(\Delta_{y+z-1}\Delta_{y+z-2}).$$

This is ready for a verification of the given formula by induction on z. That formula is clearly true when $z = 0$ and inserting it into this recurrence relation reduces the proof to a demonstration of the equality

$$\Delta_{x+y+z}\Delta_{z-1} = \Delta_{x+z}\Delta_{y+z-1} - \Delta_{y-1}\Delta_x.$$

The truth of this can however easily be checked either directly from the formula for Δ_n or using a double induction on

$$\Delta_{x+y} = \Delta_x\Delta_y - \Delta_{x-1}\Delta_{y-1}.$$

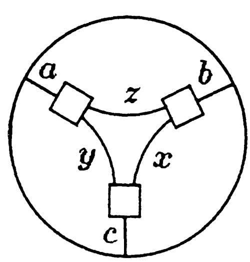


FIGURE 6

Consider the triad $T_{a,b,c}$ of Figure 6, introduced by Kauffman [2]. It is an element of $\mathbb{S}D$, the linear skein of the disc D, where now D

has an even number $a+b+c$ of specified boundary points partitioned into three subsets as shown. $T_{a,b,c}$ has the idempotents $f^{(a)}$, $f^{(b)}$, and $f^{(c)}$ placed just inside the disc adjacent to the three subsets. It is required that the integers x, y and z defined by $a=y+z$, $b=z+x$ and $c=x+y$ should all be non-negative. Suppose that D' is the disc complementary to D in S^2 with the same specified boundary points. By means of the bilinear form $\mathbb{S}D\times\mathbb{S}D'\to\mathbb{C}$, $T_{a,b,c}$ corresponds to the element $T^*_{a,b,c}$ of the dual of $\mathbb{S}D'$.

LEMMA 2. *Let A be a primitive $4r$th root of unity. Then $T^*_{a,b,c}$ is non-zero if and only if $a+b+c\le 2(r-2)$.*

Proof. $\mathbb{S}D'$ has a base consisting of all diagrams in D' with no crossing. However, for all but one of these diagrams there is an arc from a point of one of the three specified subsets (for example that with a points) to another point of the same subset. The presence of the idempotent adjacent in $T_{a,b,c}$ to that subset ensures (using $f^{(a)}e_i=0$) that $T^*_{a,b,c}$ annihilates such an element. There remains to consider the base element of $\mathbb{S}D'$ that consists of z arcs from the first boundary subset to the second such subset, x from the second to the third and y from the third to the first. Of course $T^*_{a,b,c}$ maps this element to $\Gamma(x,y,z)$. It follows from Lemma 1 that, as $x+y+z$ increases, this is non-zero until $\Delta_{x+y+z}!=0$ and that occurs when $x+y+z=r-1$.

From now onwards fix A as a primitive $4r$th root of unity.

DEFINITION. A triple (a,b,c) of non-negative integers will be called admissible if $a+b+c$ is even, $a+b+c\le 2(r-2)$, $a\le b+c$, $b\le c+a$ and $c\le a+b$. This will be written $(a,b,c)\in\mathbb{A}$.

Note that the admissibility condition is just that a triple of non-negative integers (x,y,z) should exist as above and that $x+y+z\le r-2$. In such circumstances define $\Theta(a,b,c)=\Gamma(x,y,z)$.

3. Independence in 3-manifolds. Consider now the following situation. Let M_a be a 3-manifold M in S^3 having, as specified framed points in ∂M, a points grouped in a small disc and N other points. These N points do not change in what follows. Let M'_a be the closure of the complement of M_a. Suppose that for each a, $0\le a\le r-2$, $\{X_{i,a}: i\in\mathfrak{I}(a)\}$ is a collection of elements of $\mathbb{S}M_a$, $\mathfrak{I}(a)$ being some indexing set, and that each $X_{i,a}$ has the idempotent $f^{(a)}$ placed adjacent to the a points. Let $Y_{i,a,b,c}$ be obtained from $X_{i,a}$ by adding the triad $T_{a,b,c}$ as in Figure 7 (the 2-dimensional projections will be considered only in a neighbourhood of the a points).

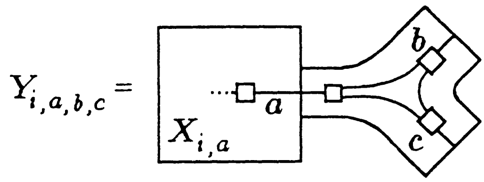


FIGURE 7

LEMMA 3. *Suppose that the* $\{X^*_{i,a}\colon i \in \mathfrak{J}(a)\}$ *are, for each* a, *independent elements of the dual of* $\mathbb{S}M'_a$. *Then, for any* b *and* c, $\{Y^*_{i,a,b,c}\colon (a, b, c) \in \mathbb{A},\ i \in \mathfrak{J}(a)\}$ *are independent.*

Proof. Suppose $\sum_{i,a} \lambda_{i,a} Y^*_{i,a,b,c} = 0$. Then $\sum_{i,a} \lambda_{i,a} Z^*_{i,a,b,c,d} = 0$ where $Z_{i,a,b,c,d}$ is as shown in Figure 8. Note that in Figure 8, *and in the remainder of the paper*, a black dot is used, as in **[2]**, to indicate the presence of a triad of the type of Figure 6.

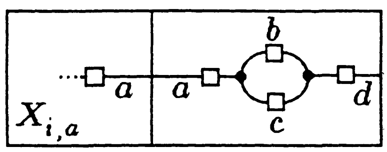


FIGURE 8

However it follows at once (see **[2]**, for example) that

$$Z^*_{i,a,b,c,d} = \delta_{a,d}\Theta(b, c, d)\Delta_d^{-1} X^*_{i,d}.$$

So, letting $d = a$, for each a such that (a, b, c) is admissible,

$$\sum_{i \in \mathfrak{J}(a)} \lambda_{i,a}\Theta(a, b, c)\Delta_a^{-1} X^*_{i,a} = 0.$$

But the admissibility means that $\Theta(a, b, c) \neq 0$ (and $\Delta_a \neq 0$) so $\lambda_{i,a} = 0$ for all a and i.

COROLLARY. *If triads are added together to form a tree*

$$X(a_1, a_2, \ldots, a_s;\ i_1, i_2, \ldots, i_{s-3})$$

as in Figure 9, *then, fixing* $a_1, a_2, \ldots, a_s$ *and allowing* $i_1, i_2, \ldots, i_{s-3}$ *to vary in all ways so that admissibility occurs at each node of the tree, the elements* $\{X(a_1, a_2, \ldots, a_s;\ i_1, i_2, \ldots, i_{s-3})^*\}$ *are independent.*

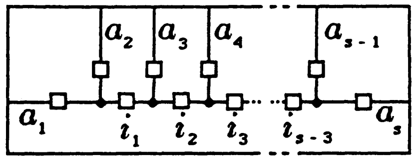


FIGURE 9

Suppose that an even number $a_1 + a_2 + \cdots + a_s$ of points are specified on the boundary of a disc D the points grouped into s subsets each containing a_i points. Then, it is easy to see, in the following way, that the duals of *all* diagrams that have, for all i, the idempotent $f^{(a_i)}$ adjacent to the ith subset, span the same subspace as $\{X(a_1, a_2, \ldots, a_s;\ i_1, i_2, \ldots, i_{s-3})^*\}$. This is because in the Temperley-Lieb algebra TL_n, $1 = f^{(n)} + R$ where R is some sum of products of the e_i. As $f^{(r-1)^*} = 0$ the dual of any diagram cutting a chord of the disc in more than $r-2$ points may be replaced by the sum of duals of diagrams cutting the chord fewer times. For $n \leq r-2$ the dual of a diagram whose arcs cross a given chord n times may be replaced by a sum of duals of diagrams, in one of which the arcs cross the chord n times but are then decorated with an $f^{(n)}$ and in the others the arcs cross fewer than n times. At any stage an inadmissible triad can be neglected as its dual is the zero map. When $s = 4$ this means there are two obvious bases for the same space. The situation is depicted in Figure 10 in which the diagrams represent elements of the dual space to $\mathbb{S}D'$ when D has $a+b+c+d$ specified points in ∂D. This is familiar from **[2]**; the summation is over all i for which (i, b, c) and (i, a, d) are admissible. The elements $\left\{\begin{smallmatrix} a & b & i \\ c & d & j \end{smallmatrix}\right\}$ of this change of base matrix are referred to as (quantised) $6j$-symbols.

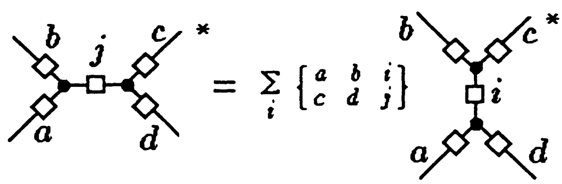


FIGURE 10

There now follows a somewhat technical little lemma that will prove useful.

LEMMA 4. *Suppose that* (b, b, a) *and* (a, c, c) *are admissible triples. Then there is some* j, $0 \leq j \leq r-2$, *such that the element of* $\mathbb{S}S^2$ *depicted by Figure* 11 *is non-zero.*

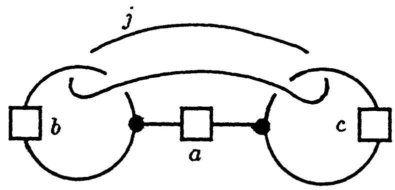


FIGURE 11

Proof. Note that the idempotent $f^{(j)}$ does *not* appear on the j strands. Suppose that, to the contrary, the diagram represents zero for every j. Then, using the above-mentioned base change, the summation shown in Figure 12 is zero.

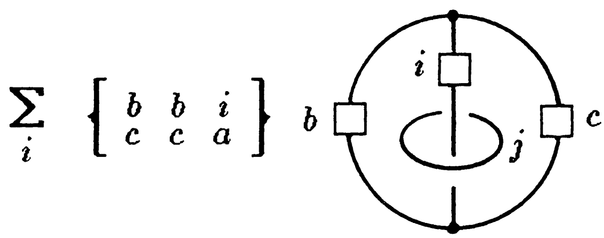


FIGURE 12

However, by Lemma 6 of [7], this means that

$$\sum_i \begin{Bmatrix} b & b & i \\ c & c & a \end{Bmatrix} (\alpha_i)^j \Theta(b, i, c) = 0,$$

where $\alpha_i = -A^{2(i+1)} - A^{-2(i+1)}$. Because A is a primitive $4r$th root of unity $\alpha_0, \alpha_1, \dots, \alpha_{r-2}$ are all distinct. Thus the Vandermonde matrix $(\alpha_i)^j$ is non-singular, and so there can be no non-trivial linear relation between its rows (compare [7]). Thus

$$\begin{Bmatrix} b & b & i \\ c & c & a \end{Bmatrix} \Theta(b, i, c) = 0$$

for all i for which (b, i, c) is admissible. But then $\Theta(b, i, c) \neq 0$ so that $\left\{\begin{smallmatrix} b & b & i \\ c & c & a \end{smallmatrix}\right\}$ is zero for all such i. That however means that a whole row of a change of base matrix is zero, a contradiction.

Note that it follows that the lemma is equally true if the idempotent $f^{(j)}$ is now inserted on the j strands.

COROLLARY. *Let* $U_{a,b,j}$ *be the element of* $\mathbb{S}D$ *in Figure* 13. *Then* $U^*_{a,b,j} = \mu_{a,b,j} T^*_{a,j,j}$, *and for each* a *and* b *with* (a, b, b)

admissible, there exists a j, *with* $0 \leq j \leq r-2$ *and* (a, j, j) *admissible, such that* $\mu_{a,b,j} \neq 0$.

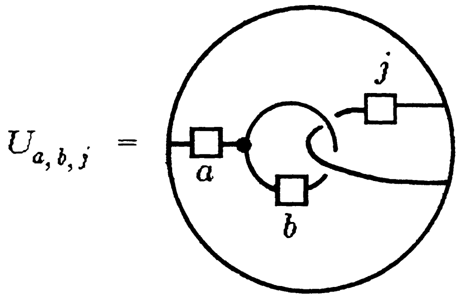


FIGURE 13

Proof. $U^*_{a,b,j}$ and $T^*_{a,j,j}$ both annihilate all but the same single base element of $\mathbb{S}D'$. Thus they differ by multiplication by a scalar $\mu_{a,b,j}$. However by Lemma 4, for each a and b, with (a, b, b) admissible, there is some j for which $U^*_{a,b,j}$ is not the zero map.

Return now to the notation at the beginning of this section in which $\{X_{i,a}: i \in \mathfrak{I}(a)\}$ is a collection of elements of $\mathbb{S}M_a$ each having the idempotent $f^{(a)}$ placed adjacent to the set of a points in ∂M. Suppose that $W_{i,a,b}$ is as shown in Figure 14.

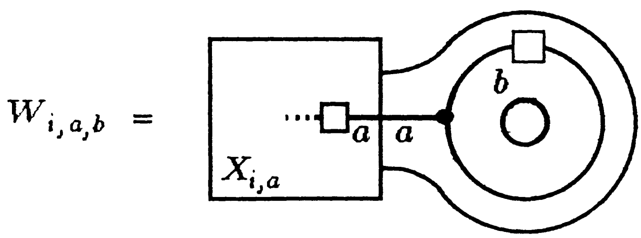


FIGURE 14

Note that $W_{i,a,b}$ is in the linear skein of a new 3-manifold in S^3 obtained by adding a 1-handle to M in a trivial manner.

LEMMA 5. *Suppose that the* $\{X^*_{i,a}: i \in \mathfrak{I}(a)\}$ *are, for each* a, *independent elements of the dual of* $\mathbb{S}M'_a$. *Then* $\{W^*_{i,a,b}: (a, b, b) \in \mathbb{A}, i \in \mathfrak{I}(a)\}$ *are independent.*

Proof. Suppose that $\sum_{i,a,b} \lambda_{i,a,b} W^*_{i,a,b} = 0$, the sum being over all (i, a, b) such that $0 \leq a \leq r-2$, $i \in \mathfrak{I}(a)$ and (a, b, b) is admissible. Then, for all j and k,

$$\sum_{i,a,b} \lambda_{i,a,b} W^{+*}_{i,a,b,j,k} = 0$$

where $W^{+}_{i,a,b,j,k}$ is shown in Figure 15.

$W^{+}_{i,a,b,j,k} =$

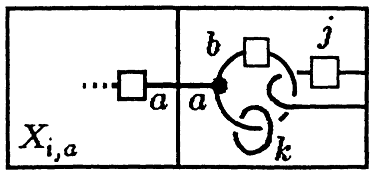


FIGURE 15

Then

$$\sum_{i,a,b} \lambda_{i,a,b} (\alpha_b)^k (X_{i,a} \cup U_{a,b,j})^* = 0$$

where, as before, the $(\alpha_b)^k$ are the elements of a non-singular Vandermonde matrix. Hence, for each b,

$$\sum_{i,a} \lambda_{i,a,b} (X_{i,a} \cup U_{a,b,j})^* = 0.$$

By the corollary to Lemma 4, for each b

$$\sum_{i,a} \lambda_{i,a,b} (X_{i,a} \cup \mu_{a,b,j} T_{a,j,j})^* = 0.$$

Then, by Lemma 3, if (a, j, j) is admissible $\lambda_{i,a,b}\mu_{a,b,j} = 0$. But the corollary to Lemma 4 states that, for each a and b, such a j can be selected so that $\mu_{a,b,j} \neq 0$. Hence $\lambda_{i,a,b} = 0$ for all (a, b, i) for which $i \in \mathfrak{I}(a)$ and (a, b, b) is admissible.

THEOREM. *Suppose that M is a handlebody in S^3 (with no specified points in its boundary), and that M', the closure of its complement, is also a handlebody. Let A be a primitive $4r$th root of unity. A base for the quotient of $\mathbb{S}M$ by the kernel of the natural map $\mathbb{S}M \to$* $\mathrm{Hom}(\mathbb{S}M', \mathbb{C})$ *is as follows. Project M to a disc with holes. The duals of all diagrams as shown in Figure* 16 *that have an admissible triad at each node form the required base.*

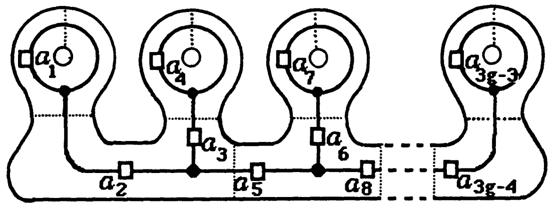


FIGURE 16

Proof. That the given set spans the quotient follows in a way similar to the remarks following Lemma 3: The infinite spanning set consisting of the duals of all diagrams with no crossing and no null-homotopic closed curve can, as there explained, be reduced to a finite set in which no diagram cuts any of the dotted chords of Figure 16 in more than $r-2$ points, and the Temperley-Lieb idempotents can likewise be inserted where the diagrams cross those chords. The only remaining diagrams with non-zero duals are those consisting of admissible triads between the dotted chords. Independence is an immediate consequence of Lemmas 3 and 5.

Note 1. Other bases can be obtained by modifying the above base at two adjacent nodes of Figure 16, using the $6j$-symbols as the change of base matrix as in Figure 10.

Note 2. The idempotents can be removed from all the triads of Figure 16 and the result is still a base (the result is clearly still a spanning set and it has the same number of elements as has the base).

Note 3. If M is a ball the dimension of the quotient space is 1. If M is a solid torus the dimension is $r-1$ and if M has genus 2 the dimension is $(r^3-r)/6$.

Note 4. If one begins with a closed connected orientable surface and specifies in it a collection of simple closed curves that separate the surface into copies of a disc with two holes ('pairs of pants'), spanning those curves with discs, then adding 3-balls, produces a handlebody bounded by the surface. The discs give a decomposition of the handlebody into triad-like pieces ready for the construction of a base of the relevant (quotient) vector space as above.

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