PACIFIC JOURNAL OF MATHEMATICS
Vol. 171, No. 2, 1995

ON H^P-SOLUTIONS OF THE BEZOUT EQUATION

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We obtain a sufficient condition on bounded holomorphic functions g_1, g_2 in the unit disk for the existence of f_1, f_2 in the Hardy space H^p such that $1 = f_1g_1 + f_2g_2$. The sharpness of this condition is also studied.

1. Let $\mathbb{D}$ be the unit disk in the complex plane, $\mathbb{T}$ its boundary. For $1 \leq p \leq \infty$, H^p denotes the Hardy-space of holomorphic functions in $\mathbb{D}$ such that

$$\|f\|_p = \sup_r \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p} < +\infty \qquad p < \infty$$

$$\|f\|_\infty = \sup_{|z|<1} |f(z)| .$$

It is well-known ([**7**, p. 57]) that if $f \in H^p$, the non-tangential maximal function

$$Mf(e^{i\theta}) = \sup\{|f(z)|;\ z \in \Gamma(\theta)\}$$

$\Gamma(\theta)$ being the Stolz angle with vertex $e^{i\theta}$, belongs to $L^p(\mathbb{T})$.

In this paper, given $g_1, g_2 \in H^\infty$, we study the Bezout equation $1 = f_1g_1 + f_2g_2$. Concretely, we are interested in knowing the precise condition on g_1, g_2 so that solutions $f_1, f_2 \in H^p$ exist.

If $|g|^2 = |g_1|^2 + |g_2|^2$, $|f|^2 = |f_1|^2 + |f_2|^2$, it follows from $1 = f_1g_1 + f_2g_2$ that $1 \leq |f|\,|g|$ and hence

$$M(|g|^{-1}) \in L^p(\mathbb{T}) . \tag{C}$$

It can be easily seen that this condition is sufficient if g_1 or g_2 is an interpolating Blaschke product. Nevertheless, we show in Section 2 that it is not sufficient in general. In fact for each $\varepsilon > 0$ we exhibit $g_1, g_2 \in H^\infty$ such that $M(|g|^{-2+\varepsilon}) \in L^p(\mathbb{T})$ but no H^p solutions exist.

In Section 3 we obtain a general sufficient condition implying in particular the following:

Theorem 1. *Assume that for some $\varepsilon > 0$*

$$M\left(|g|^{-2}\,|\log|g||^{2+\varepsilon}\right) \in L^p(\mathbb{T}) .$$

Then there exist $f_1, f_2 \in H^p$ such that $1 = f_1 g_1 + f_2 g_2$.

In Section 4, it is shown that the same method gives the following improvement on the problem considered by Wolff and also by Cegrell in [**4**].

Theorem 2. *Let $f, g_1, g_2 \in H^\infty$ be such that*

$$|f| \leq \frac{|g|^2}{|\log|g||^{2+\varepsilon}} \qquad \textit{for some} \quad \varepsilon > 0 .$$

Then there are $f_1, f_2 \in H^\infty$ such that $f = f_1 g_1 + f_2 g_2$.

The proofs rely essentially on: (a) An L^p-version of Wolff's criteria for the existence of bounded solutions of the $\overline{\partial}$-equation, already used in [**1**]. (b) An improvement of Cegrell's result in [**4**] on gradients of bounded holomorphic functions.

Both theorems hold of course for more than two generators, using the Koszul complex technique as in [**7**, p. 364]. Theorem 1 holds as well in the setting of the unit ball, but some modifications are needed (see [**2**]).

Finally, we mention that similar results to those stated here have been independently obtained by K.C. Lin in [**8**] and [**9**]. The authors thank the referee for pointing this out to them.

2. Before proceeding, we recall that a positive measure μ on $\mathbb{D}$ is called a *Carleson measure* if there exists $K > 0$ such that

$$\mu(\{z : |z - e^{i\theta}| \leq r\}) \leq Kr \qquad e^{i\theta} \in \mathbb{T}, \quad r > 0 .$$

The smallest of such K is called the Carleson norm of μ. Equivalently ([**7**, p. 32]) μ is a Carleson measure if and only if for all functions h in $\mathbb{D}$

$$\iint_{\mathbb{D}} |h| d\mu \leq c \int_0^{2\pi} M(h) d\theta .$$

In some particular cases it is quite easy to see that the condition (C) is sufficient. For instance, if g_1 is a Blaschke product with zeros z_n, the question is obviously equivalent to the interpolation problem

$$f_2(z_n) = \frac{1}{g_2(z_n)}, \quad \text{with} \quad f_2 \in H^p .$$

Indeed, $1 - f_2 g_2$ belongs then to H^p and vanishes on $\{z_n\}$, so $1 - f_2 g_2 = f_1 g_1$, $f_1 \in H^p$. In case g_1 is an interpolating Blaschke product, this interpolation problem has a solution if and only if

$$\sum_n \frac{1}{|g_2(z_n)|^p} (1 - |z_n|) < +\infty ,$$

(see [**10**] and also [**5**]). Let δ_n denote the delta-mass at the point z_n. Since $\sum(1-|z_n|)\delta_n$ is a Carleson measure, (C) implies the above condition.

Next, we give examples showing that condition (C) is far from being sufficient.

Theorem 3. *Given $1 \le p < \infty$ and $\varepsilon > 0$, there exist bounded analytic functions g_1, g_2 with $M(|g|^{-2+\varepsilon}) \in L^p(\mathbb{T})$, such that there exist no $f_1, f_2 \in H^p$ satisfying $f_1 g_1 + f_2 g_2 \equiv 1$.*

Proof. We will denote by $\rho(z,w)$ the pseudo-hyperbolic distance in the unit disc, $\rho(z,w) = |z-w|\,|1-\overline{w}z|^{-1}$, $z, w \in \mathbb{D}$ and $f^{(j)}$ the j-th derivative of a function f. Let N be a natural number such that $(N+1)\varepsilon > 1$.

Let $z_n = 1 - 2^{-n}$, $n \ge 1$, and take an H^∞-interpolating sequence $\{\alpha_n\}$, $0 < \rho(\alpha_n, z_n)$ decreasing to 0, satisfying

$$\sum_n (1-|z_n|)\rho(\alpha_n, z_n)^{-(N+1)p(2-\varepsilon)} < \infty, \tag{1}$$

$$\sum_n (1-|z_n|)\rho(\alpha_n, z_n)^{-(2N+1)p} = \infty. \tag{2}$$

Let B_1 and B_2 be the Blaschke products with zeros $\{z_n\}$ and $\{\alpha_n\}$. From now on, the letter c will denote different constants independent on n. Since B_2 is an interpolating Blaschke product, one has

$$\inf_n \rho(z, \alpha_n) \ge |B_2(z)| \ge c \inf_n \rho(z, \alpha_n), \quad |z| < 1,$$

(see [**7**, p. 404]).

Now as in [**3**] take $g_i = B_i^{N+1}$, $i = 1, 2$. Let I_n be the arc on the unit circle centered at 1 of length $2(1-|z_n|) = 2^{-n+1}$. In estimating $|g(z)|^{-1}$, for $z \in \Gamma(\theta)$, the worst case occurs when z is one of the $\{z_n\}$ or $\{\alpha_n\}$. Since $\rho(\alpha_n, z_n)$ is decreasing, for $e^{i\theta} \in I_n \setminus I_{n+1}$ one has

$$M(|g|^{-1}) \le \frac{c}{\rho(\alpha_{n+1}, z_{n+1})^{N+1}}.$$

Hence, condition (1) implies $M(|g|^{-2+\varepsilon}) \in L^p(\mathbb{T})$. Now, assume there exist $f_1, f_2 \in H^p$ satisfying $f_1 g_1 + f_2 g_2 \equiv 1$. Then,

$$f_2^{(N)}(z_n) = (B_2^{-N-1})^{(N)}(z_n), \quad n \ge 1.$$

Write

$$B_{2,n}(z) = \prod_{k \ne n} \frac{-\overline{\alpha}_k}{|\alpha_k|} \frac{z-\alpha_k}{1-\overline{\alpha}_k z}, \qquad B_2^{-N-1}(z) = h(z)k(z)$$

where $h(z) = (1-\overline{\alpha}_n z)^{N+1} B_{2,n}(z)^{-N-1}$, $k(z) = (z-\alpha_n)^{-N-1}$. Then

$$(B_2^{-N-1})^{(N)}(z) = \sum_{j=0}^{N} \binom{N}{j} h^{(j)}(z) k^{(N-j)}(z) .$$

Using Cauchy's formula on the disk of center z_n and radius $4^{-1}(1-|z_n|)$, one gets

$$|h^{(j)}(z_n)| \leq c \frac{(1-|\alpha_n|)^{N+1}}{(1-|z_n|)^j} \leq c(1-|z_n|)^{N+1-j}$$

and hence

$$\begin{aligned} |h^{(j)}(z_n)|\,|k^{(N-j)}(z_n)| &\leq c\,|z_n-\alpha_n|^{-2N+j-1}(1-|z_n|)^{N+1-j} \\ &\leq c\rho(z_n,\alpha_n)^{-2N+j-1}(1-|z_n|)^{-N} . \end{aligned}$$

For $j=0$, one gets

$$\begin{aligned} |h(z_n)|\,|k^{(N)}(z_n)| &\geq c\,(1-|z_n|)^{N+1}|z_n-\alpha_n|^{-2N-1} \\ &\geq c\rho(\alpha_n,z_n)^{-2N-1}(1-|z_n|)^{-N} . \end{aligned}$$

Therefore, for large n,

$$(3) \qquad |f_2^{(N)}(z_n)| = |(B_2^{-N-1})^{(N)}(z_n)| \geq c\,(1-|z_n|)^{-N}\rho(\alpha_n,z_n)^{-2N-1} .$$

Since $f_2 \in H^p$, the function

$$F(e^{i\theta}) = \left(\int_{\Gamma(\theta)} |f_2^{(N)}(z)|^2(1-|z|)^{2N-2}dm(z)\right)^{1/2}$$

belongs to $L^p(\mathbb{T})$ ([**11**, p. 216]). For $e^{i\theta} \in I_n \setminus I_{n+1}$, since $D_n = \{z \in \mathbb{D} : |z-z_n| < 4^{-1}(1-|z_n|)\} \subset \Gamma(\theta)$, one has

$$\begin{aligned} |F(e^{i\theta})|^2 &\geq \int_{D_n} |f_2^{(N)}(z)|^2(1-|z|)^{2N-2}dm(z) \\ &\geq c\,(1-|z_n|)^{2N}|f_2^{(N)}(z_n)|^2 , \quad e^{i\theta} \in I_n \setminus I_{n+1} . \end{aligned}$$

Using (3) and $F \in L^p(\mathbb{T})$, one gets

$$\infty > \sum (1-|z_n|)\rho(\alpha_n,z_n)^{-(2N+1)p}$$

and this contradicts (2). $\square$

3. In this section we will prove a generalization of Theorem 1 stated in the introduction.

Lemma 1. *If g is holomorphic on $\overline{\mathbb{D}}$ and $0 < p \leq 2$,*

$$\int_0^{2\pi} \{|g(e^{i\theta})|^p - |g(0)|^p\} \frac{d\theta}{2\pi} \leq 4 \int_0^{2\pi} |g(e^{i\theta}) - g(0)|^p \frac{d\theta}{2\pi}.$$

Proof. This is a general statement for a probability measure $d\mu$ on X and measurable φ

$$\int_X |\varphi|^p \, d\mu - \left| \int_X \varphi \, d\mu \right|^p \leq 4 \int_X \left| \varphi - \int_X \varphi \, d\mu \right|^p \, d\mu.$$

First notice that this is trivial for $p \leq 1$ (with constant 1) and that for $p = 2$ there is equality with constant 1, too. In general, and assuming without loss of generality that $\int_X \varphi \, d\mu = 1$, it follows for real φ integrating the inequality

$$|\varphi|^p - 1 \leq 3|\varphi - 1|^p + p(\varphi - 1).$$

For complex-valued $\varphi = \varphi_1 + i\varphi_2$, it follows from $|\varphi|^p \leq |\varphi_1|^p + |\varphi_2|^p$ (for positive φ the inequality holds with constant 1). $\square$

We start with a generalization of a result in [**4**]. Although we need it only for H^∞ functions we state it in full generality, for $BMOA$ functions (see [**7**] for definitions). We denote by $\|g\|_*$ the BMO norm of $g(e^{i\theta})$.

Let $d\mu$ be a positive measure on $[0,1)$ such that $\int_0^1 \frac{d\mu(\alpha)}{\alpha^2} < +\infty$, and write

$$\widetilde{\mu}(x) = \int_0^1 x^\alpha d\mu(\alpha), \quad x > 0.$$

Lemma 2. *If $g \in BMOA$, $\frac{|g'|^2}{|g|^2}\widetilde{\mu}(|g|^2)(1 - |z|^2)$ is a Carleson measure with Carleson norm bounded by $K\|g\|_*$, K depending on μ.*

Proof. We consider the function

$$G(z) = \int_0^1 \frac{|g(z)|^{2\alpha}}{\alpha^2} d\mu(\alpha), \qquad |z| < 1.$$

For $\alpha > 0$, a computation shows that $\Delta|g|^{2\alpha} = 4\alpha^2 |g'|^2 \, |g|^{2\alpha - 2}$ when $g \neq 0$, hence

$$\Delta G = 4|g'|^2 \, |g|^{-2} \widetilde{\mu}(|g|^2).$$

Without loss of generality we can assume that g is holomorphic on $\overline{\mathbb{D}}$. We argue like in [**7**, p. 327]. Let $z_1, \dots, z_N$ be the non-zero zeros in $\mathbb{D}$, and let Ω_ε be the domain $\mathbb{D} \setminus \bigcup_{j=0}^{N} \Delta_j$ where $\Delta_0 = \{|z| \leq \varepsilon\}$, $\Delta_j = \{|z - z_j| \leq \varepsilon\}$, $j = 1, \dots, N$.

By Green's formula applied to the function G in Ω_ε

$$4 \iint_{\Omega_\varepsilon} |g'|^2 \, |g|^{-2} \widetilde{\mu}(|g|^2) \log \frac{1}{|z|} dA(z) = \int_0^{2\pi} G(e^{i\theta}) d\theta$$

$$- \sum_{j=0}^{N} \int_{\partial \Delta_j} \left(\frac{\partial}{\partial n} G \right) \log \frac{1}{|z|} - |G| \frac{\partial}{\partial n} \left(\log \frac{1}{|z|} \right) ds \, .$$

Let $r = |z - z_j|$; then for z close to z_j,

$$|g'(z)|^2 |g(z)|^{-2} \widetilde{\mu}(|g(z)|^2) \leq C r^{-2} \widetilde{\mu}(r^2) \, .$$

Now, the hypothesis on μ translates to

$$\int_0^1 \frac{\widetilde{\mu}(x)}{x} |\log x| dx < +\infty \, .$$

Hence $|g'|^2 |g|^{-2} \widetilde{\mu}(|g|^2) \log \dfrac{1}{|z|}$ is integrable on $\mathbb{D}$. Also, for $|z - z_j| = \varepsilon$

$$|G(z)| \leq c \int_0^1 \frac{\varepsilon^{2\alpha}}{\alpha^2} \, d\mu(\alpha)$$

which tends to 0 when $\varepsilon \to 0$, and

$$|\nabla G(z)| \leq \frac{c}{\varepsilon} \int_0^1 \frac{\varepsilon^{2\alpha}}{\alpha} \, d\mu(\alpha)$$

which also tends to zero when multiplied by $\varepsilon |\log \varepsilon|$. At zero we obtain $-2\pi |G(0)|$ as limit when $\varepsilon \to 0$. Therefore

$$\frac{2}{\pi} \iint_{\mathbb{D}} |g'(z)|^2 \, |g(z)|^{-2} \widetilde{\mu}(|g(z)|^2) \log \frac{1}{|z|} \, dA(z)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \frac{|g(e^{i\theta})|^{2\alpha}}{\alpha^2} \, d\mu(\alpha) \, d\theta - \int_0^1 \frac{|g(0)|^{2\alpha}}{\alpha^2} \, d\mu(\alpha)$$

$$= \int_0^1 \frac{d\mu(\alpha)}{\alpha^2} \int_0^{2\pi} \{ |g(e^{i\theta})|^{2\alpha} - |g(0)|^{2\alpha} \} \frac{d\theta}{2\pi} \leq \quad \text{(by Lemma 1)}$$

$$\leq 4 \int_0^1 \frac{d\mu(\alpha)}{\alpha^2} \int_0^{2\pi} |g(e^{i\theta}) - g(0)|^{2\alpha} \frac{d\theta}{2\pi} \leq \int_0^1 \frac{d\mu(\alpha)}{\alpha^2} \, (c \|g\|_*)^{2\alpha} \, .$$

If ψ_w is an automorphism of the disc, applying this inequality to $g \circ \psi_w$, changing variables in the area integral and using the invariance of the BMO norm we get

$$\sup_{w\in\mathbb{D}} \iint_{\mathbb{D}} |g'(z)|^2 |g(z)|^{-2} \widetilde{\mu}(|g(z)|^2) \frac{(1-|z|^2)(1-|w|^2)}{|1-z\overline{w}|^2}\, dA(z)$$

$$\leq c \int_0^1 \frac{\|g\|_*^{2\alpha}}{\alpha^2}\, d\mu(\alpha) < +\infty\,,$$

and the result follows, by [**7**, p. 239]. □

Taking for μ a delta-mass at ε we get Cegrell's result [**4**] that $|g'|^2|g|^{\varepsilon-2}(1-|z|^2)$ is a Carleson measure. Taking $d\mu(\alpha) = \alpha^{1+\varepsilon} d\alpha$ one gets that

$$\frac{|g'|^2}{|g|^2 |\log|g||^{2+\varepsilon}} (1-|z|^2)$$

is a Carleson measure for every $\varepsilon > 0$.

Next lemma is the L^p-version of Wolff's criteria for bounded solutions of the $\overline{\partial}$-equation ([**7**, p. 322]).

Lemma 3. *Let $1 \leq p \leq \infty$, let G be a C^1 function in $\overline{\mathbb{D}}$ such that:*

(a) *$G = \varphi_1\psi_1$, where $M(\varphi_1) \in L^p$ and $|\psi_1|^2 \log \frac{1}{|z|}$ is a Carleson measure.*

(b) *$\partial G = \varphi_2\psi_2$, where $M(\varphi_2) \in L^p$ and $|\psi_2| \log \frac{1}{|z|}$ is a Carleson measure.*

Then there exists a C^1 function u in $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$ such that

$$\frac{\partial u}{\partial \overline{z}} = G$$

and

$$\int_0^{2\pi} |u(e^{i\theta})|^p \, d\theta \leq C$$

where C depends only of the L^p-norms of $M(\varphi_1), M(\varphi_2)$ and the Carleson norms of the measures in (a), (b).

Proof. We adapt Wolff's proof for the case $p = \infty$. Let q be the conjugate exponent of p, $1 < q \leq \infty$. By duality,

$$\inf\left\{\|b\|_p : \frac{\partial b}{\partial \overline{z}} = G\right\} = \sup\left\{\left|\frac{1}{2\pi}\int_0^{2\pi} F k d\theta\right| : k \in H_0^q,\ \|k\|_q \leq 1\right\}$$

where F is a priori solution, say the one given by the Cauchy kernel, which is continuous on $\overline{\mathbb{D}}$. By Green's formula

$$\frac{1}{2\pi}\int_0^{2\pi} Fk\,d\theta = \frac{1}{2\pi}\iint_{\mathbb{D}} \Delta(Fk)\log\frac{1}{|z|}dA(z)$$

$$= \frac{2}{\pi}\iint_{\mathbb{D}} k'(z)G(z)\log\frac{1}{|z|}\,dA(z) + \frac{2}{\pi}\iint_{\mathbb{D}} k(z)\frac{\partial G}{\partial z}\log\frac{1}{|z|}dA(z) = I_1 + I_2\,.$$

We will prove now that if $|\psi|^2\log\frac{1}{|z|}$ is a Carleson measure with constant K, then

$$(4) \qquad \iint_{\mathbb{D}} |k'(z)|\,|\varphi(z)|\,|\psi(z)|\,\log\frac{1}{|z|}\,dA(z) \le C\|k\|_q\|M\varphi\|_p K$$

where C is an absolute constant. This will imply the required bound for I_1. For $p=\infty$, $q=1$ this holds true as shown by Wolff reducing the situation to $k=g^2$ with $g\in H^2$. Alternatively a real-analysis proof can be obtained using the inequality, following from [**6**, Th. 1],

$$\iint_{\mathbb{D}} |k'(z)|\,|\psi(z)|\log\frac{1}{|z|}\,dA(z) \le \int_0^{2\pi} A(k)(e^{i\theta})^{1/2}C(\psi)(e^{i\theta})\,d\theta$$

where $A(k)$ is the area function of k

$$A(k)(e^{i\theta}) = \left(\iint_{\Gamma(e^{i\theta})} |k'(z)|^2\,dA(z)\right)^{1/2}$$

and $C(\psi)$ is given by

$$C(\psi)(e^{i\theta}) = \sup_{e^{i\theta}\in I}\left(\frac{1}{|I|}\iint_{\hat{I}} |\psi|^2\log\frac{1}{|z|}\,dA\right)^{1/2}$$

$\hat{I}$ being the tent over I. This method applies to situations where there is no factorization.

For $p=1$, $q=\infty$, we use Schwarz inequality to bound the left member of (4) by

$$\left(\iint_{\mathbb{D}} |\varphi|\,|k'|^2\log\frac{1}{|z|}\,dA(z)\right)^{1/2}\left(\iint_{\mathbb{D}} |\varphi|\,|\psi|^2\log\frac{1}{|z|}dA(z)\right)^{1/2}$$

If $k\in BMOA$, $|k'|^2\log\frac{1}{|z|}dA$ is a Carleson measure; since Carleson measures operate on functions with integrable non-tangential maximal function, (4) follows for $p=1$, $q=\infty$. Next, consider the operator, for fixed ψ

$$\varphi\mapsto L_\varphi$$

where $L_\varphi(k) = \iint_{\mathbb{D}} k'\varphi\psi \log \frac{1}{|z|} dA(z)$; let T^p_∞ be the tent space ([**6**])

$$T^p_\infty = \{\varphi : M(\varphi) \in L^p\}.$$

We have shown that L is bounded from T^1_∞ to $(BMOA)^*$ and from T^∞_∞ to $(H^1)^*$. By interpolation, we conclude that L is bounded from T^p_∞ to $(H^q)^*$ i.e.

$$|L_\varphi(k)| \le C\|k\|_q \, \|M\varphi\|_p K$$

(alternatively, φ can be replaced by the harmonic extension of $M\varphi$ and argue with the L^p-spaces rather than the tent spaces).

It remains to bound I_2. But

$$|I_2| \le \iint_{\mathbb{D}} |k(z)|\,|\varphi_2(z)|\,|\psi_2(z)| \log \frac{1}{|z|} dA(z)$$

and this is easier: just note that $M(k\varphi_2) \le M(k)M(\varphi_2)$ is in L^1 and use again that Carleson measures operate on such functions. $\square$

Note that the lemma holds if $G, \partial G$ are linear combinations $\sum \varphi_i\psi_i$ with φ_i, ψ_i as above.

Theorem 4. *Let $g_1, g_2 \in H^\infty$ such that $|g|^2 = |g_1|^2 + |g_2|^2 > 0$. Let μ and $\widetilde{\mu}$ be as above. Assume that*

$$M\left(\frac{1}{|g|^2} \frac{1}{\widetilde{\mu}(|g|^2)}\right) \in L^p(\mathbb{T}).$$

Then there are $f_1, f_2 \in H^p$ such that $f_1g_1 + f_2g_2 = 1$.

Proof. By a standard regularization argument we may assume that g_1, g_2 are holomorphic on $\overline{\mathbb{D}}$. The smooth solutions

$$\varphi_i = \frac{\overline{g}_i}{|g|^2}, \quad i = 1, 2$$

satisfy $M(\varphi_i) \in L^p$ and the general holomorphic solutions are given by

$$f_1 = \varphi_1 + ug_2 \qquad f_2 = \varphi_2 - ug_1$$

where u satisfies

$$\frac{\partial u}{\partial \overline{z}} = \frac{\overline{g_1'}\,\overline{g_2} - \overline{g_1}\,\overline{g_2'}}{|g|^4} \stackrel{def}{=} G.$$

We need only check that G satisfies the hypothesis of Lemma 3. For (a) we can take, by Lemma 2, ψ_1 to be

$$\psi_1 = \frac{\overline{g_i'}}{|g|}\,\widetilde{\mu}(|g|^2)^{1/2}, \quad |\psi_1| \le \frac{|g_i'|}{|g_i|}\widetilde{\mu}(|g_i|^2)^{1/2}$$

and

$$\varphi_1 = \frac{\overline{g_j}}{|g|^3 \widetilde{\mu}(|g|^2)^{1/2}}, \quad |\varphi_1| \leq \frac{1}{|g|^2 \widetilde{\mu}(|g|^2)^{1/2}}.$$

Similarly, ∂G is a linear combination of terms of type

$$\frac{\overline{g_i'}\,\overline{g_j'}}{|g|^6} g_k g_l$$

and we may take

$$\psi_2 = \frac{\overline{g_i'}\,\overline{g_j'}}{|g|^2} \widetilde{\mu}(|g|^2), \quad |\psi_2| \leq \frac{|g_i'|^2}{|g_i|^2} \widetilde{\mu}(|g_i|^2) + \frac{|g_j'|^2}{|g_j|^2} \widetilde{\mu}(|g_j|^2)$$

$$\varphi_2 = \frac{g_k g_l}{|g|^4 \widetilde{\mu}(|g|^2)}, \quad |\varphi_2| \leq \frac{1}{|g|^2 \widetilde{\mu}(|g|^2)}$$

using again Lemma 2. □

We note as a particular case of the theorem, corresponding to $d\mu(\alpha) = \alpha^{1+\varepsilon} d\alpha$, the sufficient condition

$$M\left(\frac{|\log|g||^{2+\varepsilon}}{|g|^2}\right) \in L^p$$

stated in the introduction.

4. Lemma 2 can be used as well to improve Cegrell's result on the equation $f = f_1 g_1 + f_2 g_2$:

Theorem 5. *If $f, g_1, g_2 \in H^\infty$ satisfy*

$$|f| \leq |g|^2 \widetilde{\mu}(|g|^2)$$

there exist $f_1, f_2 \in H^\infty$ such that $f = f_1 g_1 + f_2 g_2$.

Proof. In this case it must be shown that the equation $\overline{\partial} u = G$ where

$$G = f \frac{\overline{g_1'}\,\overline{g_2} - \overline{g_1}\,\overline{g_2'}}{(|g_1|^2 + |g_2|^2)^2}$$

has a bounded solution. In this case

$$|G| \leq \frac{|g_1'| + |g_2'|}{|g|} \widetilde{\mu}(|g|^2)^{1/2} \leq \frac{|g_1'|}{|g_1|} \widetilde{\mu}(|g_1|^2)^{1/2} + \frac{|g_2'|}{|g_2|} \widetilde{\mu}(|g_2|^2),$$

and $|G|^2(1 - |z|)$ is indeed a Carleson measure; similarly for ∂G.

References

[1] E. Amar, *On the corona problem*, The Journal of Geometric Analysis, Vol. 1. **4** (1991), 291–305.

[2] E. Amar and J. Bruna, *On H^p-solutions of the Bezout equation in the ball*, to appear in J. Fourier Analysis.

[3] J. Bourgain, *On finitely generated closed ideals in $H^\infty(\mathbb{D})$*, Ann. Inst. Fourier, **35** 4 (1985), 163–174.

[4] U. Cegrell, *A generalization of the corona theorem in the unit disc*, Math. Z., **203** (1990).

[5] J.A. Cima and G. Taylor, *On the equation $f_1g_1 + f_2g_2 = 1$ in H^p*, Illinois J. Math., **11** (1967), 431–438.

[6] R.R. Coiffmann, Y. Meyer and E.M. Stein, *Some new function spaces and their applications to Harmonic Analysis*, J. Funct. Analysis, **62** (1985), 304–333.

[7] J.B. Garnett, Bounded analytic functions, Academic Press, 1981.

[8] K.C. Lin, *On the constants of the corona theorem and the ideals of H^∞*, Preprint, University of Alabama.

[9] K.C. Lin, *On the H^p solutions to the corona equation*, Preprint, University of Alabama.

[10] H.S. Shapiro and A.L. Shields, *On some intepolation problems for analytic functions*, Amer. J. Math., **83** (1961), 513–532.

[11] E.M. Stein, Singular integrals and differentiability properties of functions, Princeton Univ. Press, 1970.

Received March 11, 1993 and revised November 30, 1993. This work has been done during the visit of the first-named author to the Centre de Recerca Matematica at Barcelona. He wants to thank this institution for its support. The second and third author were partially supported by DGICYT grant PB92-0804-C02-02.

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