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UNIT INDICES OF SOME IMAGINARY COMPOSITE QUADRATIC FIELDS II

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Let K be an imaginary abelian number field of type (2, 2, 2, 2) containing the 8-th cyclotomic field $\mathbf{Q}(\sqrt{-1}, \sqrt{2})$. Using the fundamental units of real quadratic subfields of K, we give a necessary and sufficient condition for the unit index Q_K of K to be equal to 2.

1. Introduction and Results.

Let K be an imaginary abelian number field and K_0 the maximal real subfield of K. Let E and E_0 be the groups of units of K and K_0, respectively, and let W be the group of roots of unity in K. Let Q_K be the unit index of K, i.e.,

$$Q_K = [E : WE_0].$$

In the previous paper [**4**] we gave a necessary and sufficient condition for Q_K to be equal to 2 when K is an imaginary abelian number field (whose Galois group is) of type $(2,2,2,2)$ not containing the 8-th cyclotomic field $\mathbf{Q}\left(\sqrt{-1}, \sqrt{2}\right)$. In this paper we give such a condition when K contains $\mathbf{Q}\left(\sqrt{-1}, \sqrt{2}\right)$.

In this paper we use the following notation, unless otherwise specified.

$\mathbf{N}, \mathbf{Z}, \mathbf{Q}$: the sets of natural numbers, rational integers and rational numbers, respectively,

$\underset{2}{=}$ (resp. $\underset{2}{=}$ in k) : the equality up to a rational quadratic factor (resp. the equality up to a square of a number of a field k),

$d_1, d_2, \cdots, d_7$: square-free positive integers such that $d_4 \underset{2}{=} d_2 d_3$, $d_5 \underset{2}{=} d_3 d_1$, $d_6 \underset{2}{=} d_1 d_2$, $d_7 \underset{2}{=} d_1 d_2 d_3$ and that $d_3 = 2$.

$K = \mathbf{Q}\left(\sqrt{-1}, \sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3}\right) = \mathbf{Q}\left(\sqrt{-1}, \sqrt{2}, \sqrt{d_1}, \sqrt{d_2}\right)$: an imaginary abelian number field of type $(2,2,2,2)$,

$K_0 = \mathbf{Q}\left(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3}\right)$,

E_0^+ : the group of totally positive units of K_0,

$K_1 = \mathbf{Q}\left(\sqrt{d_2}, \sqrt{d_3}\right)$, $K_2 = \mathbf{Q}\left(\sqrt{d_3}, \sqrt{d_1}\right)$, $K_3 = \mathbf{Q}\left(\sqrt{d_1}, \sqrt{d_2}\right)$,
$K_4 = \mathbf{Q}\left(\sqrt{d_1}, \sqrt{d_2 d_3}\right)$, $K_5 = \mathbf{Q}\left(\sqrt{d_2}, \sqrt{d_3 d_1}\right)$, $K_6 = \mathbf{Q}\left(\sqrt{d_3}, \sqrt{d_1 d_2}\right)$,
$K_7 = \mathbf{Q}\left(\sqrt{d_2 d_3}, \sqrt{d_3 d_1}\right)$,

σ_i : a generator of $\mathrm{Gal}(K_0/K_i)$, i.e., $\langle\sigma_i\rangle = \mathrm{Gal}(K_0/K_i)$ $(i = 1, 2, \cdots, 7)$,
ε_i : the fundamental unit of $k_i = \mathbf{Q}(\sqrt{d_i})$, $\varepsilon_i > 1$ $(i = 1, 2, \cdots, 7)$,
$N(x)$, $Sp(x)$: the absolute norm and the absolute trace of an algebraic number x, respectively.

For a totally positive unit η of K_0, let

$$\xi = \xi(\eta) = \eta + \eta^{\sigma_1} + 2\sqrt{\eta\eta^{\sigma_1}}, \tag{1}$$

$$\theta = \theta(\eta) = \xi + \xi^{\sigma_2} + 2\sqrt{\xi\xi^{\sigma_2}} \tag{2}$$

under the condition that

$$\sqrt{\eta\eta^{\sigma_1}} \in K_1 \quad \text{and} \quad \sqrt{\xi\xi^{\sigma_2}} \in k_3. \tag{3}$$

Let ν be the number of i for which $N(\varepsilon_i) = -1$ $(i = 1, 2, \cdots, 7)$, i.e.,

$$\nu = \#\{i \mid i = 1, 2, \cdots, 7\,; N(\varepsilon_i) = -1\}.$$

Remark 1. Using Lemmas 3 and 6 we can show that the above condition (3) follows from the equations

$$N_{K_0/K_i}(\eta) \underset{2}{=} 1 \quad \text{in } K_i \quad (i = 1, 2, 6).$$

Our result is

Theorem. (1) *If $\nu \geq 4$, then $Q_K = 1$.*

(2) *Suppose that $\nu = 3$ and that*

$$N(\varepsilon_s) = N(\varepsilon_t) = N(\varepsilon_3) = -1$$

for $s, t \in \{1, 2, \cdots, 7\}$ $(s \neq t)$ different from 3. If $d_s d_t \underset{2}{=} d_3$ does not hold, then $Q_K = 1$.

(3) *Suppose that $\nu \leq 2$ or that $\nu = 3$ and $d_s d_t \underset{2}{=} d_3$ holds for above s, t. Then $Q_K = 2$ if and only if there exists a unit η in E_0^+ such that*

$$\eta = \prod_{i=1}^{7} \varepsilon_i^{a_i} \cdot \sqrt{\prod_{N(\varepsilon_j)=+1} \varepsilon_j^{b_j}} \qquad (a_i, b_j = 0 \text{ or } 1) \tag{4}$$

satisfying the following conditions (i), (ii) :

(i)

$$N_{K_0/K_\alpha}(\eta) \underset{2}{=} 1 \quad \text{in } K_\alpha \quad (\alpha = 1, 2, 6),$$

$$N_{K_0/K_\beta}(\eta) \underset{2}{=} 1 \quad \text{in } K_0, \text{ but not in } K_\beta \quad (\beta = 3, 4, 5, 7).$$

(ii)

$$\theta = \theta(\eta) \underset{2}{=} \left(2+\sqrt{2}\right) d_1^{e_1} d_2^{e_2} \quad \textit{in } k_3 = \mathbf{Q}\left(\sqrt{2}\right)$$

for some $e_i \in \{0, 1\}$.

Moreover, in the representation (4) *of* η*, the number of* j*'s for which* $b_j = 1$ *is greater than one.*

Remark 2. When $\nu = 3$ and $d_s d_t \underset{2}{=} d_3$ holds for s, t in Theorem, we have examples of $Q_K = 1$ and $Q_K = 2$:

If $d_1 = 5, d_2 = 21$, then $Q_K = 1$, which is checked by Proposition 1.

If $d_1 = 7, d_2 = 41$, then $Q_K = 2$. Because,

$$\eta = \sqrt{\varepsilon_1}\sqrt{\varepsilon_5} = \frac{1}{2}\left(3\sqrt{2}+\sqrt{14}\right)\cdot\left(2\sqrt{2}+\sqrt{7}\right)$$

satisfies the condition (3) of Theorem. In fact,

$$\theta = \theta(\eta) \underset{2}{=} \left(2+\sqrt{2}\right) 7 \quad \text{in } k_3.$$

Remark 3. In the Theorem, when

$$\prod_{N(\varepsilon_j)=+1} \varepsilon_j^{b_j} = \varepsilon_{j_1}\varepsilon_{j_2},$$

it holds that $d_{j_1} d_{j_2} \underset{2}{=} d_3 = 2$, as seen in Lemma 5 (2).

The assertions (1) and (2) of the Theorem are easily obtained in §3 from

Proposition 1. *Let* L *be the composite of a 2-power-th cyclotomic field* $\mathbf{Q}(\zeta)$ ($\zeta = \exp(2\pi i/2^m), m \geq 2$) *and* n *independent real quadratic fields* $\mathbf{Q}\left(\sqrt{D_i}\right)$ *where* D_i *are square-free positive integers* ($i = 1, 2, \cdots, n$)*, that is,*

$$L = \mathbf{Q}\left(\zeta, \sqrt{D_1}, \sqrt{D_2}, \cdots, \sqrt{D_n}\right).$$

If $D_1 \equiv D_2 \equiv \cdots \equiv D_n \equiv 1 \pmod 4$*, then* $Q_L = 1$.

2. Characterization of $\eta \in \overline{E}_0$.

Our argument depends on

Lemma 1 (cf. [**3**, Satz 15]). $Q_K = 2$ *if and only if there exists a unit* $\eta \in E_0^+$ *such that* $K_0\left(\sqrt{\eta}\right) = K_0\left(\sqrt{2+\sqrt{2}}\right)$.

Therefore, in order to determine the alternative $Q_K = 1$ or 2, we investigate such $\eta \in E_0^+$. We replace the definition of $\overline{E}_0$ in [**4**] by

$$\overline{E}_0 = \left\{\eta \in E_0^+ \mid K_0\left(\sqrt{\eta}\right) = K_0\left(\sqrt{2+\sqrt{2}}\right)\right\}.$$

Here we note that if $\eta \in \overline{E}_0$, η is totally positive.

Lemma 2 (cf. [**4**, Lemma 1]). *For $\eta \in \overline{E}_0$, we have*

$$\eta^2 = \varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_7^{x_7}$$

for some $x_i \in \mathbf{Z}$.

Proof. For $\eta \in \overline{E}_0$, we can put

$$\eta^4 = \varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_7^{x_7} \quad (x_i \in \mathbf{Z}).$$

In fact, for a $(2,2)$-extension K/k with Galois group $\mathrm{Gal}(K/k) = \langle\sigma,\tau\rangle$ we have

$$\alpha^2 = \frac{\alpha^{1+\sigma}\alpha^{1+\tau}}{(\alpha^\sigma)^{1+\sigma\tau}}$$

for any $\alpha \in K, \alpha \neq 0$. By this formula we see that $E_0^4 \subseteq E_0^*$, where E_0^* is the subgroup of E_0 generated by $\pm\varepsilon_i$ $(i = 1, 2, \cdots, 7)$.

We show that every x_i is even.

Since $K_0(\sqrt{\eta}) = K_0\left(\sqrt{2+\sqrt{2}}\right)$, we have $\eta = \left(2+\sqrt{2}\right)\alpha_0^2$ for some $\alpha_0 \in K_0$. Then

$$\left(2+\sqrt{2}\right)^4 \alpha_0^8 = \varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_7^{x_7}. \tag{5}$$

Taking the norms N_{K_0/k_3} and N_{K_0/k_i} $(i \neq 3)$ of this equation (5) and then the positive fourth root, we have

$$\left(2+\sqrt{2}\right)^4 N_{K_0/k_3}(\alpha_0)^2 = \varepsilon_3^{x_3} \text{ and } 2^2 N_{K_0/k_i}(\alpha_0)^2 = \varepsilon_i^{x_i},$$

respectively. Here we recall that ε_3 and ε_i are positive. These equations show that $\varepsilon_i^{x_i}$ is square in k_i and hence $x_i \equiv 0 \pmod 2$ for every i. $\square$

Lemma 3 ([**2**, Satz 1]). *Let K_1 be a field with $\mathrm{char}(K_1) \neq 2$ and K_0 a quadratic extension over K_1. Let η be an element of K_0 which is not a square in K_0.*

(1) *$K_0(\sqrt{\eta})/K_1$ is Galois $\Longleftrightarrow$ $N_{K_0/K_1}(\eta) \underset{2}{=} 1$ in K_0.*

(2) *$K_0(\sqrt{\eta})/K_1$ is an extension of type $(2,2)$ $\Longleftrightarrow$ $N_{K_0/K_1}(\eta) \underset{2}{=} 1$ in K_1.*

(3) *$K_0(\sqrt{\eta})/K_1$ is cyclic $\Longleftrightarrow$ $N_{K_0/K_1}(\eta) \underset{2}{=} 1$ in K_0, but not in K_1.*

Lemma 4 (cf. [**4**, Lemma 3]). *Let $\eta \in \overline{E}_0$ and put*

$$\eta^2 = \varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_7^{x_7} \quad (x_i \in \mathbf{Z}).$$

(1) *If there exists an even* x_i, *then* $N(\varepsilon_j) = +1$ *for each odd* x_j.

(2) *If* $x_1 \equiv x_2 \equiv \cdots \equiv x_7 \equiv 1 \pmod 2$, *then* $N(\varepsilon_1) = N(\varepsilon_2) = \cdots = N(\varepsilon_7)$.

We can prove this Lemma 4 as in the same way in [**4**, Lemma 3].

Lemma 5. *Let* $\eta \in \overline{E}_0$ *and put*

$$\eta^2 = \varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_7^{x_7} \quad (x_i \in \mathbf{Z}). \tag{6}$$

(1) *There exist at least two odd integers among the* x_i*'s.*

(2) *If* x_i, x_j $(i \neq j)$ *are odd and the others* x_k *are even, then* $d_i \neq 2, d_j \neq 2$ *and* $d_i d_j \underset{2}{=} 2$.

Proof of Lemma 5. (1) First we suppose that all x_i are even. Then η is a product of some of ε_i's. Noting that η is contained in $(E_0^*)^+ = E_0^* \cap E_0^+$, we see by [**4**, Proposition 1] that η is, up to a square, a product of some of following totally positive units :

$$\begin{array}{ll} \varepsilon_i & (\text{when } N(\varepsilon_i) = +1), \\ \eta_{ij} := \varepsilon_i\varepsilon_j\varepsilon_k & (\text{when } d_i d_j \underset{2}{=} d_k \text{ and } N(\varepsilon_i) = N(\varepsilon_j) = N(\varepsilon_k) = -1), \\ \eta_{ijk} := \varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_l & (\text{when } d_i d_j d_k \underset{2}{=} d_l \text{ and } N(\varepsilon_i) = N(\varepsilon_j) = N(\varepsilon_k) = N(\varepsilon_l) \\ & = -1). \end{array}$$

For a unit ε_i with $N(\varepsilon_i) = +1$ we have

$$\eta Sp(\xi) = \xi^2$$

where $\eta = \varepsilon_i$ and $\xi = \varepsilon_i + 1$. For $\eta = \eta_{ij}$ or η_{ijk} we also have by [**5**, Proof of Zusatz 1] or by [**4**, Lemma 6] that

$$\eta Sp(\xi) = \xi^2$$

where

$$\xi = \varepsilon_i\varepsilon_j\varepsilon_k - \varepsilon_i - \varepsilon_j - \varepsilon_k$$

or

$$\xi = \varepsilon_i\varepsilon_j\varepsilon_k\varepsilon_l + 1 - (\varepsilon_i\varepsilon_j + \varepsilon_j\varepsilon_k + \varepsilon_k\varepsilon_i + \varepsilon_i\varepsilon_l + \varepsilon_j\varepsilon_l + \varepsilon_k\varepsilon_l),$$

respectively. Therefore, $K_0\left(\sqrt{\varepsilon_i}\right), K_0\left(\sqrt{\eta_{ij}}\right)$ and $K_0\left(\sqrt{\eta_{ijk}}\right)$ are 2-elementary extensions over $\mathbf{Q}$ and so is $K_0\left(\sqrt{\eta}\right)$, which contradicts $\eta \in \overline{E}_0$.

Next we suppose that x_i is odd and the other x_k are even. Choose K_j for which $\sqrt{d_i} \notin K_j$. Taking the norm N_{K_0/K_j} of the equation (6), we have

$$N_{K_0/K_j}(\eta)^2 = N(\varepsilon_i)^{x_i}\varepsilon_u^{2x_u}\varepsilon_v^{2x_v}\varepsilon_w^{2x_w}$$

where $K_j = \mathbf{Q}\left(\sqrt{d_u}, \sqrt{d_v}\right)$ and $d_w \underset{2}{=} d_u d_v$. Hence, $N(\varepsilon_i) = +1$ and so $i \neq 3$. (Then, as for above j, we can take $j = 3, 4, 5$ or 7.) Moreover, since x_u, x_v and x_w are even, we have

$$N_{K_0/K_j}(\eta) = \varepsilon_u^{x_u} \varepsilon_v^{x_v} \varepsilon_w^{x_w} \underset{2}{=} 1 \quad \text{in } K_j.$$

Therefore it follows from Lemma 3 that $K_0\left(\sqrt{\eta}\right)/K_j$ is of type (2, 2). However, the extension $K_0\left(\sqrt{\eta}\right)/K_j = K_0\left(\sqrt{2+\sqrt{2}}\right)/K_j$ is itself a cyclic extension of degree 4. Thus we get a contradiction.

(2) Choose $k \in \{1, 2, \cdots, 7\}$ for which $\sqrt{d_i} \in K_k$ and $\sqrt{d_j} \notin K_k$. Taking the norm N_{K_0/K_k} of the equation (6), we have

$$N_{K_0/K_k}(\eta)^2 = \varepsilon_i^{2x_i} N(\varepsilon_j)^{x_j} \eta_k^2$$

where η_k is a unit of K_k. Hence $N(\varepsilon_j) = +1$ and so $d_j \neq d_3 = 2$.

By exchanging i and j, we also have $N(\varepsilon_i) = +1$ and $d_i \neq d_3$.

Finally we show that $d_i d_j \underset{2}{=} 2$. Assume that this is false. Then, $K_l := \mathbf{Q}\left(\sqrt{d_i d_3}, \sqrt{d_j d_3}\right)$ contains neither $\sqrt{d_i}$ nor $\sqrt{d_j}$. Taking the norm N_{K_0/K_l} of (6) and then the positive square root, we obtain

$$N_{K_0/K_l}(\eta) = \varepsilon_\alpha^{x_\alpha} \varepsilon_\beta^{x_\beta} \varepsilon_\gamma^{x_\gamma} \underset{2}{=} 1 \quad \text{in } K_l$$

where $d_\alpha \underset{2}{=} d_i d_3, d_\beta \underset{2}{=} d_j d_3$ and $d_\gamma \underset{2}{=} d_\alpha d_\beta$, because, x_α, x_β and x_γ are even. Therefore, it follows from Lemma 3 (2) that $K_0\left(\sqrt{\eta}\right)/K_l$ is an extension of type $(2,2)$. However, by the definition of K_l, K_l does not contain $\sqrt{d_3}$ and so $K_l \neq K_1, K_2$ or K_6. Hence $K_0\left(\sqrt{\eta}\right)/K_l$ is a cyclic extension of degree 4, which is a contradiction. $\square$

3. Proofs of Proposition 1 and Theorem.

Proof of Proposition 1. Let $f(\chi)$ be the conductor of a Dirichlet character χ. For any even character χ_0 of L, we have $2 \nmid f(\chi_0)$ or $2^3 \mid f(\chi_0)$ and $2^{m+1} \nmid f(\chi_0)$. Then, from [**2**, Satz 22] it follows that $Q_L = 1$. $\square$

Remark 4. Proposition 1 is also proved in [**1** (14.7) Corollary and the comment on p. 87 - 88].

Proof of (1), (2) *of Theorem.* By the assumption we have

$$K = \mathbf{Q}\left(\sqrt{-1}, \sqrt{2}, \sqrt{d_s}, \sqrt{d_t}\right), \quad N(\varepsilon_s) = N(\varepsilon_t) = N(\varepsilon_3) = -1$$

for suitable $d_s, d_t \neq d_3$. Then for every odd prime p dividing $d_s d_t$, we have $p \equiv 1 \pmod 4$. In fact, for example, by $N(\varepsilon_s) = -1$ we have $x^2 - d_s y^2 = -4$

for some $x, y \in \mathbf{Z}$. Then, for an odd prime p dividing d_s, $x^2 \equiv -4 \pmod{p}$ and hence $(-1/p) = (-1)^{\frac{p-1}{2}} = 1$, where $(\ /\ )$ is the Legendre symbol. Thus we get $p \equiv 1 \pmod{4}$.

Therefore

$$K = \mathbf{Q}\left(\sqrt{-1}, \sqrt{2}, \sqrt{D_s}, \sqrt{D_t}\right)$$

for some $D_s, D_t \in \mathbf{N}, D_s \equiv D_t \equiv 1 \pmod{4}$. Thus Proposition 1 implies that $Q_K = 1$. $\square$

In the following we prove the assertion (3) of Theorem, for which we need

Proposition 2. *Let K and K_0 be as in the notation in §1. Let η be an element of K_0 which is not square in K_0.*

(1) *$K_0(\sqrt{\eta})/\mathbf{Q}$ is a Galois extension if and only if*

$$N_{K_0/K_i}(\eta) \underset{2}{=} 1 \quad in\ K_0 \quad (i = 1, 2, \cdots, 7). \tag{7}$$

(2) *$K_0(\sqrt{\eta})/\mathbf{Q}$ is an abelian extension of type $(2,2,2,2)$ if and only if*

$$N_{K_0/K_i}(\eta) \underset{2}{=} 1 \quad in\ K_i \quad (i = 1, 2, \cdots, 7). \tag{8}$$

(3) *$K_0(\sqrt{\eta})/\mathbf{Q}$ is an abelian extension of type $(2,2,4)$ and $K_0(\sqrt{\eta})/k_3$ of type $(2,2,2)$ if and only if*

$$\begin{cases} N_{K_0/K_\alpha}(\eta) \underset{2}{=} 1 \quad in\ K_\alpha & (\alpha = 1, 2, 6), \\ N_{K_0/K_\beta}(\eta) \underset{2}{=} 1 \quad in\ K_0, but\ not\ in\ K_\beta & (\beta = 3, 4, 5, 7). \end{cases} \tag{9}$$

Remark 5. This Proposition 2 remains valid if $K_0 = \mathbf{Q}\left(\sqrt{2}, \sqrt{d_1}, \sqrt{d_2}\right)$ is replaced by $K_0 = \mathbf{Q}\left(\sqrt{d_1}, \sqrt{d_2}, \sqrt{d_3}\right)$ with arbitrary $d_3 \in \mathbf{N}$ (d_3 : square-free, $d_3 \geq 2$). Therefore, the condition (8) leads to the condition (5) of [**4**].

For the proof of Proposition 2, we need the following two lemmas.

Lemma 6. *Let k be an algebraic number field. Let K_0/k be an abelian extension of type $(2,2)$. Let K_1, K_2 and K_3 be the intermediate fields of K_0/k. Let η be an element of K_0.*

(1) *$K_0(\sqrt{\eta})/k$ is a Galois extension if and only if*

$$N_{K_0/K_i}(\eta) \underset{2}{=} 1 \quad in\, K_0 \quad (i = 1, 2, 3).$$

(2) *Suppose that* $K_0\left(\sqrt{\eta}\right)/k$ *is a Galois extension. Let*

$$\mu = \#\{i \mid i = 1, 2, 3\,; N_{K_0/K_i}(\eta) \underset{2}{=} 1 \ \ in\, K_i\}.$$

Then, $K_0\left(\sqrt{\eta}\right)/k$ *is quaternion, abelian of type* (2, 4)*, dihedral or abelian of type* (2, 2, 2) *if and only if* $\mu = 0$, 1, 2 *or* 3*, respectively.*

Lemma 7. *Let* G *be a group of order* 16*. Assume that there exists a normal subgroup* N *of* G *of order* 2 *with quotient group* G/N *of type* $(2,2,2)$*. Then* G *is isomorphic to one of the followings* :

(a) *a* 2*-elementary group*

(b) *an abelian group of type* $(2, 2, 4)$

(c) *a central product of an abelian subgroup* A *and a dihedral or quaternion subgroup* B *of order* 8 *such that* $AB = G, A \cap B = N$*.* (*A is the center of* G*.*)

Lemma 6 is an immediate consequence of Lemma 3. Lemma 7 is a special case of [**6**, (4.16) and Theorem 4.18].

Proof of Proposition 2. (1) Suppose that $K_0\left(\sqrt{\eta}\right)/\mathbf{Q}$ is a Galois extension. Then, for any quadratic subfield k of K_0, $K_0\left(\sqrt{\eta}\right)/k$ is also a Galois extension. Hence, by Lemma 6 (1) we have

$$N_{K_0/K_i}(\eta) \underset{2}{=} 1 \quad \text{in}\, K_0$$

for every intermediate field K_i of K_0/k.

Conversely, suppose that the condition (7) is satisfied. For an automorphism σ of the algebraic closure $\overline{\mathbf{Q}}$ of $\mathbf{Q}$, the restriction $\sigma|_{K_0}$ of σ to K_0 belongs to the Galois group $\mathrm{Gal}(K_0/\mathbf{Q}) = \{\sigma_0 = 1, \sigma_1, \cdots, \sigma_7\}$. Then

$$\sigma|_{K_0} = \sigma_i$$

for some i. By the assumption, we have

$$\eta\eta^{\sigma_i} = \eta_i^2$$

for some $\eta_i \in K_0$. Therefore,

$$\sqrt{\eta}^{\sigma} = \pm\sqrt{\eta^{\sigma}} = \pm\frac{\eta_i}{\sqrt{\eta}}$$

is contained in $K_0\left(\sqrt{\eta}\right)$ and whence $K_0\left(\sqrt{\eta}\right)/\mathbf{Q}$ is a Galois extension.

(2), (3) At first, we suppose that $K_0\left(\sqrt{\eta}\right)/\mathbf{Q}$ is a Galois extension with Galois group G. Let N be the subgroup of G corresponding to K_0.

Here we assume that G is not abelian. Then, it follows from Lemma 7 that G is a central product of an abelian subgroup A and a non-abelian subgroup B of degree 8. Let k be the subfield of $K_0\left(\sqrt{\eta}\right)$ corresponding to B. Since $A \cap B = N$ and since B is of order 8, k is a quadratic subfield of K_0, i.e., $k = k_a$ for some $a \in \{1, 2, \cdots, 7\}$. Then, $K_0\left(\sqrt{\eta}\right)/k_a$ is a quaternion or dihedral extension. Let K_i' $(i = 1, 2, 3)$ be the intermediate fields of K_0/k_a and let

$$\mu = \#\{\, i \mid N_{K_0/K_i'}(\eta) \underset{2}{=} 1 \quad \text{in}\, K_i' \,\}.$$

Then, by Lemma 6 (2) we have $\mu = 0$ or 2.

Now, suppose that the condition (9) is satisfied. Then, $K_0\left(\sqrt{\eta}\right)/\mathbf{Q}$ is a Galois extension with Galois group G. If G is not abelian, then, for above μ and a, we have by the condition (9) that $\mu = 3$ or 1 according as $a = 3$ or not, which is a contradiction. Therefore G must be abelian.

Moreover, the equations

$$N_{K_0/K_\beta}(\eta) \underset{2}{=} 1 \quad \text{not in}\, K_\beta \quad (\beta = 3, 4, 5, 7)$$

imply that $K_0\left(\sqrt{\eta}\right)/K_\beta$ is cyclic. Hence it follows from Lemma 7 that $K_0\left(\sqrt{\eta}\right)/\mathbf{Q}$ is an abelian extension of type $(2, 2, 4)$. And the equations

$$N_{K_0/K_\alpha}(\eta) \underset{2}{=} 1 \quad \text{in}\, K_\alpha \quad (\alpha = 1, 2, 6)$$

imply that $K_0\left(\sqrt{\eta}\right)/k_3$ is an abelian extension of type $(2, 2, 2)$.

Next, suppose that the condition (8) is satisfied. In a similar way we see that $K_0\left(\sqrt{\eta}\right)/\mathbf{Q}$ is an abelian extension.

We show that $K_0\left(\sqrt{\eta}\right)/\mathbf{Q}$ is of type $(2, 2, 2, 2)$. Assume that this is false, i.e., assume that $K_0\left(\sqrt{\eta}\right)/\mathbf{Q}$ is of type $(2, 2, 4)$. Let, as above,

$$G = \mathrm{Gal}\left(K_0\left(\sqrt{\eta}\right)/\mathbf{Q}\right), \; N = \mathrm{Gal}\left(K_0\left(\sqrt{\eta}\right)/K_0\right).$$

Then,

$$G/N \cong \mathrm{Gal}(K_0/\mathbf{Q})$$

is of type $(2, 2, 2)$. By the assumption there exists an element σ of G of order 4. Since the order of the coset σN of G/N is at most 2, σ^2 is contained in N. Hence $N = \langle \sigma^2 \rangle$, because N has order 2. Let K_i be the subfield of K_0 corresponding to $\langle \sigma \rangle$. Then $K_0\left(\sqrt{\eta}\right)/K_i$ is cyclic. Hence, by Lemma 3 (3), we have

$$N_{K_0/K_i}(\eta) \underset{2}{=} 1 \quad \text{not in}\, K_i,$$

which is a contradiction to the condition (8).

Thus we have proved the sufficiencies of (2) and (3) of Proposition 2.

Conversely, their necessities are immediately deduced from Lemma 3 . □

For the proof of (3) of Theorem, we also need

Lemma 8 ([4, Lemma 5]). *Let K_1 be an algebraic number field and K_0 a quadratic extension of K_1. Let $K_0(\sqrt{\eta_0})$ $(\eta_0 \in K_0,\ \eta_0 \notin K_1)$ be a bi-quadratic bicyclic extension of K_1 with* $\mathrm{Gal}(K_0(\sqrt{\eta_0})/K_1) = \langle\sigma,\tau\rangle$ *and* $\mathrm{Gal}(K_0(\sqrt{\eta_0})/K_0) = \langle\tau\rangle$. *Let F be the intermediate field of $K_0(\sqrt{\eta_0})/K_1$ fixed by σ. Then we have*

$$F = K_1(\sqrt{\eta_0} + \sqrt{\eta_0{}^{\sigma}}).$$

Proof of (3) *of Theorem.* Suppose that $Q_K = 2$. Then, by Lemma 1 there exists a unit η in E_0^+ such that

$$K_0(\sqrt{\eta}) = K_0\left(\sqrt{2+\sqrt{2}}\right).$$

By Lemma 2 we have

$$\eta^2 = \varepsilon_1^{x_1}\varepsilon_2^{x_2}\cdots\varepsilon_7^{x_7}$$

for some $x_i \in \mathbf{Z}$ $(i = 1, 2, \cdots, 7)$. And we see by Lemma 5 (1) that there are at least two odd integers among x_i's.

If all x_i are odd, then it follows from Lemma 4 (2) that

$$N(\varepsilon_1) = N(\varepsilon_2) = N(\varepsilon_3) = \cdots = N(\varepsilon_7) = -1,$$

and so $\nu = 7$, which contradicts our assumption $\nu \leq 3$. Then there exists at least one even integer among x_i's. Hence Lemma 4 (1) implies that $N(\varepsilon_i) = +1$ for odd x_i. Therefore we may represent the η in question as

$$\eta = \prod_{i=1}^{7}\varepsilon_i^{a_i}\cdot\sqrt{\prod_{N(\varepsilon_j)=+1}\varepsilon_j^{b_j}} \quad (a_i, b_j = 0 \text{ or } 1),$$

and Lemma 5 (1) shows that there are at least two $b_j = 1$.

Since $K_0(\sqrt{\eta}) = K_0\left(\sqrt{2+\sqrt{2}}\right)$ is an extension of type $(2,2,4)$ over $\mathbf{Q}$ and of type $(2,2,2)$ over $k_3 = \mathbf{Q}\left(\sqrt{2}\right)$, Proposition 2 (3) implies the condition (3) (i) of Theorem.

Moreover, it follows from Lemma 8 that $K_1\left(\sqrt{\xi}\right) = K_1\left(\sqrt{\eta_0} \pm \sqrt{\eta_0{}^{\sigma}}\right)$ is the intermediate field of $K_0\left(\sqrt{\eta}\right)/K_1$ fixed by σ or $\tau\sigma$, where σ is an automorphism of $\overline{\mathbf{Q}}$ over $\mathbf{Q}$ such that $\sigma|_{K_0} = \sigma_1$, $\langle\sigma_1\rangle = \mathrm{Gal}(K_0/K_1)$ and τ is a generator of $\mathrm{Gal}\left(K_0\left(\sqrt{\eta}\right)/K_0\right)$. Consequently we have $K_1\left(\sqrt{\xi}\right) \neq K_0$. Similary we can show that $k_3\left(\sqrt{\theta}\right)$ is an intermediate field of $K_1\left(\sqrt{\xi}\right)/k_3$ and that $k_3\left(\sqrt{\theta}\right) \neq K_1$. Therefore

$$k_3\left(\sqrt{\theta}\right) = k_3\left(\sqrt{\left(2+\sqrt{2}\right) d_1^{e_1} d_2^{e_2}}\right)$$

for some $e_i \in \{0,1\}$. Thus we obtain the condition (3) (ii) of Theorem.

Conversely, suppose that there exists a unit $\eta \in E_0^+$ satisfying the conditions (3) (i), (ii) of Theorem. Then, it follows from Proposition 2 (3) that $K_0\left(\sqrt{\eta}\right)$ is of type $(2,2,4)$ over $\mathbf{Q}$ and of type $(2,2,2)$ over $k_3 = \mathbf{Q}\left(\sqrt{2}\right)$. By Lemma 8, we see that $K_1\left(\sqrt{\xi}\right)$ is an intermediate field of $K_0\left(\sqrt{\eta}\right)/K_1$ and $K_1\left(\sqrt{\xi}\right) \neq K_0$. Then we have

$$K_0\left(\sqrt{\eta}\right) = K_0\left(\sqrt{\xi}\right).$$

In the same way we get

$$K_1\left(\sqrt{\xi}\right) = K_1\left(\sqrt{\theta}\right).$$

Therefore,

$$K_0\left(\sqrt{\eta}\right) = K_0\left(\sqrt{\xi}\right) = K_0\left(\sqrt{\theta}\right).$$

By the condition (3) (ii) of Theorem we have

$$K_0\left(\sqrt{\theta}\right) = K_0\left(\sqrt{2+\sqrt{2}}\right).$$

Thus we obtain

$$K_0\left(\sqrt{\eta}\right) = K_0\left(\sqrt{2+\sqrt{2}}\right),$$

from which Lemma 1 implies $Q_K = 2$, as desired. □

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