ON THE GEVREY STRONG HYPERBOLICITY

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Abstract

In this paper we are concerned with a homogeneous differential operator \( p \) of order \( m \) of which characteristic set of order \( m \) is assumed to be a smooth manifold. We define the Gevrey strong hyperbolicity index as the largest number \( s \) such that the Cauchy problem for \( p + Q \) is well-posed in the Gevrey class of order \( s \) for any differential operator \( Q \) of order less than \( m \). We study the case of the largest index and we discuss in which way the Gevrey strong hyperbolicity index relates with the geometry of bicharacteristics of \( p \) near the characteristic manifold.

1. Introduction

Let
\[
P = D_m^0 + \sum_{|\alpha| \leq m, \alpha_0 < m} a_\alpha(x) D^\alpha = p(x, D) + P_{m-1}(x, D) + \cdots
\]
be a differential operator of order \( m \) defined near the origin of \( \mathbb{R}^{n+1} \) where \( x = (x_0, \ldots, x_n) = (x_0, x') \) and
\[
D_j = -i\partial/\partial x_j, \quad D = (D_0, D'), \quad D' = (D_1, \ldots, D_n).
\]
Here \( p(x, \xi) \) is the principal symbol of \( P \);
\[
p(x, \xi) = \xi_m^0 + \sum_{|\alpha| = m, \alpha_0 < m} a_\alpha(x) \xi^\alpha.
\]
We assume that the coefficients \( a_\alpha(x) \) are in the Gevrey class of order \( s > 1 \), sufficiently close to 1, which are constant outside \( |x'| \leq R \). We say that \( f(x) \in \gamma^s(\mathbb{R}^{n+1}) \), the Gevrey class of order \( s \), if for any compact set \( K \subset \mathbb{R}^{n+1} \) there exist \( C > 0, A > 0 \) such that we have
\[
|D^\alpha f(x)| \leq CA^{(|\alpha|)!}, \quad x \in K, \quad \forall \alpha \in \mathbb{N}^{n+1}.
\]

Definition 1.1. We say that the Cauchy problem for \( P \) is \( \gamma^s \) well-posed at the origin if for any \( \Phi = (u_0, u_1, \ldots, u_{m-1}) \in (\gamma^s(\mathbb{R}^n))^m \) there exists a neighborhood \( U_\Phi \) of the origin such that the Cauchy problem
\[
\begin{cases}
    Pu = 0 & \text{in } U_\Phi, \\
    D^j_0 u(0, x') = u_j(x'), \quad j = 0, 1, \ldots, m - 1, \quad x' \in U_\Phi \cap \{x_0 = 0\}
\end{cases}
\]
has a unique solution \( u(x) \in C^\infty(U_\Phi) \).
It is a fundamental fact that if \( p(x, \xi) \) is strictly hyperbolic near the origin, that is \( p(x, \xi_0, \xi') = 0 \) has \( m \) real distinct roots for any \( x \), near the origin and any \( \xi' \neq 0 \) then the Cauchy problem for \( p + Q \) with any differential operator \( Q \) of order less than \( m \) is \( C^\infty \) well-posed near the origin. In particular, \( \gamma(s) \) well-posed for any \( s > 1 \). On the other hand the Lax-Mizohata theorem in the Gevrey classes asserts:

**Proposition 1.1** ([16, Theorem 2.2]). If the Cauchy problem for \( P \) is \( \gamma(s) \) (\( s > 1 \)) well-posed at the origin then \( p(0, \xi_0, \xi') = 0 \) has only real roots \( \xi_0 \) for any \( \xi' \in \mathbb{R}^n \).

Taking this result into account we assume, throughout the paper, that \( p(x, \xi_0, \xi') = 0 \) has only real roots for any \( x \) near the origin and any \( \xi' \in \mathbb{R}^n \).

**Definition 1.2.** We define \( G(p) \) (the Gevrey strong hyperbolicity index) by

\[
G(p) = \sup \left\{ 1 \leq s \left| \text{Cauchy problem for } p + Q \text{ is } \gamma(s) \text{ well-posed at the origin for any differential operator } Q \text{ of order } < m \right. \right\}.
\]

We first recall a basic result of Bronshtein [4].

**Theorem 1.1** ([4, Theorem 1]). Let \( p \) be a homogeneous differential operator of order \( m \) with real characteristic roots. Then for any differential operator \( Q \) of order less than \( m \), the Cauchy problem for \( p + Q \) is \( \gamma(m/(m-1)) \) well-posed.

This implies that for differential operators \( p \) of order \( m \) with real characteristic roots we have

\[
G(p) \geq m/(m-1).
\]

We also recall a result which bounds \( G(p) \) from above. The following result is a special case of Ivrii [10, Theorem 1]. Recall that \( (x, \xi) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\}) \) is called a characteristic of order \( r \) of \( p \) if

\[
\partial_x^\alpha \partial_\xi^\beta p(x, \xi) = 0, \quad \forall |\alpha + \beta| < r.
\]

**Theorem 1.2** ([10, Theorem 1]). Let \( p \) be a homogeneous differential operator of order \( m \) with real analytic coefficients and let \( (0, \tilde{\xi}), \tilde{\xi} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1} \) be a characteristic of order \( m \). If the Cauchy problem for \( P = p + P_{m-1} + \cdots \) is \( \gamma(s) \) well-posed at the origin we have

\[
\partial_x^\alpha \partial_\xi^\beta p_{m-1}(0, \tilde{\xi}) = 0
\]

for any \( |\alpha + \beta| \leq m - 2k/(k-1) \).

Assume that \( p \) has a characteristic \((0, \tilde{\xi})\) of order \( m \) and that the Cauchy problem for \( p + P_{m-1} + \cdots \) is \( \gamma(s) \) well-posed for any \( P_{m-1} \). Then from Theorem 1.2 it follows that \( m - 2k/(k-1) < 0 \), that is \( k < m/(m-2) \) which yields

\[
G(p) \leq m/(m-2).
\]

Let \( \rho \) be a characteristic of order \( m \). Then the localization \( p_\rho(X) \) of \( p \) at \( \rho \) is defined by

\[
p(\rho + \mu X) = \mu^\rho (p_\rho(X) + o(1)) \text{ with } X = (x, \xi) \text{ as } \mu \to 0 \text{ which is nothing but the first non-vanishing term of the Taylor expansion of } p \text{ around } \rho.
\]

Note that \( p_\rho \) is a hyperbolic polynomial in \( X \) in the direction \((0, \theta) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \) where \( \theta = (1, \ldots, 0) \in \mathbb{R}^{n+1} \) (for
example [7, Lemma 8.7.2]). The hyperbolic cone $\Gamma_\rho$ of $p_\rho$ is the connected component of $(0, \theta)$ in the set (for example [7, Lemma 8.7.3])

$$\Gamma_\rho = \{ X \in \mathbb{R}^{2(n+1)} | p_\rho(X) \neq 0 \}$$

and the propagation cone $C_\rho$ of the localization $p_\rho$ is the dual cone with respect to the symplectic two form $\sigma = d\xi \wedge dx = \sum_{j=0}^n d\xi_j \wedge dx_j$.

$$C_\rho = \{ X \in \mathbb{R}^{2(n+1)} | (d\xi \wedge dx)(X, Y) \leq 0, \forall Y \in \Gamma_\rho \}.$$  

Let

$$H_\rho = \sum_{j=0}^n (\partial p/\partial \xi_j) \partial/\partial x_j - (\partial p/\partial x_j) \partial/\partial \xi_j$$

be the Hamilton vector field of $p$ then integral curves of $H_\rho$, along which $p = 0$, are called bicharacteristics of $p$. We note that $C_\rho$ is the minimal cone including every bicharacteristic which has $\rho$ as a limit point in the following sense:

**Lemma 1.1** ([12, Lemma 1.1.1]). Let $\rho \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\})$ be a multiple characteristic of $p$. Assume that there are simple characteristics $\rho_j$ and non-zero real numbers $\gamma_j$ with $\gamma_j p_{\rho_j}(0, \theta) > 0$ such that

$$\rho_j \to \rho \text{ and } \gamma_j H_\rho(\rho_j) \to X, \ j \to \infty.$$  

Then $X \in C_\rho$.

We now introduce assumptions of which motivation will be discussed in the next section. Denote by $\Sigma$ the set of characteristics of order $m$ of $p(x, \xi)$:

$$\Sigma = \{(x, \xi) \in \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\}) | \partial^\alpha p(x, \xi) = 0, \forall |\alpha| < m\}$$

which is assumed to be a $\gamma^{(s)}$ manifold. Note that $p_\rho$ is a function on $\mathbb{R}^{2(n+1)}/T_\rho \Sigma$ because $p_\rho(X + Y) = p_\rho(Y)$ for any $X \in T_\rho \Sigma$ and any $Y \in \mathbb{R}^{2(n+1)}$ where $T_\rho \Sigma$ denotes the tangent space of $\Sigma$ at $\rho \in \Sigma$. We assume that

(1.1) $p_\rho$ is a strictly hyperbolic polynomial on $\mathbb{R}^{2(n+1)}/T_\rho \Sigma$, $\rho \in \Sigma$.

We also assume that the propagation cone $C_\rho$ is transversal to the characteristic manifold $\Sigma$;

(1.2) $C_\rho \cap T_\rho \Sigma = \{0\}$, $\rho \in \Sigma$.

Denoting $(T_\rho \Sigma)^\nu = \{ X \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} | (d\xi \wedge dx)(X, Y) = 0, \forall Y \in T_\rho \Sigma \}$ we note that (1.2) is equivalent to $\Gamma_\rho \cap (T_\rho \Sigma)^\nu \neq \emptyset$.

Our aim in this paper is to prove

**Theorem 1.3.** Assume (1.1) and (1.2). Then the Cauchy problem for $p + Q$ is $\gamma^{(s)}$ well-posed at the origin for any differential operator $Q$ of order less than $m$ and for any $1 < s < m/(m-2)$. In particular we have $G(p) = m/(m-2)$.

**Example 1.1.** Let

$$q(\xi) = \zeta_0^m + \sum_{|\alpha|=m-n_0 \leq m-2} c_\alpha \zeta^\alpha, \quad \xi = (\zeta_0, \xi_1, \ldots, \xi_k)$$
be a strictly hyperbolic polynomial in the direction $\xi_0$ where $k \leq n$. Let $b_j(x, \xi^0)$, $j = 1, \ldots, k$ be smooth functions in a conic neighborhood of $(0, \xi^0)$ which are homogeneous of degree 1 in $\xi^0$ with linearly independent differentials at $(0, \xi^0)$. We define
\[ p(x, \xi) = q(b(x, \xi)), \quad b = (b_0, b_1, \ldots, b_k) \]
where we set $b_0(x, \xi) = \xi_0$ for notational convenience. Then it is easy to see that $p(x, \xi)$ verifies the condition (1.1) near $\rho = (0, 0, \xi^0)$ with $\Sigma = \{ (x, \xi) \mid b_j(x, \xi) = 0, \ j = 0, \ldots, k \}$ and $p_\rho(x, \xi) = q(db_\rho(x, \xi))$, that is
\[ p_\rho(x, \xi) = q(\hat{b}(x, \xi)), \quad \hat{b} = (\hat{b}_0, \hat{b}_1, \ldots, \hat{b}_k) \]
where $\hat{b}_j(x, \xi)$ is the linear part of $b_j(x, \xi)$ at $\rho$. Therefore $\Gamma_\rho = \{ X \mid \hat{b}(X) \in \Gamma \}$ where $\Gamma$ is the hyperbolic cone of $q$. If
\[ (\{ b_i, b_j \})_{0 \leq i, j \leq k} \]
is non-singular at $\rho$
then $p(x, \xi)$ verifies the condition (1.2) near $\rho$ where $\{ b_i, b_j \}$ denotes the Poisson bracket
\[ \sum_{\mu=0}^{n} \left( \partial b_\rho / \partial \xi_\mu \right) (\partial b_j / \partial x_\mu) - \left( \partial b_\rho / \partial x_\mu \right) (\partial b_j / \partial \xi_\mu). \]
Indeed since $(T_\rho \Sigma)^\nu$ is spanned by $H_{b_0}(\rho), H_{b_1}(\rho), \ldots, H_{b_k}(\rho)$ it suffices to show that there are $c_j$ such that $0 \neq X = \sum_{j=0}^{k} c_j H_{b_j}(\rho) \in \Gamma_\rho$. From
\[ \hat{b}_j(X) = \sum_{i=0}^{k} c_i \{ b_i, b_j \}(\rho), \quad j = 0, \ldots, k \]
one can choose $c_j$ so that $\hat{b}(X) = (1, 0, \ldots, 0)$ by assumption (1.3) and hence the result.

**Example 1.2.** Consider
\[ q(\xi_0, \xi_1, \xi_2) = \prod_{j=1}^{\ell} (\xi_0 - c_j(\xi_1^2 + \xi_2^2)) \]
where $c_j$ are real positive constants different from each other and $2\ell = m$. Take $b_1 = (x_0 - x_1)\xi_n, b_2 = \xi_1$ and consider
\[ p(x, \xi) = \prod_{j=1}^{\ell} (\xi_0^2 - c_j((x_0 - x_1)^2 \xi_n^2 + \xi_1^2)) \]
in a conic neighborhood of $\rho = (0, 0, \ldots, 0, 1)$. The $3 \times 3$ anti-symmetric matrix $(\{ b_i, b_j \})$ is obviously singular. If $\max(c_j) = c < 1$ then $C_\rho \cap T_\rho \Sigma = \{ 0 \}$. To see this take any $X = (t, t, x_2, \ldots, x_n, 0, 0, \xi_2, \ldots, \xi_n) \in T_\rho \Sigma$. Assume $X \in C_\rho$ so that $(d\xi \wedge dx)(X, Y) \leq 0$ for any $Y = (y, \eta) \in \Gamma_\rho$, that is for any $(y, \eta) \in \mathbb{R}^{2(n+1)}$ with $\eta_0^2 > c((y_0 - y_1)^2 + \eta_1^2)$ and $\eta_0 > 0$. This implies that $x_2 = \cdots = x_n = 0, \xi_2 = \cdots = \xi_n = 0$ and $-t(\eta_0 + \eta_1) \leq 0$ for any $\eta_0 > \sqrt{c} |\eta_1|$. Since $c < 1$ this gives $t = 0$ so that $X = 0$.

On the other hand if $\max(c_j) = c \geq 1$ then $C_\rho \cap T_\rho \Sigma \neq \{ 0 \}$. Indeed let $X = (1, 1, 0, \ldots, 0, 0, \ldots, 0) \in T_\rho \Sigma$. Noting that $\eta_0 > \sqrt{c} |\eta_1|$ if $Y = (y, \eta) \in \Gamma_\rho$ we see $(d\xi \wedge dx)(X, Y) = -\eta_0 - \eta_1 < 0$ for any $Y \in \Gamma_\rho$, which proves $X \in C_\rho$.  

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Example 1.3. Take \( q \) in (1.4) and choose \( b_1 = x_0\xi_n, b_2 = \xi_1 \) and consider

\[
p(x, \xi) = \prod_{j=1}^{\ell} \left( \xi_0^2 - c_j (x_0\xi_n^2 + \xi_1^2) \right)
\]

near \( \rho = (0, 0, \ldots, 0, 1) \). As remarked in Example 1.2 the matrix \((b, b_j)\) is singular. Suppose \( X = (0, x_1, \ldots, x_n, 0, \xi_2, \ldots, \xi_n) \in T_p\Sigma \cap C_\rho \). As in Example 1.2 we conclude \( x_2 = \cdots = x_n = 0, \xi_1 = \cdots = \xi_n = 0 \) and \( -x_1\eta_1 < 0 \) for any \( \eta_0^2 > c(y_0^2 + \eta_1^2) \). This gives \( x_1 = 0 \) so that \( X = 0 \). Thus we conclude \( C_\rho \cap T_p\Sigma = \emptyset \).

Example 1.4. We specialize Example 1.1 with

\[
q(\zeta_0, \xi_1) = \prod_{j=1}^{m} (\zeta_0 - \alpha_j \xi_1), \quad q(\zeta_0, \xi_1) = \prod_{j=1}^{\ell} (\zeta_0^2 - c_j \xi_1^2)
\]

where \( \alpha_j \) are real constants different from each other such that \( \sum_{j=1}^{m} \alpha_j = 0 \) and \( c_j \) are positive constant different from each other and \( m = 2\ell \). For these \( q \) choosing \( b_1 = x_0\xi_1 \) and \( b_1 = x_0|\xi'| \) respectively we get

\[
p(x, \xi) = \prod_{j=1}^{m} (\zeta_0 - \alpha_j x_0\xi_1), \quad p(x, \xi) = \prod_{j=1}^{\ell} (\zeta_0^2 - c_j x_0^2 |\xi'|^2).
\]

It is clear that \( \{b_0, b_1\} = \xi_1 \neq 0 \) and \( \{b_0, b_1\} = |\xi'| \neq 0 \) respectively and hence \( C_\rho \cap T_p\Sigma = \emptyset \). We find these examples in [5] where they studied Levi type conditions for differential operators of order \( m \) with coefficients depending only on the time variable.

2. Motivation, the doubly characteristic case

In this section we provide the motivation to introduce \( G(p) \) and assumptions (1.1), (1.2). Let \( m = 2 \) and we consider differential operators of second order

\[
P(x, D) = p(x, D) + P_1(x, D) + P_0(x)
\]

of principal symbol \( p(x, \xi) \). Let \( \rho \) be a double characteristic of \( p \) and hence singular (stationary) point of \( H_p \). We linearize the Hamilton equation \( \dot{X} = H_p(X) \) at \( \rho \), the linearized equation turns to be \( \dot{Y} = F_p(\rho)Y \) where \( F_p(\rho) \) is given by

\[
F_p(\rho) = \left( \begin{array}{cc}
\frac{\partial^2 p}{\partial x^2} (\rho) & \frac{\partial^2 p}{\partial \xi \partial \xi} (\rho) \\
-\frac{\partial^2 p}{\partial \xi \partial x} (\rho) & -\frac{\partial^2 p}{\partial \xi^2} (\rho)
\end{array} \right)
\]

and called the Hamilton map (fundamental matrix) of \( p \) at \( \rho \).

The following special structure of \( F_p(\rho) \) results from the fact that \( p(x, \xi_0, \xi') = 0 \) has only real roots \( \xi_0 \) for any \( (x, \xi') \).

Lemma 2.1 ([9, Lemma 9.2, 9.4]). All eigenvalues of the Hamilton map \( F_p(\rho) \) are on the imaginary axis, possibly one exception of a pair of non-zero real eigenvalues.

We assume that the doubly characteristic set \( \Sigma = \{(x, \xi) \mid \partial^2_{\xi} \partial_x^2 p(x, \xi) = 0, \forall |\alpha + \beta| < 2\} \)
verifies the following conditions:

\[
\begin{align*}
\Sigma & \text{ is a } \gamma(s) \text{ manifold}, \\
\rho & \text{ vanishes on } \Sigma \text{ of order exactly } 2, \\
\text{rank} \ (d\xi \wedge dx) & = \text{const. on } \Sigma.
\end{align*}
\]

Note that \( p, (X) \) is always a strictly hyperbolic polynomial on \( \mathbb{R}^{2(n+1)/T_p} \Sigma \) as far as \( \rho \) vanishes on \( \Sigma \) of order exactly 2. We also assume that the codimension \( \Sigma \) is 3 and no transition of spectral type of \( F_p \) occur on \( \Sigma \), that is we assume

\[
(2.2) \quad \text{either } \ker F_p^2 \cap \text{im } F_p^2 = \{0\} \text{ or } \ker F_p^2 \cap \text{im } F_p^2 \neq \{0\}
\]

throughout \( \Sigma \). The following table sums up a general picture of the Gevrey strong hyperbolicity for differential operators with double characteristics ([2, 3, 17, 11]) where \( W = \ker F_p^2 \cap \text{im } F_p^2 \).

<table>
<thead>
<tr>
<th>Spectrum of ( F_p )</th>
<th>( W )</th>
<th>Geometry of bicharacteristics near ( \Sigma )</th>
<th>( G(p) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exists non-zero real eigenvalue</td>
<td>( W = {0} )</td>
<td>At every point on ( \Sigma ) exactly two bicharacteristics intersect ( \Sigma ) transversally</td>
<td>( G(p) = \infty )</td>
</tr>
<tr>
<td>No non-zero real eigenvalue</td>
<td>( W \neq {0} )</td>
<td>No bicharacteristic intersects ( \Sigma )</td>
<td>( G(p) = 4 )</td>
</tr>
<tr>
<td>( W = {0} )</td>
<td>Exists a bicharacteristic tangent to ( \Sigma )</td>
<td>( G(p) = 3 )</td>
<td></td>
</tr>
<tr>
<td>( W = {0} )</td>
<td>No bicharacteristic intersects ( \Sigma )</td>
<td>( G(p) = 2 )</td>
<td></td>
</tr>
</tbody>
</table>

This table shows that, assuming (2.1), (2.2) and the codimension \( \Sigma \) is 3, the Gevrey strong hyperbolicity index \( G(p) \) takes only the values 2, 3, 4 and \( \infty \) and that these values completely determine the structure of the Hamilton map and the geometry of bicharacteristics near \( \Sigma \) and vice versa.

**Lemma 2.2** ([6, Corollary 1.4.7], [12, Lemma 1.1.3]). Let \( \rho \) be a double characteristic. Then the following two conditions are equivalent.

(i) \( F_p(\rho) \) has non-zero real eigenvalues,

(ii) \( C_\rho \cap T_\rho \Sigma = \{0\} \).

Note that the condition (ii) is well defined for characteristics of any order while \( F_p(\rho) \equiv 0 \) if \( \rho \) is a characteristic of order larger than 2.

**Remark 2.1.** Based on the table, it is quite natural to ask whether the converse of Theorem 1.3 is true. That is if \( G(p) = m/(m-2) \) then (1.1) and (1.2) hold?

**Remark 2.2.** Consider the case \( C_\rho \subset T_\rho \Sigma \) that would be considered as a opposite case to \( C_\rho \cap T_\rho \Sigma = \{0\} \). Here we note

**Lemma 2.3** [18, Lemma 2.11]. We have \( C_\rho \subset T_\rho \Sigma \) if and only if \( T_\rho \Sigma \) is involutive, that is \( (T_\rho \Sigma)^\circ \subset T_\rho \Sigma \).
It is also natural to ask whether $G(p) = m/(m-1)$ if $C_p \subset T_ρΣ, ρ ∈ Σ$. When Σ is involutive one can choose homogeneous symplectic coordinates $x, ξ$ in a conic neighborhood of $ρ ∈ Σ$ such that Σ is defined by ([8, Theorem 21.2.4], for example)

$$ξ_0 = ξ_1 = \cdots = ξ_k = 0.$$ 

Thus by conjugation of a Fourier integral operator $p(x, ξ)$ can be written

$$p(x, ξ) = ξ_0^m + \sum_{α_0 ≤ m-2, |β| = m} a_α(x, ξ)ξ^α$$

where $\tilde{ξ} = (ξ_0, ξ_1, \ldots, ξ_k)$. Thus $∂^α_ξ∂^β_xp(0, \tilde{ξ}) = 0$ for $|α| < m$ and any $β$. If the resulting $p(x, D)$ is a differential operator so that $a_α(x, ξ) = a_α(x)$ then from [10, Theorem 1] we conclude that if the Cauchy problem for $p + P_{m-1} + \cdots$ is $γ^{(κ)}$ well-posed then

$$p^x_ξp^0_ξ(0, \tilde{ξ}) = 0$$

for any $|α| ≤ m - κ/(κ - 1)$ and any $β$. This proves $G(p) ≤ m/(m-1)$ and hence $G(p) = m/(m-1)$.

**Example 2.1.** When $m \geq 3$ the geometry of $p$ with the limit point $ρ$ becomes to be complicated comparing with the case $m = 2$, even (1.1) and (1.2) are satisfied. We give an example. Let us consider

$$p(x, ξ) = ξ_0^3 - 3a((x_0^2 + x_1^2)ξ_2^2 + ξ_1^2)ξ_0 - 2bx_0x_1ξ_0ξ_1$$

near $ρ = (0, \ldots, 0, 1)$ which is obtained from Example 1.2 with

$$q(ξ_0, ξ_1, ξ_2, ξ_3) = ξ_0^3 - 3a(ξ_1^2 + ξ_2^2 + ξ_3^2)ξ_0 - 2bξ_1ξ_2ξ_3$$

and $b_1 = x_0ξ_n, b_2 = x_1ξ_n$, $b_3 = ξ_1$ where $a > 0, b$ are real constants. Choosing $b = δa^{3/2}$ with $|δ| < 1$ and repeating similar arguments as in Example 1.3 it is easily seen that $p(x, ξ)$ satisfies (1.1) and (1.2).

Consider the Hamilton equations

$$\dot{x}_j = ∂p/∂ξ_j, \quad \dot{ξ}_j = -∂p/∂x_j, \quad j = 0, \ldots, n.$$ 

Since $ξ_n = 0$ we take $ξ_n = 1$ and $x_1 = ξ_0 = 0$ in (2.3) so that the resulting equations reduce to:

$$\dot{x}_0 = -3a(x_0^2 + ξ_1^2), \quad \dot{ξ}_1 = 2bx_0ξ_1.$$ 

We fix $-1 < δ < 0$ and take $a > 0$ so that $2b/(3a) < -1$. Then any integral curve of (2.4) passing a point in the cone $|ξ_1| < |1 + (2b/3a)|^{1/2}|x_0|, x_0 < 0$ arrives at the origin inside the cone (see, for example [20]). In particular there are infinitely many bicharacteristics with the limit point $ρ$.

**3. Preliminaries**

Choosing a new system of local coordinates leaving $x_0 = \text{const.}$ to be invariant one can assume that

$$p(x, ξ) = ξ_0^m + a_2(x, ξ^r)ξ_0^{m-2} + \cdots + a_m(x, ξ^r)$$

and hence $Σ ⊂ \{ξ_0 = 0\}$. Thus near $ρ$ we may assume that $Σ$ is defined by $b_0(x, ξ) = \cdots =
homogeneous hyperbolic polynomial of degree 2 at \( \rho' \) where \( \rho' \) stands for \( (\hat{x}, \hat{\xi}) \) when \( \rho = (x, \xi) \). Recall that the localization \( p_\rho(x, \xi) \) is a homogeneous hyperbolic polynomial in the direction \((0, \theta) \in \mathbb{R}^{n+1} \times \mathbb{R}^n \).

**Lemma 3.1** ([12, Lemma 1.1.3]). The next two conditions are equivalent.

(i) \( C_\rho \cap T_\rho \Sigma = \{0\} \).

(ii) \( \Gamma_\rho \cap (T_\rho \Sigma)^\circ \cap \langle (0, \theta) \rangle^\circ \neq \emptyset \) where \( \langle (0, \theta) \rangle = \{ t(0, \theta) \mid t \in \mathbb{R} \} \).

Assume \( C_\rho \cap T_\rho \Sigma = \{0\} \) then thanks to Lemma 3.1 there exists \( 0 \neq X \in \Gamma_\rho \cap (T_\rho \Sigma)^\circ \cap \langle (0, \theta) \rangle^\circ \). Since \( (T_\rho \Sigma)^\circ \) is spanned by \( H_\rho(p, \rho) \), \( j = 0, \ldots, k \) one can write

\[
X = \sum_{j=0}^k \alpha_j H_\rho(p, \rho)
\]

where \( \alpha_0 = 0 \) because \( X \in \langle (0, \theta) \rangle^\circ \). This proves \( \partial_{\xi_0} b_j(\rho') \neq 0 \) with some \( 1 \leq j \leq k \).

Indeed if not we would have \( X = (x, 0, \xi') \) while denoting \( p_\rho(x, \xi) = \prod_{j=1}^m (\xi_0 - \Lambda_j(x, \xi')) \) we see \( \Gamma_\rho = \{(x, \xi) \mid \xi_0 > \max_j \Lambda_j(x, \xi') \} \) (for example [7, Lemma 8.7.3]) and we would have \( \Lambda_j(x, \xi') < 0 \) which contradicts \( \sum_{j=1}^m \Lambda_j(x, \xi') = 0 \). Renumbering, if necessary, one can assume \( \partial_{\xi_0} b_1(\rho') \neq 0 \) so that

\[
b_1(x, \xi') = (x_0 - f_1(x, \xi'))e_1(x, \xi'), \quad e_1(x, \xi') \neq 0.
\]

Writing \( b_j(x, \xi') = b_j(f_1(x', \xi'), x_1, \ldots, x_n, \xi') + c_j(x, \xi')b_1(x, \xi') \) we may assume \( b_j(x, \xi') \), \( 2 \leq j \leq k \) are independent of \( x_0 \). Since \( p(x, \xi) \) vanishes on \( \Sigma \) of order \( m \) one can write with \( b = (b_0, b_1, \ldots, b_k) = (b_0, b') \)

\[
p(x, \xi) = b_0^m + \sum_{|\alpha|=m, \alpha_0 \leq m-2} \tilde{a}_\alpha(x, \xi')b(x, \xi')^\alpha.
\]

Let \( \hat{b} \) be defined by \( b_j(\rho + \mu X) = \mu \hat{b}_j(X) + O(\mu^2) \) and with \( \hat{b} = (\xi_0, \hat{b}_1, \ldots, \hat{b}_k) \) we have \( p_\rho(X) = q(\hat{b}(X)) \) where

\[
q(\zeta) = \xi_0^m + \sum_{|\alpha|=m, \alpha_0 \leq m-2} \tilde{a}_\alpha(\rho')\zeta^\alpha, \quad \zeta = (\xi_0, \xi_1, \ldots, \xi_k) = (\xi_0, \zeta')
\]

is a strictly hyperbolic polynomial in the direction \((1, 0, \ldots, 0) \in \mathbb{R}^{k+1} \) by (1.1). Denote \( \tilde{q}(\xi, x, \xi') = q(\xi) + \sum a_\alpha(x, \xi')\zeta^\alpha \) with \( a_\alpha(x, \xi') = \tilde{a}_\alpha(x, \xi') - \tilde{a}(\rho') \) and hence we have \( p(x, \xi) = \tilde{q}(b(x, \xi); x, \xi') \).

**Lemma 3.2.** There are \( m \) real valued functions \( \lambda_1(x, \xi') \leq \lambda_2(x, \xi') \leq \cdots \leq \lambda_m(x, \xi') \) defined in a conic neighborhood of \( \rho' \) such that

\[
p(x, \xi) = \prod_{j=1}^m (\xi_0 - \lambda_j(x, \xi')), \quad |\lambda_j(x, \xi')| \leq C|b'(x, \xi')|, \quad |\lambda_i(x, \xi') - \lambda_j(x, \xi')| \geq c|b'(x, \xi')|, \quad (i \neq j)
\]

with some \( c > 0, C > 0 \).

Proof. The first assertion is clear because \( p(x, \xi) \) is a hyperbolic polynomial in the direction \( \xi_0 \). Note that \( \tilde{q}(\xi; \rho') = 0 \) has \( m \) real distinct roots for \( \xi' \neq 0 \) then by Rouché's
Since (4.1), set (4.3) \( H \) Here we recall (3.1), that is \( b_j \) for any \( C \)
are of homogeneous of degree 1 in \( \xi \). It is easy to check that \( |\lambda_j(\xi'; x, \xi')| \leq C|\xi'| \)
and \( |\lambda_j(\xi'; x, \xi') - \lambda_j(\xi'; x, \xi')| \geq c|\xi'| \) (\( i \neq j \)) with some \( c > 0, C > 0 \).
Since \( |\xi'| = 1 \) is compact we end the proof. \( \square \)

4. Basic weights \( \text{(energy estimates)} \)

We first introduce symbol classes of pseudodifferential operators which will be used in
this paper. Denote \( (\xi')_r = r^2 + |\xi'|^2 \) where \( r \geq 1 \) is a positive parameter.

Definition 4.1. Let \( W = W(x, \xi; \gamma) > 0 \) be a positive function and let \( s > 0 \), \( 0 \leq \delta \leq \rho \leq 1 \).
We define \( S^{(\rho)}(W) \) to be the set of all \( a(x, \xi; \gamma) \in C^\infty(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}) \) such that one can find \( A, C > 0 \) so that

\[
(4.1) \quad |\tilde{\partial}_\xi^r \tilde{\partial}_\xi^s a(x, \xi; \gamma)| \leq CA^{[r+\beta]}|\alpha + \beta|^{|s|}W(\xi')_{\gamma^{-\delta}|\alpha|+|\beta|}, \quad \forall \alpha, \beta \in \mathbb{N}^{n+1}
\]

holds with some \( A, C > 0 \) independent of \( \gamma \geq 1 \) and \( S^{(\rho)}(W) \) to be the set of all \( a(x, \xi; \gamma) \)
satisfying (4.1) with \( C_{\rho, \delta} \) in place of \( CA^{[r+\beta]}|\alpha + \beta|^{|s|} \) which may depend on \( \alpha, \beta \) but not
on \( \gamma \geq 1 \). We denote \( S^{(\rho)}(W), S^{1,0}(W), S(W) \) simply by \( S^{(\rho)}(W), S(W) \) respectively. We define
\( S^{(\rho)}(W) \) to be the set of all \( a(x, \xi; \gamma) \) such that we have

\[
(4.2) \quad |\tilde{\partial}_\xi^r \tilde{\partial}_\xi^s a| \leq CA^{[r+\beta]}W^{|\alpha + \beta| + |s|}(\xi')_{\gamma^{-\delta}(|\alpha|+|\beta|)}
\]

for any \( \alpha, \beta \in \mathbb{N}^{n+1} \) with positive constants \( C, A > 0 \) independent of \( \gamma \geq 1 \). If \( a(x, \xi; \gamma) \)
satisfies (4.1) (resp. (4.2)) in a conic open set \( U \subset \mathbb{R}^{n+1} \times (\mathbb{R}^{n+1} \setminus \{0\}) \) we say \( a(x, \xi; \gamma) \in S^{(\rho)}(W) \) (resp. \( S^{(1,0)}(W) \)) in \( U \). We often write \( a(x, \xi) \) for \( a(x, \xi; \gamma) \) dropping \( \gamma \).

It is clear \( S^{(\rho)}(W) \subset S^{(1,0)}(W) \) if \( 1 - \rho \geq \delta/2 \). It is also clear that one may replace
\( |\alpha + \beta| + |s| \) by \( |\alpha + \beta| + |s| \) in (4.2), still defining the
same symbol class.

Since \( p(x, \xi) \) is a polynomial in \( \xi \) of degree \( m \) it is clear that \( p(x, \xi) \in S^{(\rho)}(\langle \xi \rangle^m) \).
Since \( b_j(x, \xi') \) are defined only in a conic neighborhood of \( \rho' = (\tilde{x}, \tilde{\xi}) \) we extend such symbols to \( \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \). Let \( \chi(t) \in \chi^{(\rho)}(\mathbb{R}) \) be 1 for \( |t| < c/2 \) and 0 for \( |t| > c \) with small \( 0 < c < 1/2 \) and
set

\[
\left\{ \begin{array}{l}
y(x) = \chi(|x - \bar{x}|)(x - \bar{x}) + \bar{x}, \\
\eta'(\xi') = \chi(\langle \xi' - \bar{\xi}' \rangle (\xi' - \langle \xi' \rangle \bar{\xi}') + \langle \xi' \rangle \bar{\xi}'.
\end{array} \right.
\]

Then it is easy to see \( \eta', b_j(y, \eta') \in S^{(\rho)}(\langle \xi \rangle \gamma) \) and \( \langle \xi \rangle^{\gamma}/C \leq |\eta'| \leq C\langle \xi \rangle^{\gamma} \) with some \( C > 0 \).
In what follows we denote \( b_j(y, \eta') \) by \( b_j(x, \xi) \).

We now define \( u(x, \xi), \omega(x, \xi) \) by

\[
\left\{ \begin{array}{l}
u(x, \xi) = \left( \sum_{j=1}^k b_j(x, \xi)^2 \langle \xi \rangle^{2\gamma} + \langle \xi \rangle^{-2\gamma} \right)^{1/2}, \\
\omega(x, \xi) = \phi(x, \xi)^2 + \langle \xi \rangle^{-2\gamma} \right)^{1/2}, \quad \phi(x, \xi) = \sum_{j=1}^k \alpha_j b_j(x, \xi) \langle \xi \rangle^{-1}.
\end{array} \right.
\]

Here we recall (3.1), that is

\[
(3.1) \quad H_\phi(\rho) \in \Gamma_{\rho}.
\]

In what follows we always assume that
Let $M > 0$ be such that $2(1 + 4 \sum_{j=0}^{\infty} (j + 1)^{-2})M \leq 1/2$ and $\Gamma_1(k) = M k! / k^3$, $k \in \mathbb{N}$ where $\Gamma(0) = M$. Then we have

$$\sum_{\alpha + \alpha = \alpha'} (\alpha / \alpha') \Gamma_1(|\alpha'|) \Gamma_1(|\alpha''|) \leq \Gamma_1(|\alpha|)/2.$$  

Proof of Lemma 4.1. It suffices to prove the assertion for $\varepsilon w$ with small $\varepsilon > 0$ so that one can assume $|w| \leq 1$. Thus with $w^2 = F$ there is $A_1 > 0$ such that

$$|\partial_\xi^a \partial_\xi^b F| \leq A_1^{[\alpha + \beta]} \Gamma_1(|\alpha + \beta|) \langle \xi \rangle^{-|\alpha + \beta|}$$

holds for any $\alpha, \beta$. Noting $|\partial_\xi^a \partial_\xi^b w| \leq C_{a b} w^{-1}|\alpha + \beta| \langle \xi \rangle^{-|\alpha + \beta|}$ for any $\alpha, \beta$ we choose $A \geq 2 A_1$ so that $C_{a b} \leq A^{[\alpha + \beta]} \Gamma_1(|\alpha + \beta|)$ for $|\alpha + \beta| > 4$ then we have

$$|\partial_\xi^a \partial_\xi^b F| \leq A^{[\alpha + \beta]} \Gamma_1(|\alpha + \beta|) \langle \xi \rangle^{-|\alpha + \beta|} \langle \xi \rangle^{-|\alpha + \beta|} w^{-1} \langle \xi \rangle^{-|\alpha + \beta|} \langle \xi \rangle^{-|\alpha + \beta|}.$$  

(4.5)

Suppose that (4.5) holds for $|\alpha + \beta| \leq k$, $4 \leq k$ and let $|\alpha + \beta| = k + 1 \geq 4$. Noting

$$2 w^2 \partial_\xi^a \partial_\xi^b w = - \sum_{1 \leq |\alpha + \beta| \leq k} \left( \begin{array}{c|c} \alpha & \beta \\ \hline \alpha' & \beta' \end{array} \right) \partial_\xi^a \partial_\xi^b w \partial_\xi^a \partial_\xi^b F$$

and $w^{-1} \geq 1$, applying Lemma 4.2 we see that $w |\partial_\xi^a \partial_\xi^b w|$ is bounded by

$$\frac{1}{2} A_1^{[\alpha + \beta]} \Gamma_1(|\alpha + \beta|) w^{-2} \langle \xi \rangle^{-|\alpha + \beta|} \langle \xi \rangle^{-|\alpha + \beta|} + A_1^{[\alpha + \beta]} \Gamma_1(|\alpha + \beta|) w^{-2} \langle \xi \rangle^{-|\alpha + \beta|} \langle \xi \rangle^{-|\alpha + \beta|}.$$  

Since we have $w^{-2} \langle \xi \rangle^{-|\alpha + \beta|} \leq \langle \xi \rangle^{-|\alpha + \beta|}$ we conclude that (4.5) holds for $|\alpha + \beta| = k + 1$. Therefore noting $w^{-1} \langle \xi \rangle^{-|\alpha + \beta|} \leq 1$ we get

$$|\partial_\xi^a \partial_\xi^b w| \leq A_1^{[\alpha + \beta]} \Gamma_1(|\alpha + \beta|) w^{-2} \langle \xi \rangle^{-|\alpha + \beta|} (1 + |\alpha + \beta|^{-1} \langle \xi \rangle^{-|\alpha + \beta|}).$$

The assertion for $\omega$ is proved similarly. As for $\omega^{-1}$, using

$$\omega |\partial_\xi^a \partial_\xi^b \omega^{-1}| \leq \sum \left( \begin{array}{c|c} \alpha & \beta \\ \hline \alpha' & \beta' \end{array} \right) A_1^{[\alpha + \beta]} \Gamma_1(|\alpha' + \beta'|) \Gamma_1(|\alpha + \beta| - |\alpha' + \beta'|) \langle \xi \rangle^{-|\alpha + \beta|} \langle \xi \rangle^{-|\alpha + \beta|} \langle \xi \rangle^{-|\alpha + \beta|}$$

the proof follows from induction on $|\alpha + \beta|$.  

\(\square\)
We now introduce a basic weight symbol which plays a key role in obtaining energy estimates:

\[ (4.6) \quad \psi = \langle \xi \rangle^k \log (\phi + \omega), \quad k = \rho - \delta. \]

**Lemma 4.3.** We have \((\phi + \omega)^{\pm 1} \in S_{\rho, \delta}^{(1, \gamma)}((\phi + \omega)^{\pm 1})\). We have also \(\psi \in S_{\rho, \delta}^{(1, \gamma)}(\langle \xi \rangle^k \log (\xi)_{\gamma})\).

Moreover \(\partial^{\beta}_{\xi} \partial^{\beta}_{\omega} W \in S_{\rho, \delta}^{(1, \gamma)}((\phi + \omega)^{\pm 1})\) for \(|\alpha + \beta| = 1\).

Proof. With \(W = \phi + \omega\) we put for \(|\alpha + \beta| = 1\)

\[ (4.7) \quad \partial^{\beta}_{\xi} \partial^{\beta}_{\omega} W = \frac{\partial^{\beta}_{\xi} \partial^{\beta}_{\omega} \phi}{\omega} + \frac{\partial^{\beta}_{\xi} \partial^{\beta}_{\omega} (\langle \xi \rangle^k \log (\xi)_{\gamma})^{\pm 2}}{2\omega} = \Phi^{\beta}_{\phi} W + \Psi^{\beta}_{\phi}. \]

We examine \(\partial^{\beta}_{\xi} \partial^{\beta}_{\omega} \phi \in S_{\rho, \delta}^{(1, \gamma)}((\phi + \omega)^{\pm 1})\) for \(|\alpha + \beta| = 1\). Indeed noting \(\omega^{-1} \langle \xi \rangle^{\delta}_{\gamma} \leq 1\) we have

\[ |\partial^{\beta}_{\xi} \partial^{\beta}_{\omega} \phi| \leq CA^{[\mu + \nu + 1]}(\phi + \omega) + |\mu + \nu|^{[\rho + \sigma + \nu + \mu + \beta]}(\langle \xi \rangle^{\delta - 2})_{\gamma}^{[\mu + \nu + \nu + \mu + \beta]} \]

\[ \leq CA^{[\mu + \nu + 1]}(\phi + \omega) + |\mu + \nu|^{[\rho + \sigma + \nu + \mu + \beta]}(\langle \xi \rangle^{\delta - 2})_{\gamma}^{[\mu + \nu + \nu + \mu + \beta]}, \quad \forall \mu, \nu. \]

Since \(\omega^{-1} \in S_{\rho, \delta}^{(1, \gamma)}((\omega)^{\pm 1})\) one can find \(A_1 > 0\) such that

\[ |\partial^{\alpha}_{\xi} \partial^{\beta}_{\omega} \phi| \leq A_1^{[\mu + \nu + 1]}(\langle \xi \rangle^{\delta - 2})_{\gamma}^{[\mu + \nu + \nu + \mu + \beta]}|\mu + \nu|^{[\rho + \sigma + \nu + \mu + \beta]} \]

holds for \(|\alpha + \beta| = 1\). Since \(\langle \xi \rangle^{\delta - 2} \leq W\) similar arguments prove

\[ |\partial^{\alpha}_{\xi} \partial^{\beta}_{\omega} \psi| \leq A_1^{[\mu + \nu + 1]} W(\langle \xi \rangle^{\delta - 2})_{\gamma}^{[\mu + \nu + \nu + \mu + \beta]}|\mu + \nu|^{[\rho + \sigma + \nu + \mu + \beta]}, \quad \forall \mu, \nu. \]

Now suppose

\[ (4.9) \quad |\partial^{\beta}_{\xi} \partial^{\beta}_{\omega} W| \leq CA_1^{[\mu + \nu + \nu + \mu + \beta]}|\mu + \nu|^{[\rho + \sigma + \nu + \mu + \beta]} \]

holds for \(|\alpha + \beta| = \ell\) and letting \(|\alpha + \beta + e_1 + e_2| = \ell + 1\) we see

\[ |\partial^{\beta}_{\xi} \partial^{\beta}_{\omega} W| \leq C \sum A_1^{[\mu + \nu + \nu + \mu + \beta]} \]

\[ \times |\alpha + \beta|^{[\rho + \sigma + \nu + \mu + \beta]}|\mu + \nu|^{[\rho + \sigma + \nu + \mu + \beta]}. \]

Thus, it suffices to choose \(A_2\) so that \(A_1(A_2 - 1)^{-1} + C^{-1}(A_1 A_2^{-1}) \leq 1\) to conclude \(W + \omega \in S_{\rho, \delta}^{(1, \gamma)}(\phi + \omega)\). For \((\phi + \omega)^{-1}\) it suffices to repeat the proof of Lemma 4.1.

We turn to the next assertion. From \(\langle \xi \rangle^{\delta - 2} / C \leq \phi + \omega \leq \xi \log (\xi)_{\gamma}\) it is clear \(|\phi| \leq \langle \xi \rangle^{\delta}_{\gamma} \log (\xi)_{\gamma}\).

Since \(\partial^{\beta}_{\xi} \partial^{\beta}_{\omega} \log (\phi + \omega) = \partial^{\beta}_{\xi} \partial^{\beta}_{\omega} \log (\phi + \omega)\) for \(|\alpha + \beta| = 1\) and \((\phi + \omega)^{\pm 1} \in S_{\rho, \delta}^{(1, \gamma)}((\phi + \omega)^{\pm 1})\) we see \(\psi \in S_{\rho, \delta}^{(1, \gamma)}((\langle \xi \rangle^{\delta}_{\gamma} \log (\xi)_{\gamma})\). Since \(\partial^{\beta}_{\xi} \partial^{\beta}_{\omega} \phi \in S_{\rho, \delta}^{(1, \gamma)}((\langle \xi \rangle^{\delta}_{\gamma})\) for \(|\alpha + \beta| = 1\) and \(\omega^{-1} \in S_{\rho, \delta}^{(1, \gamma)}((\omega)^{\pm 1})\) it follows from (4.7) and (4.9) that

\[ |\partial^{\beta}_{\xi} \partial^{\beta}_{\omega} (\partial^{\beta}_{\xi} \partial^{\beta}_{\omega} W) | \leq CA_{\mu + \nu + \nu + \mu + \beta} W(\langle \xi \rangle^{\delta - 2})_{\gamma}^{[\mu + \nu + \nu + \mu + \beta]} \]

\[ \times (1 + |\mu + \nu|^{[\rho + \sigma + \nu + \mu + \beta]}(\langle \xi \rangle^{\delta - 2})_{\gamma}^{[\mu + \nu + \nu + \mu + \beta]}). \]
which proves the second assertion. □

5. Composition formula (energy estimates)

In studying $\text{Op}(e^{\psi})P\text{Op}(e^{-\psi}) = \text{Op}(e^{\psi}P e^{-\psi})$, if $\psi \in S^{(1,\lambda)}_{\rho,\delta}(\xi)^{k'}$ with $k' < \rho - \delta$ one can apply the calculus obtained in [19] to get an asymptotic formula of $e^{\psi}P e^{-\psi}$, where the proof is based on the almost analytic extension of symbols and the Stokes’ formula using a space $\rho - \delta - k' > 0$. In the present case $\psi \in S^{(1,\lambda)}_{\rho,\delta}(\xi)^{k}$ there is no space between $\kappa$ and $\rho - \delta$ and then, introducing a small parameter $\epsilon > 0$, we carefully estimate $e^{\psi}P e^{-\psi}$ directly to obtain the composition formula in Theorem 5.1 below.

We denote $a(x,\xi;\gamma,\epsilon) \in e^{\epsilon}S^{(1,\lambda)}_{\rho,\delta}(W)$ if $e^{-\epsilon}a \in S^{(1,\lambda)}_{\rho,\delta}(W)$ uniformly in $0 < \epsilon \ll 1$. Our aim in this section is to give a sketch of the proof of

**Theorem 5.1.** Let $p(x,\xi) \in S^{(1)}(\xi^{m})$. Then there exists $\epsilon_0 > 0$ such that one can find $K = 1 + r$, $r \in \sqrt{E}S_{\rho,\delta}(1)$ and $\gamma_0(\epsilon) > 0$ so that we have for $\gamma \geq \gamma_0(\epsilon)$

\[ e^{\psi}P e^{-\psi} = K + R \]

(5.1)

where $p_{(\alpha)}^{(\beta)} = \partial_{\xi}^{\alpha}p_{(\alpha)}^{(\beta)}$ and $c^\alpha_{(\beta)} \in S_{\rho,\delta}(\xi^{(\beta - \delta)}\rho_{(\beta)}^{(\delta)})$, $R \in S_{\rho,\delta}(\xi^{m-\delta(\mu+1)})$. Moreover we have $c^0_{(\beta)} \in S_{\rho,\delta}(\omega^{-1}(\xi)^{m-\delta(\mu+1)})$ for $|\alpha + \beta| = 1$. In particular $e^{\psi}P e^{-\psi} \in S_{\rho,\delta}(\xi^{m})$.

Denote

\[ p_{(\alpha)}^{(\beta)}(x,\xi) = \sum_{|\alpha + \beta| \leq m} \frac{\epsilon^{|\alpha + \beta|}}{\alpha!\beta!} p_{(\alpha)}^{(\beta)}(-i\nabla_{\xi}\psi)^\alpha(i\nabla_{\xi}\psi)^\beta \]

(5.2)

which will be the principal part of $e^{\psi}P e^{-\psi}K$ and differs from the second term on the right-hand side of (5.1) by multiplicative factor $O(\epsilon^{1/2})$.

**5.1. Estimates of symbol $e^{\psi}$.** Let $H = H(x,\xi;\gamma) > 0$ be a positive function. Assume that $f$ satisfies

\[ |\partial_{\xi}^{\nu}\partial_{\xi}^{\mu}f| \leq C_0A_0^{\mu+1}(|\mu + \nu| - 1)! \times (1 + (|\mu + \nu| - 1)^{1-1}(\xi)^{1-1}|\mu + \nu| - 1) \]

for $|\mu + \nu| \geq 1$. Set $\Omega_{\mu}^{\nu} = e^{-f}d_{\xi}^{\beta}d_{\xi}^{\alpha}e_{\xi}^{\epsilon}$ then we have

**Lemma 5.1.** Notations being as above. There exist $A_i, C > 0$ such that the following estimate holds for $|\alpha + \beta| \geq 1$:

\[ |\partial_{\xi}^{\alpha}\partial_{\xi}^{\beta}e_{\xi}^{\epsilon}| \leq CA_i^{|\alpha + \beta|}A_2^{\mu+1}(\xi)^{\alpha+1}(\xi)^{\beta+1} \times \sum_{j=1}^{\mu+1} \{H^{\alpha+\beta-j+1}(\xi - (|\alpha + \beta| + 1))^{\alpha+\beta-j+1} \}

Corollary 5.1. We have with some $A, C > 0$

\[ |\partial_{\xi}^{\alpha}\partial_{\xi}^{\beta}e_{\xi}^{\epsilon}| \leq CA_i^{|\alpha + \beta|}(\xi)^{\alpha+1}(\xi)^{\beta+1} \times (H + |\alpha + \beta| + |\alpha + \beta|) \}

\]
Moreover for \(|\alpha + \beta| \geq 1, |\partial^\alpha Y^\beta| \) is bounded by

\[
\mathcal{C}A^{1 + |\beta|}(\xi)'_{\gamma}^{\delta + \rho\beta}(H + |\alpha + \beta| + |\alpha + \beta|^2(\xi)'_{\gamma}^{-\delta/2})|\alpha + \beta|^{-1}e^{\lambda}.
\]

**Corollary 5.2.** Notations being as above. We have for \(|\alpha + \beta| \geq 1
\]

\[
\Omega^\beta_{\alpha} \in S^{(1,\gamma)}(H(H + |\alpha + \beta| + |\alpha + \beta|^2(\xi)'_{\gamma}^{-\delta/2})|\alpha + \beta|^{-1}e^{\lambda}.
\]

**Corollary 5.3.** Let \(\omega^\alpha_{\beta} = e^{-\lambda} \partial^\alpha Y^\beta Y^\gamma\). Then there exists \(\gamma_0(\epsilon) > 0\) such that \(\omega^\alpha_{\beta} \in e^{1 + \lambda} S^{(1,\gamma)}((\xi)'_{\gamma}^{\delta + \rho\beta})\) for \(\gamma \geq \gamma_0(\epsilon)\).

**5.2. Estimates of \((\omega Y^\gamma)\# e^{-\lambda}\).** Let \(\chi(r) \in \gamma(r)(\mathbb{R})\) be 1 in \(|r| \leq 1/4\) and 0 outside \(|r| \leq 1/2\). Let \(b \in S^{(1,\gamma)}(\omega Y^\gamma)\) and consider

\[
(b e^{-\lambda})\# e^{-\lambda} = \int e^{-2i(\eta - \gamma)Y} b(X + Y) e^{\lambda(Y + Y)} dYdZ
\]

\[
= b(X) + \int e^{-2i(\eta - \gamma)Y} b(X + Y) (e^{\lambda(Y + Y) - \lambda(Y + Z) - 1}) dYdZ
\]

where \(Y = (y, \eta), Z = (z, \xi)\). Denoting \(\hat{\chi} = \chi(|\eta| / 4)(\xi)_{\gamma}^{-1}\), \(\tilde{\chi} = \chi(|y| / 4)(|z| / 4)\) we write

\[
\int e^{-2i(\eta - \gamma)Y} b(X + Y) (e^{\lambda(Y + Y) - \lambda(Y + Z) - 1}) (1 - \tilde{\chi}) dYdZ
\]

After the change of variables \(Z \to Z + Y\) the first integral turns to

\[
\int e^{-2i(\eta - \gamma)Y} b(X + Y) (e^{\lambda(Y + Y) - \lambda(Y + Z) - 1}) \hat{\chi}_0 dYdZ
\]

where we have set \(\hat{\chi}_0 = \hat{\chi}(y, z) \chi(\xi)_{\gamma}^{-1}\).

**Lemma 5.2.** Let \(\Psi(X, Y, Z) = \psi(X + Y) - \psi(X + Y + Z)\) then on the support of \(\hat{\chi}_0\) one has

\[
|\Psi(X, Y, Z)| \leq C(\xi)'_{\gamma} g^1_{\gamma} (Z)
\]

where \(g(X, \xi)(y, \eta) = (\xi)'_{\gamma}^{2\eta} |\eta|^2 + (\xi)'_{\gamma}^{-2\eta} |\eta|^2\) and for \(|\alpha + \beta| \geq 1
\]

\[
|\partial^\alpha Y^\beta Y^\gamma| e^{\lambda} \leq C A^{1 + |\beta|}(\xi)'_{\gamma}^{\delta + \rho\beta}(\xi)'_{\gamma}^{1/2}(\xi)'_{\gamma}^{2\eta} g^1_{\gamma}(Z)
\]

\[
\times (e^{\xi(\xi)'_{\gamma}^{2\eta} |\eta|^2} + |\alpha + \beta|^2(\xi)'_{\gamma}^{2\eta} |\eta|^2 + g_{\gamma}(Y)).
\]

Proof. The assertions follow from Lemma 4.2 and Corollary 5.2.

Introducing the following differential operators and symbols

\[
L = 1 + 4^{-1}(\xi)'_{\gamma}^{2\eta} |D_{\eta}|^2 + 4^{-1}(\xi)'_{\gamma}^{-2\eta} |D_{\eta}|^2,
\]

\[
M = 1 + 4^{-1}(\xi)'_{\gamma}^{1/2} |D_{\eta}|^2 + 4^{-1}(\xi)'_{\gamma}^{-1/2} |D_{\eta}|^2,
\]

\[
\Phi = 1 + (\xi)'_{\gamma}^{2\eta} |\eta|^2 + (\xi)'_{\gamma}^{-2\eta} |\eta|^2 = 1 + (\xi)'_{\gamma}^{2\eta} g_{\gamma}(Y),
\]

\[
\Theta = 1 + (\xi)'_{\gamma}^{1/2} |\eta|^2 + (\xi)'_{\gamma}^{-1/2} |\eta|^2 = 1 + g_{\gamma}(Y)
\]

so that \(\Phi^{-1} L^n e^{-2i(\eta - \gamma)Y} = e^{-2i(\eta - \gamma)Y}, \Theta^{-1} M^n e^{-2i(\eta - \gamma)Y} = e^{-2i(\eta - \gamma)Y}\) we make integration
by parts in (5.3). Let $F = b(X + Y)(e^{\Psi} - 1),$ $\chi^* = \chi(\epsilon \Phi),$ $\chi_* = 1 - \chi^*$ and note $|(\xi)_{\gamma}^{0\rho} \partial_{\xi}^\rho ((\xi)_{\gamma}^0 \partial_{\xi}^\gamma \chi^*| \leq C_{\alpha\beta}$ with $C_{\alpha\beta}$ independent of $\epsilon.$ Consider

$$\int e^{-2i(\xi - y_{\gamma})} \chi^* \partial_{\xi}^\alpha F_{\chi^*} dYd\zeta$$

(5.4)

$$= \int e^{-2i(\xi - y_{\gamma})} \Phi^{-N} L^N \Theta^{-\varepsilon} M^\varepsilon (\chi^* \partial_{\xi}^\alpha F_{\chi^*}) dYd\zeta.$$

Applying Corollary 5.1 we can estimate the integrand of the right-hand side of (5.4):

$$|\Phi^{-N} L^N \Theta^{-\varepsilon} M^\varepsilon (\chi^* \partial_{\xi}^\alpha F_{\chi^*})| \leq C \epsilon^{-1} \Phi^{-N} L^N \Theta^{-\varepsilon} \left( \epsilon (\xi)_{\gamma}^{0\rho} g_{\chi^*}^{1/2}(Z) \right)$$

(5.5)

$$\times (\epsilon (\xi)_{\gamma}^{0\rho} g_{\chi^*}^{1/2}(Z) + 2N + |\alpha + \beta| + (2N + |\alpha + \beta|) \epsilon (\xi)_{\gamma}^{0\rho} g_{\chi^*}^{1/2}(Z) + 2N + (2N + |\alpha + \beta|) \epsilon (\xi)_{\gamma}^{0\rho} g_{\chi^*}^{1/2}(Z)$$

$$+ \epsilon (\xi)_{\gamma}^{0\rho} g_{\chi^*}^{1/2}(Z) + 2N + |\alpha + \beta| + (2N + |\alpha + \beta|) \epsilon (\xi)_{\gamma}^{0\rho} g_{\chi^*}^{1/2}(Z) \omega'(X + Y) (\xi)_{\gamma}^m.$$

Here we remark the following easy lemma.

**Lemma 5.3.** Let $A \geq 0, B \geq 0.$ Then there exists $C > 0$ independent of $n, m \in \mathbb{N}, A, B$ such that

$$(A + n + m + (n + m)^{2}B)^{nm} \leq C^{n+m}(A + n + m^2B)^n(A + m + n^2B)^m.$$  

Since $|e^{\Psi} - 1| \leq C |e^\Psi| \leq C e^{\Phi^{1/2}} \leq C \sqrt{\epsilon}$ on the support of $\chi^*$, the right-hand side of (5.5) can be estimated by

$$C \epsilon^{-1} \Phi^{-N} L^N \Theta^{-\varepsilon} \left( \epsilon (\xi)_{\gamma}^{0\rho} g_{\chi^*}^{1/2}(Z) + 2N - 1 \right)$$

$$\times (\epsilon (\xi)_{\gamma}^{0\rho} g_{\chi^*}^{1/2}(Z) + 2N - 1 + (2N + |\alpha + \beta|) \epsilon (\xi)_{\gamma}^{0\rho} g_{\chi^*}^{1/2}(Z))$$

$$\times (2N + |\alpha + \beta|) \epsilon (\xi)_{\gamma}^{0\rho} g_{\chi^*}^{1/2}(Z) \omega'(X + Y) (\xi)_{\gamma}^m.$$

where we remark $\omega^{1/2}(X + Y) \leq C \omega^{1/2}(X)(1 + g_{\chi^*}^{1/2}(Y))$ on the support of $\hat{\chi}_0$ and hence we have $\omega'(X + Y) \leq C \omega'(X)\Theta^{1/2}$ with some $t'$. Noting

$$A_1^{2N} \Phi^{-N} (\xi)_{\gamma}^{0\rho} g_{\chi^*}^{1/2}(Z) + 2N + (2N + (2N + |\alpha + \beta|) \epsilon (\xi)_{\gamma}^{0\rho} g_{\chi^*}^{1/2}(Z))$$

$$= (\epsilon A_1 (\xi)_{\gamma}^{0\rho} g_{\chi^*}^{1/2}(Z)) + 2A_1 N \Phi^{1/2} + A_1 (2N + (\xi)_{\gamma}^{0\rho} g_{\chi^*}^{1/2}(Z))$$

$$\omega'(X + Y) (\xi)_{\gamma}^m.$$
choosing $\tilde{c}$ small and $\gamma \geq \gamma_0(\epsilon)$ large. On the other hand since $\langle \xi \rangle^{\epsilon}_{\gamma} \varphi_{X}^{1/2}(Z) \leq \Phi^{1/2}$ it is clear
\[
(\epsilon(\xi)^{\epsilon}_{\gamma} \varphi_{X}^{1/2}(Z) + |\alpha + \beta| + |\alpha + \beta\langle \xi \rangle^{\epsilon - \delta/2}_{\gamma} + \epsilon\Phi^{1/2} = \bar{C}A^{[r + \beta]}(\alpha + \beta| + |\alpha + \beta\langle \xi \rangle^{\epsilon - \delta/2}_{\gamma} + \epsilon\Phi^{1/2}.
\]
Set $\ell = \ell - \ell$. Then noting $e^{-\epsilon\Phi^{1/2}} \leq C\Phi^{-\ell}$ we have
\[
|\Phi^{-N}L^{N}\Theta^{-\ell}M'(\chi, \partial_x \partial_{\xi}^2 F_{\tilde{X}_0})| \leq \sqrt{\epsilon} \bar{C}A^{[r + \beta]}(\alpha + \beta| + |\alpha + \beta\langle \xi \rangle^{\epsilon - \delta/2}_{\gamma} + \epsilon\Phi^{1/2} \Theta^{-\ell} \Phi^{-\ell}.
\]
Finally choosing $\ell > \ell + (n + 1)/2$ and recalling $\int \Theta^{-\ell} \Phi^{-\ell} dYdZ = C$ we conclude
\[
\int e^{-2i(\pi-x-y)\chi}\partial_x \partial_{\xi}^2 F_{\tilde{X}_0}dYdZ \leq \sqrt{\epsilon} \bar{C}A^{[r + \beta]}(\alpha + \beta| + |\alpha + \beta\langle \xi \rangle^{\epsilon - \delta/2}_{\gamma} + \epsilon\Phi^{1/2}.
\]
We next consider
\[
\int e^{-2i(\pi-x-y)\chi}\partial_x \partial_{\xi}^2 F_{\tilde{X}_0}dYdZ = \int e^{-2i(\pi-x-y)\chi}F_{\tilde{X}_0}dYdZ.
\]
Similar arguments obtaining (5.6) show that
\[
|\Phi^{-N}L^{N}\Theta^{-\ell}M'(\chi, \partial_x \partial_{\xi}^2 F_{\tilde{X}_0})| \leq \bar{C}A^{[r + \beta]}(\alpha + \beta| + |\alpha + \beta\langle \xi \rangle^{\epsilon - \delta/2}_{\gamma} + \epsilon\Phi^{1/2} \Theta^{-\ell} \Phi^{-\ell}.
\]
Since $\Phi^{1/2} \geq e^{-1/2}$ on the support of $\chi$, we see $e^{-\epsilon\Phi^{1/2}} \leq e^{-\epsilon e^{-1/2}} \leq C \sqrt{\epsilon}$ and this proves
\[
\int e^{-2i(\pi-x-y)\chi}\partial_x \partial_{\xi}^2 F_{\tilde{X}_0}dYdZ \leq \sqrt{\epsilon} \bar{C}A^{[r + \beta]}(\alpha + \beta| + |\alpha + \beta\langle \xi \rangle^{\epsilon - \delta/2}_{\gamma} + \epsilon\Phi^{1/2}.
\]
We then consider
\[
\int e^{-2i(\pi-x-y)\chi}(|\Delta|^2 + |\Delta|^2)^{-N}(D_{\xi}^2 + [D_{\eta}]^2)^{2N}F_{\tilde{X}_0}dYdZ
\]
where $F = b(X + Y)(e^{\epsilon\Phi(X+Y)} - e^{\Phi(X+Z)} - 1)$ and $\tilde{\chi}_1 = (1 - \tilde{\chi})\chi$. Let $\kappa < \kappa_1 < \rho$ then since $|\varphi(X+Y)| + |\varphi(X+Z)|$ is bounded by $C(\xi)^{\epsilon_1}$ and $C^{-1} < \langle \xi + \eta\rangle/\langle \xi \rangle^{\gamma}, (\xi + \zeta\rangle/\langle \xi \rangle^{\gamma} \leq C$ with some $C > 0$ on the support of $\chi$ thanks to Corollary 5.1 it is not difficult to show
\[
\int (D_{\xi}^2 + [D_{\eta}]^2)^{2N}F_{\tilde{X}_0} \leq \bar{C}\left(\begin{array}{c}
\langle \xi \rangle^{2\rho N}(\langle \xi \rangle^{\epsilon_1} + |\alpha + \beta| + |\alpha + \beta\langle \xi \rangle^{\epsilon - \delta/2}_{\gamma} + \epsilon\Phi^{1/2} \\
\times \langle \xi \rangle^{2\rho N}(\langle \xi \rangle^{\epsilon_1} + 2N + (2N)^{\gamma} + (\langle \xi \rangle^{\epsilon - \delta/2}_{\gamma})^{2N}e^{\epsilon\Phi^{1/2}}
\end{array}\right).
\]
Choose $N = c_1(\xi)^{\epsilon_1}$ with small $c_1 > 0$ so that
\[
A^{2N}(\xi)^{\epsilon_1} + 2N + (2N)^{\gamma} \leq C e^{\epsilon\Phi^{1/2}} e^{-\epsilon \langle \xi \rangle^{\epsilon_1}}
\]
is bounded by $C e^{\epsilon\Phi^{1/2}} e^{-\epsilon \langle \xi \rangle^{\epsilon_1}}$ is bounded by $C A^{[r + \beta]} e^{\epsilon\Phi^{1/2}}$. Then noting $\omega'(X +
\[ Y \leq C\omega'(X)(\xi)^{\varepsilon'} \text{ and } e^{-c(\xi)^{\varepsilon'}} \leq \sqrt{t} C(\xi)^{-\gamma_{n+1}^T} \text{ for } \gamma \geq \gamma_0(\varepsilon) \text{ and that } \langle \xi \rangle_{\gamma}^{-\gamma_{n+1}^T} \int (|y|^2 + |z|^2)^{-N} \chi \cdot dYdZ \leq C \text{ we conclude} \]

**Lemma 5.4.** Let \( \hat{\chi} = \chi((\eta)_{\gamma}^{1/4}) \chi((\xi)_{\gamma}^{-1}) \). Then we have for \( \gamma \geq \gamma_0(\varepsilon) \)

\[
\left| \frac{d^\alpha \beta}{\xi} \int e^{-2(\eta^2 - \psi)(b(X + y)(e^{\psi(X + y)} - \psi(X + z)) - 1)} \hat{\chi} \cdot dYdZ \right| \\
\leq \sqrt{t} C A^{\varepsilon + \beta}(|\alpha + \beta| + |\alpha + \beta|^{\delta/2})^{\varepsilon + \beta} \omega'(X)(\xi)^{\varepsilon + \beta} e^{-c(\eta)^{\varepsilon'}}.
\]

Let us write

\[
1 - \hat{\chi} = (1 - \chi((\eta))_{\gamma}^{1/4})(1 - \chi((\xi))_{\gamma}^{-1}) + (1 - \chi((\eta))_{\gamma}^{-1})((1 - \chi((\xi))_{\gamma}^{-1})}
\]

\[
+ (1 - \chi((\xi))_{\gamma}^{-1})\chi((\eta))_{\gamma}^{-1} = \hat{\chi}_2 + \hat{\chi}_3 + \hat{\chi}_4.
\]

Denoting \( F = b(X + y)(e^{\psi(X + y)} - \psi(X + z)) - 1 \) again we consider

\[
\int e^{-2(\eta^2 - \psi)(\eta)^{-N}} \langle \xi \rangle^{-2N} \langle D_{\eta} \rangle^{-2N} \langle D_{\xi} \rangle^{-2N} (y)^{-2\ell}(z)^{-2\ell} (\alpha + \beta)|^{\delta/2} F \cdot \hat{\chi} \cdot dYdZ
\]

where \( \chi' \) is either \( \chi((\xi))_{\gamma}^{-1}/4 \) or \( 1 - \chi((\xi))_{\gamma}^{-1}/4 \). If \( \chi' = \chi((\xi))_{\gamma}^{-1}/4 \) we choose \( N_1 = \ell, N_2 = N \) and noting \( \omega'(X + Y) \leq C(\eta)^{\varepsilon'} \) with some \( \varepsilon' \geq 0 \) and \( \psi(X + Y) + |\psi(X + Z)| \leq C(\eta)^{\varepsilon} \) with \( \kappa < \kappa_1 < \rho \) on the support of \( \hat{\chi} \cdot \chi' \) it is not difficult to see that

\[
\langle \eta \rangle^{-2N} \langle \xi \rangle^{-2\ell} \langle D_{\eta} \rangle^{-2\ell} \langle D_{\xi} \rangle^{-2\ell} (y)^{-2\ell}(z)^{-2\ell} (\alpha + \beta)|^{\delta/2} F \cdot \hat{\chi} \cdot \chi' \]

is bounded by

\[
C A^{2N + \varepsilon + \beta} \langle \eta \rangle^{-2N} \langle \xi \rangle^{-2\ell} (y)^{-2\ell}(z)^{-2\ell} (\eta)^{m + \varepsilon} e^{-c(\eta)^{\varepsilon'}}
\]

\[
\times ((\eta)^{\varepsilon_1} + 2N(\eta)^{\delta} + N^2(\eta)^{\delta/2})^{2N}
\]

\[
\times ((\eta)^{\varepsilon_1} + (\eta)^{\delta} |\alpha + \beta| + (\eta)^{\delta/2} |\alpha + \beta|^{\delta/2}) e^{C(\eta)^{\varepsilon}}.
\]

Here writing

\[
A^{2N}(\eta)^{-2N}(\eta)^{\varepsilon_1} + 2N(\eta)^{\delta} + N^2(\eta)^{\delta/2})^{2N}
\]

\[
= \left( A(\eta)^{\varepsilon_1} + 2AN + AN^2(\eta)^{\delta/2} \right)^{2N}
\]

we take \( 2N = c_1(\eta)^{\varepsilon} \) with small \( c_1 > 0 \) so that the right-hand side is bounded by \( A e^{-c(\eta)^{\varepsilon}} \).

Noting \( (\eta)^{\delta} |\alpha + \beta| e^{-c(\eta)^{\varepsilon}} \leq CA^{\varepsilon + \beta} |\alpha + \beta|^{\delta/2} e^{-c(\eta)^{\varepsilon}} \) and \( (\eta)^{\varepsilon_1} + (\eta)^{\delta/2} |\alpha + \beta|^{\delta/2} e^{C(\eta)^{\varepsilon}} \leq CA^{\varepsilon + \beta} |\alpha + \beta|^{\delta/2} e^{-c(\eta)^{\varepsilon}} \) one sees that (5.7) is bounded by

\[
C A^{\varepsilon + \beta} (\xi)^{-2\ell} (y)^{-2\ell}(z)^{-2\ell} (|\alpha + \beta|^{1+\delta/\rho} + |\alpha + \beta|^{s+\delta/2}) e^{-c(\eta)^{\varepsilon}}.
\]

Similarly if \( \chi' = 1 - \chi((\xi))_{\gamma}^{-1}/4 \) choosing \( N_1 = N, N_2 = \ell \) it is proved that (5.7) is estimated by

\[
C A^{\varepsilon + \beta} (\eta)^{-2\ell} (y)^{-2\ell}(z)^{-2\ell} (|\alpha + \beta|^{1+\delta/\rho} + |\alpha + \beta|^{s+\delta/2}) e^{-c(\eta)^{\varepsilon}}.
\]

Thus taking \( 1 + \delta/\rho = 1/\rho \) and \( s + \delta/2 \rho \leq 1/\rho \) into account and recalling that \( (\xi)_\gamma \leq (\eta), (\xi)_\gamma \leq (\xi) \) on the support of \( \hat{\chi}_2 \) we get
Lemma 5.5. We have
\[
\left| \partial_x^2 \partial_y^2 \int e^{-2i(\xi y - \eta z)} b(X + Y)(e^{\psi(X+Y)} - e^{\psi(X+Z)} - 1) \, dYdZ \right|
\leq CA^{\alpha + \beta} |x + \beta|^{\alpha + \beta + \rho} e^{-c_1 |\xi|^\rho}.
\]

Repeating similar arguments we can prove
\[
\left| \partial_x^2 \partial_y^2 \int e^{-2i(\xi y - \eta z)} b(X + Y)(e^{\psi(X+Y)} - e^{\psi(X+Z)} - 1) \, dYdZ \right|
\leq CA^{\alpha + \beta} |x + \beta|^{\alpha + \beta + \rho} e^{-c_i |\xi|^\rho}
\]
for \( i = 3, 4 \). We summarize what we have proved in

Proposition 5.1. Let \( b \in S^{(1,\alpha)}_{\rho,\delta} (\xi)^n \) then we have
\[
(be^{\psi})# = b + \omega \hat{b} + R
\]
where \( \hat{b} \in \sqrt{e} S^{(1,\alpha)}_{\rho,\delta} (\xi)^n \) and \( R \in S^{(1,\rho)}_{0,0} (e^{-c_i |\xi|^\rho}) \), that is
\[
|\partial_x^2 \partial_y^2 R| \leq CA^{\alpha + \beta} |x + \beta|^{\alpha + \beta + \rho} e^{-c_i |\xi|^\rho}.
\]

5.3. Proof of Theorem 5.1. We start with the next lemma which is proved repeating similar arguments in the preceding subsection.

Lemma 5.6. Let \( a \in S^{(1)}(\xi)^n \) and \( b \in S^{(1,\alpha)}_{\rho,\delta} (\xi)^n \). Then we have
\[
(be^{\psi})#a = \sum_{|\alpha| + |\beta| < N} \frac{(-1)^{|\beta|}}{(2i)^{\alpha + \beta} |\alpha| \beta!} a^{(\alpha)}(\psi)^{(\beta)} + b_N e^{\psi} + R
\]
where \( b_N \in S^{(1,\alpha)}_{\rho,\delta} (\xi)^n \), \( R \in S^{(1,\rho)}_{0,0} (e^{-c_i |\xi|^\rho}) \). For all \((be^{\psi})#\) similar assertion holds, where \((-1)^{\beta}\) is replaced by \((-1)^{|\beta|}\).

We can also prove

Lemma 5.7. Let \( R \in S^{(1,\rho)}_{0,0} (e^{-c_i |\xi|^\rho}) \). Then we have
\[
R#e^{\psi}, \quad e^{\psi}#R \in S^{(1,\rho)}_{0,0} (e^{-c_i |\xi|^\rho}).
\]

Corollary 5.4. Let \( R \in S^{(1,\rho)}_{0,0} (e^{-c_i |\xi|^\rho}) \). Then for any \( t \in \mathbb{R} \) we have
\[
e^{\psi}#R#e^{\psi} \in S((\xi)^n).
\]

Lemma 5.8. Let \( p \in S^{(1)}((\xi)^n) \). Then one can write
\[
(e^{\psi})#p = \sum_{|\alpha| + |\beta| < N} \frac{(-1)^{|\beta|}}{(2i)^{|\alpha| + |\beta|} |\alpha| \beta!} p^{(\beta)}(\omega^{\alpha}_\beta e^{\psi}) + r_N e^{\psi} + R
\]
where \( r_N \in S^{(1,\alpha)}_{\rho,\delta} ((\xi)^n) \), \( R \in S^{(1,\rho)}_{0,0} (e^{-c_i |\xi|^\rho}) \) and \( \omega^{\alpha}_\beta = e^{-\psi} \partial_x^2 \partial_y^2 e^{\psi} \).

Proof. We first examine
\[
p^{(\beta)}(\omega^{\alpha}_\beta e^{\psi}) = \sum_{|r| + |s| < N} \frac{(-1)^{|r|}}{(2i)^{|r| + |s|} |r| \! |s|!} p^{(\beta + r)}(\alpha + s) #(\omega^{\alpha + s}_\beta e^{\psi}) = r_N e^{\psi} + R
\]
with \( r_{N;\alpha,\beta} \in S^{(1,\alpha)}_\rho(\langle \xi \rangle^{d-\delta N}) \). Indeed since \( \partial^{\gamma}_{\xi} \partial^{\mu}_{\xi} (\omega^{\alpha}_{\rho} e^{\beta}) = \omega^{\alpha}_{\rho} e^{\beta} \) thanks to Lemma 5.6 one can write
\[
\sum_{|\gamma|+|\delta|<N} \frac{(-1)^{|\gamma|}}{(2i)^{|\gamma|+|\delta|} \gamma! \delta!} P_{(\alpha,\beta)}^{(\gamma,\delta)} (\omega^{\alpha}_{\rho} e^{\beta})
\]
\[
= \sum_{|\gamma|+|\delta|<N} \frac{(-1)^{|\gamma|}}{(2i)^{|\gamma|+|\delta|} \gamma! \delta!} \left( \sum_{\mu} \left( \begin{array}{c} \gamma' \\ \mu \end{array} \right) \left( \begin{array}{c} \delta' \\ \nu \end{array} \right) (1) \right) P_{(\alpha,\beta)}^{(\gamma',\delta')} (\omega^{\alpha}_{\rho} e^{\beta})
\]
\[+ r_{N;\alpha,\beta} e^{\gamma} + R \]
where \( \sum \left( \begin{array}{c} \gamma' \\ \mu \end{array} \right) (1) = 0 \) if \(|\gamma'| + |\delta'| > 0\) so that the right-hand side is
\[
P_{(\alpha,\beta)}^{(\gamma,\delta)} (\omega^{\alpha}_{\rho} e^{\beta}) + r_{N;\alpha,\beta} e^{\gamma} + R,
\]
which follows (5.8). Now insert the expression of \( p_{(\alpha,\beta)}^{(\gamma,\delta)} (\omega^{\alpha}_{\rho} e^{\beta}) \) in (5.8) into
\[
(e^{\gamma})^{#} p = \sum_{|\alpha|+|\beta|<N} \frac{(-1)^{|\beta|}}{(2i)^{|\alpha|+|\beta|+1} \alpha! \beta!} P_{(\alpha,\beta)}^{(\gamma,\delta)} (\omega^{\alpha}_{\rho} e^{\beta}) + r_{N} e^{\gamma} + R
\]
which follows from Lemma 5.6 to get
\[
\sum_{|\alpha|+|\beta|<N} \frac{(-1)^{||\beta|}}{(2i)^{|\alpha|+|\beta|+1} \alpha! \beta!} \left( \sum_{\mu} \left( \begin{array}{c} \gamma' \\ \mu \end{array} \right) \left( \begin{array}{c} \delta' \\ \nu \end{array} \right) (1) \right) P_{(\alpha,\beta)}^{(\gamma',\delta')} (\omega^{\alpha}_{\rho} e^{\beta}) + r_{N} e^{\gamma} + R
\]
where \( \tilde{r}_{N} \in S^{(1,\alpha)}_\rho(\langle \xi \rangle^{d-\delta N}) \). Here we note \( \sum \left( \begin{array}{c} \gamma' \\ \mu \end{array} \right) (1) = 2^{|\gamma|+|\delta|} \). It is clear that \( p_{(\alpha,\beta)}^{(\gamma,\delta)} (\omega^{\alpha}_{\rho} e^{\beta}) = r_{N} e^{\gamma} + R \) with \( r_{N} \in S^{(1,\alpha)}_\rho(\langle \xi \rangle^{d-\delta N}) \) for \(|\alpha|+|\beta| \geq N\) and hence we get the assertion.

Proof of Theorem 5.1. From Lemma 5.8 we see
\[
(e^{\gamma})^{#} p^{#} e^{\gamma} = \sum_{|\alpha|+|\beta|<m} \frac{(-1)^{|\beta|}}{\alpha! \beta!} P_{(\alpha,\beta)}^{(\gamma,\delta)} ((\omega^{\alpha}_{\rho} e^{\beta})^{#} e^{\gamma} + (r_{m} e^{\gamma} + R)^{#} e^{\gamma}
\]
where \( (r_{m} e^{\gamma} + R)^{#} e^{\gamma} \in S_{\rho,\delta}(\langle \xi \rangle^{m-\delta(m+1)}) \) which follows from Propositions 5.1 and Lemma 5.7. Therefore Proposition 5.1 together with Corollary 5.3 gives
\[
(e^{\gamma})^{#} p^{#} e^{\gamma} = \sum_{|\alpha|+|\beta|<m} \frac{(-1)^{|\beta|}}{\alpha! \beta!} P_{(\alpha,\beta)}^{(\gamma,\delta)} (\omega^{\alpha}_{\rho} e^{\beta} + \tilde{\omega}_{\rho}^{\gamma}) + R
\]
where \( \tilde{\omega}_{\rho}^{\gamma} \in e^{1+|\beta|+1/2} S(1,\alpha)_{\rho,\delta}^{(1)}(\langle \xi \rangle^{m-\delta(m+1)}) \) and
\[
\tilde{\omega}_{\rho}^{\gamma} \in e^{1/2} S(1,\alpha)_{\rho,\delta}^{(1)}(\langle \xi \rangle^{m-\delta(m+1)})
\]
Since \( e^{\gamma} \) is a polynomial of \( e^{\gamma} \) in \( S_{\rho,\delta}(1) \) by Proposition 5.1 there exists \( K = 1 + r_{1} \), \( r_{1} \in \sqrt{e} S_{\rho,\delta}(1) \) such that \( e^{\gamma} \# e^{\gamma} \# K = 1 \) if \( 0 < \epsilon \leq \epsilon_{0} \) is small ([1, Theorem 3.2] and [15, Theorem 2.6.27] for example). Thus we have
\[
(e^{\gamma})^{#} p^{#} e^{\gamma} \# K = p + \sum_{1,|\alpha|+|\beta|<m} \frac{(-1)^{|\beta|}}{\alpha! \beta!} P_{(\alpha,\beta)}^{(\gamma,\delta)} (\omega^{\alpha}_{\rho} e^{\beta} + \tilde{\omega}_{\rho}^{\gamma}) + R
\]
On the other hand it is clear \( (\omega^{\alpha}_{\rho} e^{\beta} + \tilde{\omega}_{\rho}^{\gamma})^{#}(1 + r) = \omega^{\alpha}_{\rho} e^{\beta} + \tilde{\omega}_{\rho}^{\gamma} \) with \( \tilde{\omega}_{\rho}^{\gamma} \in e^{1+|\beta|+1/2} S_{\rho,\delta}(\langle \xi \rangle^{m-\delta(m+1)}) \).
Since $\omega_{\beta}^{\alpha} \in eS_{P,\delta}(\omega^{-1}(\xi)^{x-\nu})$ for $|\alpha + \beta| = 1$ it is also clear that $\tilde{\omega}_{\beta}^{\alpha} \in e^{3/2}S_{P,\delta}(\omega^{-1}(\xi)^{x-\nu})$ for $|\alpha + \beta| = 1$. Note

$$p^{(p)}_{(\alpha)}(\omega_{\beta}^{\alpha} + \tilde{\omega}_{\beta}^{\alpha}) - \sum_{|\mu| = m} (\frac{-1}{|\mu|!})^{\mu} p^{(p+\mu)}_{(\alpha+\nu)}(\omega_{\beta}^{\alpha} + \tilde{\omega}_{\beta}^{\alpha})^\nu \in S_{P,\delta}(\omega^{-1}(\xi)^{x-\nu})$$

$$\in S_{P,\delta}(\omega^{-1}(\xi)^{x-\nu})$$

and $(\omega_{\beta}^{\alpha} + \tilde{\omega}_{\beta}^{\alpha})^\nu \in \gamma^{-x}\nu p^{(p)}_{(\alpha)}(\omega_{\beta}^{\alpha} + \tilde{\omega}_{\beta}^{\alpha})^\nu \in S_{P,\delta}(\omega^{-1}(\xi)^{x-\nu})$ which is contained in $e^{1/2 + |\nu| + |\mu|}S_{P,\delta}(\omega^{-1}(\xi)^{x-\nu})$ if $|\mu + \nu| \geq 1, \gamma \geq \gamma_0(\epsilon)$ so that

$$(e^{\nu})^{p}# e^{-\nu}# K = p + \sum_{1 \leq |\alpha + \beta| \leq m} (\frac{-1}{|\alpha + \beta|!})^{\alpha + \beta} p^{(p)}_{(\alpha)}(\omega_{\beta}^{\alpha} + \tilde{\omega}_{\beta}^{\alpha}) + R$$

where $\omega_{\beta}^{\alpha} \in e^{1/2 + |\nu|}S_{P,\delta}(\omega^{-1}(\xi)^{x-\nu})$ and $\tilde{\omega}_{\beta}^{\alpha} \in e^{3/2}S_{P,\delta}(\omega^{-1}(\xi)^{x-\nu})$ for $|\alpha + \beta| = 1$. Now check $\omega_{\beta}^{\alpha}$. For $|\alpha + \beta| = 1$ we have $\omega_{\beta}^{\alpha} = e^{-i\nabla_\psi}\psi^{\alpha}(i\nabla_\psi)^{\beta}$. Let $|\alpha + \beta| \geq 2$ then $\alpha^{(\beta)}$ is a linear combination of terms $(e^{(\psi)}_{(\alpha)}) \cdots (e^{(\psi)}_{(\beta)})$ with $\alpha_1 + \cdots + \alpha_s = \alpha, \beta_1 + \cdots + \beta_s = \beta, |\alpha_j + \beta_j| \geq 1$. If $|\alpha_j + \beta_j| = 1$ for all $j$ it is clear $\omega_{\beta}^{\alpha} = e^{X_{\psi}}(-i\nabla_\psi)^{\beta}(i\nabla_\psi)^{\beta}$. If $|\alpha_j + \beta_j| \geq 2$ for some $j$ so that $s \leq |\alpha + \beta| - 2$ then one has

$$(e^{\psi}_{(\alpha)}) \cdots (e^{\psi}_{(\beta)}) \in S_{P,\delta}(\omega^{-1}(\xi)^{x-\nu}) \subset \gamma^{-2}S_{P,\delta}(\omega^{-1}(\xi)^{x-\nu})$$

Since we can assume $\gamma^{-2} \leq e^{1|\nu| + 1/2}$ for $\gamma \geq \gamma_0(\epsilon)$ we get the assertion. 

\hspace{1cm} \Box

6. Energy estimates

To obtain energy estimates we follow [13] where the main point is to derive microlocal energy estimates. We sketch how to get microlocal energy estimates. Let us denote

$$P_{(\psi)} = Op(e^{\psi})Op(e^{-\psi})Op(K)$$

of which principal symbol is given by $p_{(\psi)} = e^{\psi}# p # e^{-\psi}# K$. In this section we say $a(x, \xi; \gamma, \epsilon) \in S_{P,\delta}(W)$ if $a \in S_{P,\delta}(W)$ for each fixed $0 < \epsilon \ll 1$. Let $a \in S_{P,\delta}(W)$ and let $N \in \mathbb{N}$ be given. Then with a fixed small $0 < 2 \epsilon \ll \epsilon - \delta$ we have

$$|\partial_\xi^{\alpha}\partial_\xi^{\beta}| \leq C_{a\beta}(\epsilon)W(\xi)^{-\alpha - \beta - \delta} \leq C_{a\beta} \gamma^{-2(\alpha + \beta)}W(\xi)^{-\alpha - \beta - \delta}$$

where one can assume that $C_{a\beta}(\epsilon) \gamma^{-2(\alpha + \beta)}$ are arbitrarily small for $1 \leq |\alpha + \beta| \leq N$ taking $\gamma$ large.

6.1. Symbol of $P_{(\psi)}$. Define $h_{f}(x, \xi)$ by

$$h_{f}(x, \xi) = \sum_{1 \leq |\alpha| \leq m} |q_{\alpha}|^{2} \cdot |q_{\beta}|^{2}, \quad q_{\beta} = \gamma^{2} - \lambda_{f}(x, \xi).$$

Lemma 6.1. There exists $c > 0$ such that

$$h_{m-k}(x, \xi - i\epsilon \omega^{-1}(\xi)^{x}) \geq c(\epsilon \omega)^{2(1-k)}(\xi)^{2(1-k)}h_{m-j}(x, \xi - i\epsilon \omega^{-1}(\xi)^{x})$$

for $j = k, \ldots, m$ where $\epsilon_{0} = 1$ and $1 \leq k \leq m$.

Proof. We show the case $k = 1$. By definition $h_{m-1}(x, \xi - i\epsilon \omega^{-1}(\xi)^{x})$ is bounded from below by
\[ 2^{-1}(|q_i(x, \xi)|^2 + |q_j(x, \xi)|^2 + e^{2\omega^{-2}(\xi)^2} \prod_{k \neq i, j} |q_k(x, \xi - i\epsilon \omega^{-1}(\xi)^\epsilon \theta)|^2). \]

From Lemma 3.2 we have \(|q_i(x, \xi)|^2 + |q_j(x, \xi)|^2 \geq c|b'(x, \xi)|^2\). Since
\[
|b'(x, \xi)|^2 + e^{2\omega^{-2}(\xi)^2} = e^{2\omega^{-2}(\xi)^2}(e^{-2|b'(x, \xi)|^2}\omega^{-2}(\xi)^2 + (\xi)^{-2\delta}) \geq c e^{2\omega^2(\xi)^2}
\]
with some \(c > 0\) because \(C|b'(x, \xi)|^2(\xi)^{-2} \geq \phi^2\) and \(\omega^2 \geq \phi^2\) then it is clear that \(h_{m-1}(x, \xi - i\epsilon \omega^{-1}(\xi)^\epsilon \theta)\) is bounded from below by
\[
ce^{2\epsilon\omega^2(\xi)^2} \prod_{k \neq i, j} |q_k(x, \xi - i\epsilon \omega^{-1}(\xi)^\epsilon \theta)|^2.
\]

Summing up over all pair \(i, j\) \((i \neq j)\) we get the assertion for the case \(j = 2\). Continuing this argument one can prove the case \(j \geq 3\). \(\square\)

Let us put
\[
h(x, \xi) = h_{m-1}(x, \xi - i\epsilon \omega^{-1}(\xi)^\epsilon \theta)^{1/2}.
\]

**Lemma 6.2.** There exists \(C > 0\) such that we have
\[
\begin{align*}
|p^{(\alpha)}_{(\beta)}| &\leq C(\epsilon \omega)^{1-|\alpha|+|\beta|}(\xi)^{1-|\alpha|}h, & 1 \leq |\alpha + \beta| \leq m, \\
|p p^{(\alpha)}_{(\beta)}| &\leq C(\epsilon \omega)^{2-|\alpha|+|\beta|}(\xi)^{2-|\alpha|}h^2, & 2 \leq |\alpha + \beta| \leq m.
\end{align*}
\]

Proof. From [4, Proposition 3] one has
\[
|p^{(\alpha)}_{(\beta)}(x, \xi)| \leq Ch_{m-|\alpha|+|\beta|}(x, \xi)^{1/2}\xi^{|\beta|}
\]
for \(|\alpha + \beta| \leq m\) which is bounded by \(Ch_{m-|\alpha|+|\beta|}(x, \xi - i\epsilon \omega^{-1}(\xi)^\epsilon \theta)^{1/2}\xi^{|\beta|}\) clearly. On the other hand it follows from Lemma 6.1
\[
Ch(x, \xi) \geq (\epsilon \omega)^{|\alpha|+|\beta|-1}(\xi)^{|\alpha|+|\beta|-1}h_{m-|\alpha|+|\beta|}(x, \xi - i\epsilon \omega^{-1}(\xi)^\epsilon \theta)^{1/2}
\]
for \(1 \leq |\alpha + \beta| \leq m\) which proves the assertion. The proof of the second assertion is similar. \(\square\)

**Lemma 6.3.** Assume that \(c^\alpha_{\beta} \in S_{p, \delta}(\xi)(\xi)^{0-|\alpha|}\) and \(c^\alpha_{\beta} \in S_{p, \delta}(\omega^{-1}(\xi)^{-|\alpha|})\) for \(|\alpha + \beta| = 1\).
Then for \(1 \leq |\alpha + \beta| \leq m\) we have \(p^{(\alpha)}_{(\beta)} c_{\beta}^\alpha \in S_{p, \delta}(\omega^{-1}(\xi)^\epsilon \theta)\) and \(\epsilon^{x+\beta}p^{(\alpha)}_{(\beta)} c_{\beta}^\alpha \leq C\epsilon\omega^{-1}(\xi)^\epsilon \theta\) with \(C > 0\) independent of \(\epsilon\).

Proof. Let \(2 \leq |\alpha + \beta| \leq m\) then since \(1 = \kappa + 2\delta\) by (4.4) and (4.6) we see by Lemma 6.2
\[
e^{x+\beta}p^{(\alpha)}_{(\beta)} c_{\beta}^\alpha \leq C\epsilon\omega^{-1}(\xi)^{1-|\alpha|+|\beta|}\xi^{0-|\alpha|+|\beta|}\theta \leq C\epsilon\omega^{-1}(\xi)^{x+\beta-2}\theta \leq C\epsilon\omega^{-1}(\xi)^\epsilon \theta.
\]
When \(|\alpha + \beta| = 1\) noting \(c^\alpha_{\beta} \in S_{p, \delta}(\omega^{-1}(\xi)^{0-|\beta|})\) we get the same assertion. We next estimate \(\sum p^{(\alpha+\mu')}_{(\beta+\nu')} c_{\beta}^\alpha c_{\beta}^{\mu'} c_{\beta}^{\nu'}\). If \(|\alpha + \mu' + \beta + \nu'| \geq m\) we have
\[
\left| P_{(\beta+\nu')}(\alpha^{\nu}) \right| \leq \langle \xi \rangle^m C_{\omega^{-|\alpha|}} \langle t \rangle^{|\nu|} \langle \xi \rangle^{|\beta|} \langle |\nu| + |\beta| \rangle
\]
\[
\leq C_{\omega^{-|\alpha|}} \langle \xi \rangle^{|\beta|} \langle t \rangle^{|\nu|} \langle \xi \rangle^{|\nu|} \langle |\nu| + |\beta| \rangle
\]
\[
\leq C_{\omega^{-|\alpha|}} \langle \xi \rangle^{|\beta|} \langle t \rangle^{|\nu|} \langle \xi \rangle^{|\nu|} \langle |\nu| + |\beta| \rangle
\]
where the right-hand side is bounded by \( C_{\omega^{-|\beta|}} \langle \xi \rangle^{\nu} \langle |\nu| + |\beta| \rangle \). We turn to the case \(|\alpha + \mu' + \beta + \nu'| \leq m\). From Lemma 6.2 it follows
\[
\left| P_{(\beta+\nu')}(\alpha^{\nu}) \right| \leq C_{\omega^{-|\alpha|}} \langle \xi \rangle^{|\beta|} \langle t \rangle^{|\nu|} \langle \xi \rangle^{|\nu|} \langle |\nu| + |\beta| \rangle
\]
therefore for \(|\alpha + \beta| \geq 2\) we see easily
\[
\left| P_{(\beta+\nu')}(\alpha^{\nu}) \right| \leq C_{\omega^{-|\alpha|}} \langle \xi \rangle^{|\beta|} \langle t \rangle^{|\nu|} \langle \xi \rangle^{|\nu|} \langle |\nu| + |\beta| \rangle
\]
which also holds for \(|\alpha + \beta| = 1\) because \( \epsilon_a^0 \in S_{p,\beta}(\omega^{-|\xi|}) \).

\[\square\]

6.2. Definition of \( Q(z) \) which separates \( P_\varphi(z) \). We follow the arguments in [13]. Let us define \( \tilde{p}(x + iy, \xi + i\eta) \) by
\[
\tilde{p}(x + iy, \xi + i\eta) = \sum_{|\alpha| + |\beta| \leq m} \frac{1}{\alpha! \beta!} \partial_\alpha \partial_\beta p(x, \xi)(iy)^\alpha (i\eta)^\beta.
\]
Then \( p_\varphi \) given in (5.2) is expressed as \( \tilde{p}(z - i\epsilon H_\varphi) \), which one can also write as
\[
\tilde{p}(z - i\epsilon H_\varphi) = \sum_{j=0}^{m} \left( \frac{\partial}{\partial t} \right)^j p(z - i\epsilon H_\varphi) / j! \bigg|_{t=0}.
\]
Using this expression we define \( Q(z) \) which separates \( \tilde{p}(z - i\epsilon H_\varphi) \) by
\[
Q(z) = \epsilon^{-1} \langle \tilde{H}_\varphi \rangle^{-1} \left( \frac{\partial}{\partial t} \right)^j \sum_{j=0}^{m} \left( \frac{\partial}{\partial t} \right)^j p(z - i\epsilon H_\varphi) / j! \bigg|_{t=0}
\]
where \( \tilde{H}_\varphi = (\langle \xi \rangle, \nabla \xi, -\nabla \xi) \). By the homogeneity it is clear that
\[
\tilde{p}(z - i\epsilon H_\varphi) = \langle \xi \rangle^m \tilde{p}(z) \frac{\partial}{\partial \xi} / \langle \xi \rangle^{m-1}
\]
where \( \tilde{z} = (x, \xi(\xi^{-1}) \rangle^{-1} \)). It is not difficult to check \( \tilde{p}(z - i\epsilon H_\varphi) \in S_{p,\beta}(\langle \xi \rangle^m) \) and \( Q \in S_{p,\beta}(\langle \xi \rangle^{m-1}) \). We study \( \tilde{p}(z - i\epsilon H_\varphi) \) and \( Q(z) \) in a conic neighborhood of \( \rho \). We first recall

**Proposition 6.1** ([14, Lemma 5.8]). Let \( \rho \) be a characteristic of \( p \) of order \( m \) and let \( K \subset \Gamma_{\rho} \) be a compact set. Then one can find a conic neighborhood \( V \) of \( \rho \) and positive \( C > 0 \) such that for any \((x, \xi) \in V \cdot \xi \in K \) and small \( s \in \mathbb{R} \) one can write
\[
p(z - s\xi) = \epsilon(z, \xi, s) \sum_{j=1}^{m} (s - \mu_j(z, \xi))
\]
where \( \mu_j(z, \xi) \) are real valued and \( \epsilon(z, \xi, s) \neq 0 \) for \((z, \xi, s) \in V \times K \times |s| < s_0 \). Moreover there exists \( C > 0 \) such that we have
for any \((x, \xi) \in V, \xi \in K\).

Writing \(\prod_{j=1}^{m} (t - \mu_j) = \sum_{\ell=0}^{m} p_{\ell} t^{\ell}\) we see that \(\tilde{p}(z - is\xi)\) is written

\[
\sum_{j=0}^{m} \frac{1}{j!} \left( i s \frac{\partial}{\partial t} \right)^j \left( e^{\sum_{j=1}^{m} (t - \mu_j)} \right)_{t=0} = \sum_{\ell=0}^{m} \sum_{k=0}^{m-\ell} \frac{1}{k!} \left( i s \frac{\partial}{\partial t} \right)^k \left|_{t=0} p_{\ell}(is)^{\ell} \right.
\]

which is equal to

\[
\sum_{k=0}^{m} \frac{1}{k!} \left( i s \frac{\partial}{\partial t} \right)^k e^{\sum_{j=1}^{m} (t - \mu_j)}_{t=0} = \sum_{k=0}^{m} \frac{1}{k!} \left( i s \frac{\partial}{\partial t} \right)^k e^{\sum_{j=1}^{m} (is - \mu_j)}_{t=0} + O(s^{m+1})
\]

which proves

\[
\tilde{p}(z - is\xi) = e_0(z, \xi, s) \prod_{k=1}^{m} (is - \mu_j(z, \xi)) + O(s^{m+1}).
\]

Note that \(e_0(z, \xi, s) = \sum_{k=0}^{m} (is\partial / \partial t)^k e(z, \xi, t)_{t=0}\) and hence we have \(e_0(z, \xi, 0) = e(z, \xi, 0) \neq 0\).

**Lemma 6.4.** There exist a conic neighborhood \(U\) of \(\phi\) and a compact convex set \(K \subset \Gamma_{\rho}\) such that \(\tilde{H}_{\phi}/|H_{\phi}| \in K\) for \((x, \xi) \in U, \xi \geq \gamma_0\).

**Proof.** From \(\psi = (\xi)^{\gamma}_0 \log (\phi + \omega)\) it is easy to see

\[
\left\{
\begin{array}{l}
\nabla_{\xi} \psi = \omega^{-1}(\xi)^{\gamma}_0 \nabla_{\phi} \psi,
\nabla_{\phi} \psi = \omega^{-1}(\xi)^{\gamma}_0 \nabla_{\phi} \psi + O((\xi)^{\gamma^{-1}}_0) \log (\phi + \omega) + O((\xi)^{\gamma^{-1}}_0).
\end{array}
\right.
\]

Therefore one has

\[
\tilde{H}_{\phi} = \omega^{-1}(\xi)^{\gamma}_0 (\tilde{H}_{\phi} + (O(1) \log (\phi + \omega), 0)).
\]

Since \(|\phi + \omega| \leq 2\omega\) we can assume \(\omega \log (\phi + \omega)\) is enough small taking \(U\) small. In particular we have \(\omega^{-1}(\xi)^{\gamma}_0 / C \leq |\tilde{H}_{\phi}| \leq C \omega^{-1}(\xi)^{\gamma}_0\). Then noting \(\tilde{H}_{\phi}(\rho)/|\tilde{H}_{\phi}(\rho)| \in \Gamma_{\rho}\) which follows from (4.3) we get the assertion. \(\square\)

We rewrite \(Q(z)\) according to (6.2).

**Lemma 6.5.** Let \(\bar{\omega} = \tilde{H}_{\phi}/|H_{\phi}|\). Then we have

\[
Q(z) = (\xi)^{\gamma^{-1}}_0 \left\{ -i \delta e_0(\bar{z}, \bar{\omega}, \lambda)/\partial \lambda \prod_{j=1}^{m} (i\lambda - \mu_j(\bar{z}, \bar{\omega})) \right.
\]

\[
+ e_0(\bar{z}, \bar{\omega}, \lambda) \sum_{j=1}^{m} \prod_{k=1, k \neq j}^{m} (i\lambda - \mu_k(\bar{z}, \bar{\omega})) + O(\lambda^m) \right\}.
\]

**Proof.** Noting \(\lambda(\bar{z})|\tilde{H}_{\phi}|^{-1}(\xi)^{\gamma}_0 = \epsilon\) one can write

\[
Q(z) = (\xi)^{\gamma^{-1}}_0 \left\{ -i \delta e_0(\bar{z}, \bar{\omega}, \lambda)/\partial \lambda \prod_{j=1}^{m} (i\lambda - \mu_j(\bar{z}, \bar{\omega})) \right.
\]

\[
+ e_0(\bar{z}, \bar{\omega}, \lambda) \sum_{j=1}^{m} \prod_{k=1, k \neq j}^{m} (i\lambda - \mu_k(\bar{z}, \bar{\omega})) + O(\lambda^m) \right\}.
\]
where and (6.2) it follows that

\[ \rho Vo f \]

modulo

\[ (\tilde{\psi} \chi) \]

are led to consider

\[ \langle \rho, 2 \rangle \]

(\tilde{\psi} \lambda)

\[ \xi \]

\[ (\tilde{\psi} - t\tilde{\omega}) \]

Thus from Lemma 6.5

\[ \delta = (1 - \epsilon\gamma / m, \mu = (m - 1 + \epsilon\gamma) / m, \kappa = \rho - \delta \]

where \( \rho + \delta = 1 \).

**Lemma 6.6.** Let \( S_0(z) = \text{Im}(\tilde{\rho}(z - i\epsilon H_\theta)Q(z)) \). Then one can find a conic neighborhood \( V \) of \( \rho \) and \( C > 0 \) such that we have

\[ \epsilon \omega^{-1}(\xi^{\gamma\xi}h^2(z)) / C \leq S_0(z) \leq C \epsilon \omega^{-1}(\xi^{\gamma\xi}h^2(z)) \]

Proof. Write \( e_0(\tilde{z}, \tilde{\omega}, \lambda) = e(\tilde{z}, \tilde{\omega}, 0) + i\lambda(\partial e / \partial \lambda)(\tilde{z}, \tilde{\omega}, 0) + O(\lambda^2) \) then it is clear \( |e_0|^2 = |e(\tilde{z}, \tilde{\omega}, 0)|^2 + O(\lambda^2) \) so that \( \text{Re}(\partial e_0 / \partial \lambda)e_0 = 2^{-1}|e_0|^2 / \partial \lambda = O(\lambda) \). Thus from Lemma 6.5 and (6.2) it follows that

\[ \text{Im}(\tilde{\rho}(z - i\epsilon H_\theta)Q(z)) = \langle \xi \rangle^{2m-1} |e_0|^2 \lambda \left\{ \sum_{j=1}^{m} \prod_{k=1, k \neq j}^{m} (\lambda^2 + \mu_k^2) \left( 1 + O \left( \frac{m}{\lambda} + \sum_{j=1}^{m} |\mu_j| \right) \right) \right\} \]

Since \( \mu_j(\rho, \tilde{\omega}) = 0, j = 1, 2, \ldots, m \) one obtains

\[ C^{-1} \leq \left\{ \langle \xi \rangle^{2m-1} \lambda \sum_{j=1}^{m} \prod_{k=1, k \neq j}^{m} (\lambda^2 + \mu_k(\tilde{z}, \tilde{\omega})^2) \right\} \leq S_0(z) \leq C. \]

On the other hand noting \( C^{-1} \leq \lambda(z) / (\epsilon \omega^{-1}(\xi^{\gamma\xi})^{m-1}) \) ≤ C and
\[ C^{-1} \leq \left( \langle \xi \rangle^{2m-2} \sum_{j=1}^{m} \prod_{k=1}^{m} (e^{i\omega^{-2}\langle \xi \rangle^{2e^{-2}} + \mu(\bar{z},(0,\theta)))} \right) \times h_{m-1}(x, \xi - i\epsilon \omega^{-1}\langle \xi \rangle \theta)^{-1} \leq C \]

we conclude the assertion from Lemma 6.4 and Proposition 6.1. \( \square \)

**Lemma 6.7.** We have \( Q \in \tilde{S}_{\rho,\delta}(h) \) and \( S_{\delta}^{-1} \in \tilde{S}_{\rho,\delta}((\omega^{-1}\langle \xi \rangle^{-1})^{2}) \) in \( U \) for \( \gamma \geq \gamma_{0}(\epsilon) \). Moreover \( |Q| \leq C \) with \( C > 0 \) independent of \( \epsilon \).

Proof. From the definition one can see easily that \( Q \) is a sum of terms, up to constant factor;

\[(6.5) e^{(\alpha + \beta - 1)P_{\rho,\delta}(z)(\nabla_{\xi} \psi)^{\alpha}(\nabla_{\xi} \psi)^{\alpha} / (\langle \xi \rangle^{2}\nabla_{\xi} \psi^{2} + |\nabla_{\xi} \psi|^{2})^{1/2}}\]

with \( 1 \leq |\alpha + \beta| \leq m + 1 \). We also note that \( \text{Im} Q \) is a sum of such terms (6.5) with \( 2 \leq |\alpha + \beta| \leq m + 1 \). From Lemma 4.3 it follows

\[\nabla_{\xi} \psi \in S_{\rho,\delta}(\omega^{-1}\langle \xi \rangle^{\alpha-1}), \quad \nabla_{\xi} \psi \in S_{\rho,\delta}(\omega^{-1}\langle \xi \rangle^{\gamma})\]

in \( U \) and hence \( (\langle \xi \rangle^{2}\nabla_{\xi} \psi^{2} + |\nabla_{\xi} \psi|^{2})^{-1/2} \in S_{\rho,\delta}((\omega^{-1}\langle \xi \rangle^{\alpha})^{-1}) \) then we have

\[V_{\alpha}^{\beta} = (\nabla_{\xi} \psi)^{\beta}(\langle \xi \rangle^{\alpha} |\nabla_{\xi} \psi|^{2} + |\nabla_{\xi} \psi|^{2})^{-1/2} \in S_{\rho,\delta}(\omega(\langle \xi \rangle^{\gamma})^{-1})\]

Noting \( \omega^{-2}\langle \xi \rangle^{\alpha-1} \leq 1 \) it suffices to repeat the proof of Lemma 6.3 to conclude \( p_{\rho,\delta}V_{\alpha}^{\beta} \in S_{\rho,\delta}(h) \) and \( e^{(\alpha + \beta - 1)P_{\rho,\delta}V_{\alpha}^{\beta}} \leq C \) with \( C \) independent of \( \epsilon \) for \( 1 \leq |\alpha + \beta| \leq m + 1 \). From Lemma 6.3 it follows that \( \tilde{p}(z - i\epsilon H_{\rho}) - p(z) \in \tilde{S}_{\rho,\delta}(\omega^{-1}\langle \xi \rangle^{\alpha})h \). Since from Lemma 6.2 one can check that \( p(z)\text{Im} Q \in \tilde{S}_{\rho,\delta}(\omega^{-1}\langle \xi \rangle^{\alpha})h^{2} \) we get the assertion for \( S_{\delta}^{-1}(z) \). The assertion for \( S_{\delta}^{-1}(z) \) follows from Lemma 6.6 and \( S_{\delta}(z)S_{\delta}^{-1}(z) = 1 \). \( \square \)

From Theorem 5.1 and Lemma 6.3 one can write

\[ p_{\rho} - \tilde{p}(z - i\epsilon H_{\rho}) = \sqrt{\epsilon} r + r_{0} \]

where \( r \in \tilde{S}_{\rho,\delta}(\omega^{-1}\langle \xi \rangle^{\alpha})h \) with \( |r| \leq C \epsilon \omega^{-1}\langle \xi \rangle^{\alpha}h \) and \( r_{0} \in S_{\rho,\delta}(\langle \xi \rangle^{\alpha-\delta(m+1)}) \). Thus \( r\# Q \in \tilde{S}_{\rho,\delta}(\omega^{-1}\langle \xi \rangle^{\alpha})h^{2} \) and \( |r_{0}Q| \leq C \epsilon \omega^{-1}\langle \xi \rangle^{\alpha}h^{2} \). On the other hand Lemma 6.1 shows

\[\langle \xi \rangle^{\gamma - \delta(m+1)} = \langle \xi \rangle^{m-1+k - \delta(m+1)} \leq C \epsilon^{1-m}\omega^{-1}\langle \xi \rangle^{\alpha}h \]

so that we see \( r_{0} \in S_{\rho,\delta}(\omega^{-1}\langle \xi \rangle^{\alpha})h \) and \( |r_{0}| \leq C \epsilon^{1-m}\omega^{-1}\langle \xi \rangle^{\alpha}h \) for \( \gamma \geq \gamma_{0}(\epsilon) \). Therefore in virtue of Lemma 6.7 there is \( \tilde{r} \in \tilde{S}_{\rho,\delta}(1) \) with \( |\tilde{r}| \leq C \sqrt{\epsilon} \) such that

\[\text{Im}(p_{\rho}\# Q) = S_{\delta}(1 - \tilde{r}) \]

in some conic neighborhood of \( \rho \).

We turn to \( R = \sum_{j=0}^{m-1}P_{j}p_{\rho} \in S_{\rho,\delta}(\langle \xi \rangle^{m-1}) \). From Lemma 6.1 again we have

\[\langle \xi \rangle^{m-1} \leq C \epsilon^{1-m}\omega^{-1}\langle \xi \rangle^{m-1}h = C \epsilon^{1-m}\omega^{-1}\langle \xi \rangle^{m-1}h^{2}(\omega^{-1}\langle \xi \rangle^{\alpha}) \]

\[\leq C \epsilon^{1-m}\omega^{-1}\langle \xi \rangle^{m-2}\omega^{-1}\langle \xi \rangle^{\alpha}h. \]

Recalling that \( k - \delta(m - 2) = \epsilon^{*} > 0 \) and hence we can assume \( C \epsilon^{1-m}\gamma^{-\epsilon^{*}} \leq C \epsilon^{3/2} \) for
\[ \gamma \geq \gamma_0(\epsilon) \] so that there exists \( \hat{r} \in \tilde{S}_{\rho,\delta}(1) \) with \( |\hat{r}| \leq C \sqrt{\epsilon} \) such that \( \text{Im}(R\# \hat{Q}) = S_0(1 - \hat{r}) \) in a conic neighborhood of \( \rho \). Thus we conclude
\[
\begin{cases}
\text{Im}(P_\theta \# \hat{Q}) = E^2, \quad E \in \tilde{S}_{\rho,\delta}(\omega^{-1/2}(\xi)_{\gamma}^{s/2}h), \\
\epsilon^{1/2} \omega^{-1/2}(\xi)_{\gamma}^{s/2}h/C \leq |E| \leq C \epsilon^{1/2} \omega^{-1/2}(\xi)_{\gamma}^{s/2}h
\end{cases}
\]
in a conic neighborhood of \( \rho \) with \( C \) independent of \( \epsilon \). The rest of the proof of deriving microlocal energy estimates is just a repetition of the arguments in [13] and we conclude that the Cauchy problem for \( p + P_{m-1} + \cdots \) is \( \gamma^{1/\alpha} \) well-posed at the origin. Note that \( 1/\kappa = m/(m - 2 + 2\epsilon^*) \) and \( \epsilon^* > 0 \) is arbitrarily small so that \( 1/\kappa \) is as close to \( m/(m - 2) \) as we please.

References


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