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A NOTE ON THE EXPONENTIAL DECAY FOR THE NONLINEAR SCHRÖDINGER EQUATION

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Abstract

We prove new results about exponential decay rates associated with the two dimensional Schrödinger equation with critical nonlinearity and localized damping. Our article improve incomplete previous results established in [4].

1. Introduction

Along the years, several researchers have been interested in proving important results of controllability/stabilization related to dispersive models posed on bounded and unbounded domains. The results have been improved and sufficient conditions were obtained to insure control/stability related to these evolution equations. One of the most important equation in this development concerns to the nonlinear Schrödinger equation

$$iu_t + \Delta u + \lambda |u|^{\alpha-1}u + g(x, u)u = 0, \tag{1.1}$$

where $\lambda = \pm 1$, $\alpha \geq 1$ and $u = u(x, t)$, $(x, t) \in \mathcal{O} \times (0, +\infty)$, is a complex-valued function and $\mathcal{O}$ is a convenient subset of $\mathbb{R}^n$, $n \geq 1$. Here, function g is a dissipation term which satisfies the following condition

$$\mathrm{Im}(g(x, u(x, t))) \geq 0, \quad \forall (x, t) \in \mathcal{O} \times (0, +\infty),$$

which is responsible for the dissipative mechanism in L^2-level whether we assume convenient boundary conditions. For instance, by supposing $\mathcal{O} = \mathbb{R}^n$, we can multiply equation (1.1) by $\bar{u}$ in order to get, after integration over $\mathbb{R}^n$,

$$\frac{d}{dt}\int_{\mathbb{R}^n} |u(x, t)|^2\,dx = -2\int_{\mathbb{R}^n} \mathrm{Im}(g(x, u(x, t)))|u(x, t)|^2\,dx \leq 0. \tag{1.2}$$

So, from inequality in (1.2) it makes sense to find decay rates on the energy in L^2-level for the nonlinear equation (1.1). A particular example of the equation (1.1) which possesses interesting results of controllability/stabilization in current literature is the

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following nonlinear equation

$$iu_t + \Delta u + \lambda |u|^2 u + ia(x)u = 0, \quad (x,t) \in \mathbb{R}^2 \times (0, +\infty), \tag{1.3}$$

that is, $g(x,u) = a(x)$, $\alpha = 3$ and $n = 2$ in equation (1.1). The energy identity obtained from (1.2), after integrating the result over $[0,t)$, is given by

$$\begin{aligned} E_0(t) &:= \int_{\mathbb{R}^2} |u(x,t)|^2\, dx \\ &= -2\int_0^t \int_{\mathbb{R}^2} a(x)|u(x,s)|^2\, dx\, ds + \int_{\mathbb{R}^2} |u_0(x)|^2\, dx. \end{aligned} \tag{1.4}$$

In what follows, in whole this paper, we assume the following set of assumptions:
(H1) $a \in L^\infty(\mathbb{R}^2)$ and $a(x) \geq 0$ a.e. in $\mathbb{R}^2$.
(H2) $a(x) \geq \alpha_0 > 0$ a.e. in $\mathbb{R}^2 \setminus B_R(0)$.

If one considers equation (1.1) with $g \equiv 0$ and $\mathcal{O}$ being a bounded domain with smooth boundary, the authors in [19] established exact controllability results in H^s-level with solution u satisfying either Dirichlet or Neumann boundary conditions. On periodic domains, we have the work [10]. In this case, the author established controllability/stabilization for the equation (1.3) when the space $\mathbb{R}^2$ is replaced by the torus $\mathbb{T}$. The main ingredient is to use of some multilinear estimate in some appropriate Bourgain periodic space.

Next, in unbounded domains we have the work [4] where it was presented exponential decay rates of the energy, in L^2-level related to the equation (1.3) with $\lambda = -1$ (defocusing nonlinearity) and function a satisfying similar assumptions as in (H1) and (H2). Since a produces a localized dissipative effect in L^2-level, a result of unique continuation was proved in order to obtain the desired exponential stability result. However, we mention here that some points in that work are not clear. Therefore, the goal is to give a positive and definitive answer for this question jointly with the proof of the exponential decay of the energy for the case $\lambda = 1$ (focusing nonlinearity). A correct proof was determined in [3] for the case $n = 1$, where we have used results of unique continuation in [24] combined with the $H^{1/2}$ smoothing effect for the linear Schrödinger equation. This last point was not mentioned in [4]. In [23], it was studied the asymptotic behavior in time of small solutions for the problem (1.1) with $g(x,u) = \mu |u|^{2/n}$, $\mu \in \mathbb{R} \setminus \{0\}$, and $n = 1, 2, 3$. He showed that if $\mu > 0$, there exists a unique global solution which decays like $(t \log t)^{-n/2}$ as $t \to +\infty$ in $L^\infty(\mathbb{R}^n)$ for small initial data.

Other dispersive equation with a huge quantity of contributors in this subject concerns the generalized Korteweg–de Vries,

$$u_t + u_x + u^p u_x + u_{xxx} + a(x)u = 0, \tag{1.5}$$

where $p \geq 1$ is an integer, $u = u(x,t)$ is a real valued function defined in $\mathcal{J} \times (0, +\infty)$ and a is a nonnegative real function which depends on $x \in \mathcal{J}$. Eventually, function a

can satisfy localized properties of dissipation or it can be zero to guarantee results of control/stability of the energy. Some contributors deserve to be mentioned for the case $p = 1$ as [16] and [17]. Regarding equations of KdV-type having general nonlinearities, we can mention [13] and [21]. Periodic problems were studied in [9], [11] and [22]. A good review in these problems is [20]. In unbounded domains we can cite [2] and [12].

Next, we shall give a brief outline of our work. To do so, we employ the unique continuation principle determined in [7] (see also [1] and [15] for additional references) combined with the local smoothing effect for the Schrödinger equation in $H^{1/2}$. These facts allow us to establish the exponential decay rate for the L^2-critical equation (1.3) given by

$$E_0(t) \leq ce^{-\omega t}, \quad t \gg 1, \tag{1.6}$$

where c and ω are positive constants, provided that the initial data in L^2 is small enough. This last fact makes necessary in order to guarantee an uniform bound of the solution u related to the equation (2.1) in a convenient L^p space, $p > 1$.

Our paper is organized as follows. In Section 2 we present some preliminary and useful results used in paper. Exponential decay rates associated with the equation (1.3) is presented in Section 3.

2. Preliminaries results

In what follows, we consider the initial value problem (IVP henceforth) related to the equation (1.3) as

$$\begin{cases} iu_t + \Delta u + \lambda |u|^2 u + ia(x)u = 0, & \text{in} \quad \mathbb{R}^2 \times (0, +\infty), \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^2. \end{cases} \tag{2.1}$$

Our first result concerns to enunciate the local solvability of the IVP (2.1) for initial data in $L^2(\mathbb{R}^2)$.

Theorem 2.1. *Consider $u_0 \in L^2(\mathbb{R}^2)$ and $a \in L^\infty(\mathbb{R}^2)$. There is a unique local solution u for the Cauchy problem* (2.1) *which belongs to*

$$C([0, T']; L^2(\mathbb{R}^2)) \cap L^4([0, T']; L^4(\mathbb{R}^2)),$$

for all $0 < T' < T_{\max}$. In addition, the local solution satisfies identity (1.4)*, for all $t \in [0, T_{\max}]$, and the map*

$$u_0 \in L^2(\mathbb{R}^2) \mapsto u \in C([0, T']; L^2(\mathbb{R}^2)),$$

is continuous for all $0 < T' < T_{\max}$. In addition, if $\|u_0\|_{L^2}$ is small enough the solution *u extends to any interval* $[0, T]$, *that is,*

$$C([0, T]; L^2(\mathbb{R}^2)) \cap L^4([0, T]; L^4(\mathbb{R}^2)),$$

for all $T > 0$.

Proof. The proof of the theorem is a slight adaptation of Theorem 4.7.1 in [5] and because of this, we shall omit its proof. The second part can be found in [8] (see Corollary 5.2). □

REMARK 2.1. 1) In Theorem 2.1, the local time T' is assumed to satisfy $0 < T' < T_{\max}$ because we could have a blowup alternative, that is, $\|u(t)\|_X \to \infty$ as $t \uparrow T_{\max}$, if $T_{\max} < \infty$. If $T_{\max} = +\infty$ solution u is global in class above. The space X indicates $X = L^2(\mathbb{R}^2)$.
2) Global solutions in $L^2(\mathbb{R}^2)$ to the equation (2.1) (for arbitrary initial data) can be obtained by using the energy identity in (1.4) jointly with the fact that function a in assumption (H1) is non-negative. See arguments in Remark 2.2 in order to justify the validity of the computations to deduce the refereed identity.

The assumption that $\|u_0\|_{L^2}$ is small enough is useful to establish the following local smoothing effect result.

Lemma 2.2. *Suppose that there is* $L_0 > 0$ *such that* $\|u_0\|_{L^2} \leq L_0$. *Let* u *be the corresponding solution obtained in* Theorem 2.1. *Then we have the following estimate*

$$\int_0^T \int_{B_R} |D_x^{1/2} u(x, t)|^2 \, dx \, dt \leq c(\lambda, R, T, \|a\|_{L^\infty}), \quad \text{for all} \quad T > 0. \tag{2.2}$$

Proof. The arguments in order to establish this result can be found in [6]. In fact, solution $u(\cdot)$ must satisfies the integral equation

$$u(t) = S(t)\left(u(0) + i \int_0^t S(-\tau)(\lambda|u|^2 u + ia(\cdot)u)(\tau)\, d\tau\right), \tag{2.3}$$

where $S(t)$, $t \geq 0$, denotes the semigroup related to Schrödinger equation. Let us define,

$$I = \left(\int_0^T \int_{B_R} |D_x^{1/2} u(x, t)|^2 \, dx \, dt\right)^{1/2}.$$

The next step is to use the well known Strichartz estimates associated with the Schrödinger equation and the smoothing effect in $H^{1/2}$-norm associated to linear Schrödinger equation (see [6] and [8] for more details) in order to deduce from Hölder inequality that

$$
\begin{aligned}
I &\le cR\left(\|u(0)\|_{L^2} + \sup_{[0,T]} \left\| \int_0^t S(t)(\lambda |u|^2 u + ia(\cdot)u)(t)\,dt \right\|_{L^2} \right) \\
&\le cR\left(c\|u(0)\|_{L^2} + c\lambda \left(\int_0^T \|u(t)\|_{L^4}^4\,dt \right)^{3/4} \right).
\end{aligned} \tag{2.4}
$$

The right-hand side of the estimate (2.4) is bounded for all $T > 0$ provided that the initial data is small enough (see Corollary 5.2 in [8]). The result is now proved. □

REMARK 2.2. Equality in (1.4) can be rigourously deduced by supposing a regular solution for equation in (2.1). For instance, if we take an initial data u_0 in $H^2(\mathbb{R}^2) \hookrightarrow H^1(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$, the local solution u given by Theorem 2.1, for some $T' > 0$, coincides with the classical solution which can be determined by using classical semigroup theory or H^2 local theory on p. 152 in [5]. Global solutions in $H^2(\mathbb{R}^2)$ is determined for arbitrary large initial data whether $\lambda = -1$ or small initial data in L^2 if $\lambda = 1$. By density arguments, we deduce the validity of computations to conclude equality (1.4). In that case,

$$u \in C([0, T]; H^2(\mathbb{R}^2)) \cap C^1([0, T]; L^2(\mathbb{R}^2)).$$

In order to guarantee the existence of smooth solutions related to the nonlinear Schrödinger equation (1.3) with $a \equiv 0$, we need to present some basic spaces and useful notations. We follow the arguments in [14]. Indeed, for any 2-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}^2$, we define $J^\alpha(t) = M(t)(2it\nabla)^\alpha M(-t)$, where $M(t) = e^{i|x|^2/(4t)}$, $t > 0$. Let us introduce the following basic space

$$X_0 = L^4([0, T]; L^4(\mathbb{R}^2)) \cap L^\infty([0, T]; L^2(\mathbb{R}^2)).$$

We treat equation (1.3) with $a \equiv 0$ in the following function space with $b > 0$:

$$G_0^b(J) = \left\{ u \in X_0,\ \|u\|_{G_0^b(J)} := \sum_{\alpha \ge 0} \frac{b^{|\alpha|}}{\alpha!} \|J^\alpha u\|_{X_0} \right\},$$

where $|\alpha| = \alpha_1 + \alpha_2$, $\alpha! = \alpha_1!\,\alpha_2!$ and, $\alpha \ge 0$ provided that $\alpha_i \ge 0$, $i = 1, 2$.

Next, for $\rho > 0$, we define

$$B_0^b(\rho) = \left\{ u_0 \in G_0^b(x),\ \|u_0\|_{G_0^b(x)} := \sum_{\alpha \ge 0} \frac{b^{|\alpha|}}{\alpha!} \|x^\alpha u_0\|_{L^2} \right\}.$$

We enunciate one of the main results in [14].

Theorem 2.3. *There exists a constant $\rho > 0$ such that for any $b > 0$ and $u_0 \in B_0^b(\rho)$ equation* (1.3) *with $a \equiv 0$ has a unique solution $u \in G_0^b(J)$.*

REMARK 2.3. In our case, Theorem 2.3 works satisfactorily because the result states that we have analytic smoothing properties of solutions since functions in $G_0^b(J)$ are analytic in $\mathbb{R}^2 \times (0, +\infty)$. Moreover, if one considers an initial data u_0 with compact support, it easy to see that $u_0 \in B_0^b(\rho)$. Thus, we can deduce that solution $u \in C^\infty(\mathbb{R}^2 \times (0, +\infty))$.

Next, we have the following unique continuation theorem for regular solutions of the nonlinear equation in (2.1) with $a \equiv 0$. This result is more general in the sense that it deserves for nonlinear Schrödinger equation in the domain $(x, t) \in \mathbb{R}^n \times [0, T]$, $n \geq 1$, with a general nonlinearity $F(v, \bar{v})$. In such case, we must consider $k \in \mathbb{Z}^+$ satisfying $k > n/2 + 1$.

Let us define the weighted Sobolev space $H^1(e^{\beta|x|^\rho} dx)$, as

$$H^1(e^{\beta|x|^\rho}\, dx) = \left\{ f\, ;\ \int_{\mathbb{R}} |f(x)|^2 e^{\beta|x|^\rho}\, dx + \int_{\mathbb{R}} |f'(x)|^2 e^{\beta|x|^\rho}\, dx < \infty \right\}.$$

Theorem 2.4. *Let $w \in C([0, T]; H^k(\mathbb{R}))$, $k \in \mathbb{Z}^+$, $k > 2$ be a strong solution of the equation in* (2.1) *with $a \equiv 0$ in the domain $(x, t) \in \mathbb{R}^2 \times [0, T]$. If there exist $t_1, t_2 \in [0, T]$, $t_1 \neq t_2$, $\rho > 2$ and $\beta > 0$ such that*

$$w(\cdot\,, t_1),\ w(\cdot\,, t_2) \in H^1(e^{\beta|x|^\rho} dx), \tag{2.5}$$

then $w \equiv 0$.

Proof. See Theorem 2.1 in [7]. □

3. Exponential decay

In this section, we are interested in obtaining exponential decay rate for the energy in L^2-level related to the equation (1.3). First, we assume that assumptions (H1) and (H2) are verified. Multiplying the first equation in (1.3) by $\bar{u}$ and integrate the result over $\mathbb{R}^2$ and then, over $[0, t)$ to get

$$E_0(t) = -2 \int_0^t \int_{\mathbb{R}^2} a(x)|u(x, t)|^2\, dx\, ds + \|u_0\|_{L^2}^2 \leq \|u_0\|_{L^2}^2. \tag{3.1}$$

From equation (1.3) we have that $(d/dt)E_0(t) := (d/dt)\int_{\mathbb{R}^2} |u(x, t)|^2\, dx = -2\int_{\mathbb{R}^2} a(x)|u(x, t)|^2\, dx$ and, consequently, we have the following estimate:

$$\int_0^T E_0(t)\, dt \leq -2\alpha_0^{-1} \left[\int_{\mathbb{R}^2} |u(x, t)|^2\, dx \right]_0^T + \underbrace{\int_0^T \int_{B_R} |u(x, t)|^2\, dx\, dt}_{\mathcal{I}}, \tag{3.2}$$

where $B_R := \{x \in \mathbb{R}^2;\ |x| \leq R\}$

Theorem 3.1. *Consider the potential* $a \in L^\infty(\mathbb{R}^2)$ *satisfying assumptions* (H1) *and* (H2). *We suppose that there exists* $L_0 > 0$ *such that* $\|u_0\|_{L^2} \leq L_0$. *Thus, there are positive constants* $\alpha = \alpha(L_0)$ *and* $\omega = \omega(L_0)$ *such that,*

$$E_0(t) \leq \alpha e^{-\omega t},$$

for all $t \geq 0$ *large enough and for any solution of* (1.3) *given in* Theorem 2.1.

A preliminary result makes necessary to determine good bounds for the integral equation $\mathcal{I}$ in (3.2). Consider $a \in L^\infty(\mathbb{R}^2)$ and suppose that the initial data belongs to a (small enough) bounded set of L^2, according with Theorem 3.1. We are enable to prove the following lemma:

Lemma 3.2. *Let* u *be a solution associated to the equation* (1.3) *with initial data* u_0 *satisfying the smallness condition on the initial data in* L^2 *as required in* Theorem 3.1. *Then, for all* $T \gg 1$ *there exists a positive constant* $c > 0$ *which depends on* T *and* L_0 *such that the following inequality holds,*

$$\int_0^T \int_{B_R} |u(x,t)|^2\, dx\, dt \leq c \int_0^T \int_{\mathbb{R}^2} a(x)|u(x,t)|^2\, dx\, dt. \tag{3.3}$$

Proof. We denote $B_R := B_R(0)$ to simplify the notation. We argue by contradiction. Let us suppose that (3.3) is not true and let $\{u_k(0)\}_{k\in\mathbb{N}}$ be a sequence of initial data where the corresponding solutions $\{u_k\}_{k\in\mathbb{N}}$ of (1.3) with $E_0^k(0)$, defined in (3.1) for all $k \in \mathbb{N}$, is assumed to be small enough in L^2. Thus,

$$\lim_{k\to+\infty} \frac{\int_0^T \|u_k(t)\|^2_{L^2(B_R)}\, dt}{\int_0^T \int_{\mathbb{R}^2} (a(x)\|u_k(x,t)\|^2)\, dx\, dt} = +\infty. \tag{3.4}$$

In other words,

$$\lim_{k\to+\infty} \frac{\int_0^T \int_{\mathbb{R}^2} (a(x)\|u_k(x,t)\|^2)\, dx\, dt}{\int_0^T \|u_k(t)\|^2_{L^2(B_R)}\, dt} = 0. \tag{3.5}$$

Since,

$$E_0^k(t) \leq E_0^k(0) \leq L_0,$$

we obtain a subsequence of $\{u_k\}_{k\in\mathbb{N}}$, still denoted by $\{u_k\}_{k\in\mathbb{N}}$ from now on, which verifies the convergence:

$$u_k \rightharpoonup u \quad \text{weakly in} \quad L^2([0,T]; L^2(\mathbb{R}^2)). \tag{3.6}$$

So, we deduce

$$\lim_{k\to+\infty}\int_0^T\int_{\mathbb{R}^2} a(x)|u_k(x,t)|^2\,dx\,dt = 0, \tag{3.7}$$

consequently, from hypothesis (H2) one has

$$\lim_{k\to+\infty}\int_0^T\int_{\mathbb{R}^2\setminus B_R} |u_k(x,t)|^2\,dx\,dt = 0. \tag{3.8}$$

On the other hand, from Lemma 2.2 we guarantee that $\{u_k\}_{n\in\mathbb{N}}$ is bounded in $L^2([0,T]; H^{1/2}(B_R))$. Further, since $H^{1/2}(B_R)$ is compactly embedded in $L^2(B_R)$, we guarantee from Aubin–Lions' lemma that there is a subsequence, still denoted by $\{u_k\}_{k\in\mathbb{N}}$, such that

$$u_k \to u \quad \text{strong in} \quad L^2(B_R\times[0,T]). \tag{3.9}$$

Therefore,

$$u_k \to u \quad \text{a.e. in} \quad B_R\times[0,T]. \tag{3.10}$$

Statements (3.8) and (3.10) enable us to deduce the following convergence:

$$u_k \to \tilde{u} \quad \text{a.e. in} \quad \mathbb{R}^2\times[0,T], \tag{3.11}$$

where

$$\tilde{u} = \begin{cases} u, & \text{a.e. in} \quad B_R\times[0,T],\\ 0, & \text{a.e. in} \quad \mathbb{R}^2\setminus B_R\times[0,T]. \end{cases}$$

At this point we will divide the proof into two cases.

CASE (I): $u \neq 0$.

By using convergence in (3.11), the fact that $\{|u_k|^2u_k\}_{k\in\mathbb{N}}$ is bounded in $L^{4/3}(B_R\times[0,T])$, and Lions' lemma, we can pass to the limit to deduce that u is a solution of the problem

$$\begin{cases} iu_t + \Delta u + \lambda|u|^2u = 0, & \text{in} \quad \mathbb{R}^2\times[0,T],\\ u = 0, & \text{a.e. in} \quad \mathbb{R}^2\setminus B_R\times[0,T]. \end{cases} \tag{3.12}$$

Moreover, since $u\in L^2([0,T];L^2(\mathbb{R}^2))$ there exists $t_0\in[0,T]$ such that $u(\cdot,t_0)\in L^2(\mathbb{R}^2)$ and consequently from Theorem 2.1 one has (a unique) $u\in C([t_0,T];L^2(\mathbb{R}^2))$. From the continuous dependence of the initial data and the uniqueness of the solution u, we see that u is a mild solution with initial data $u_0 := u(\cdot,0)$ (see arguments in [10]) having compact support. Thus, we can use Theorem 2.3 (see also Remark 2.3) to conclude that u is smooth with $u(x,t) = 0$ a.e. $(x,t) \in \mathbb{R}^2 \setminus B_R \times [0,T]$. Therefore, from Theorem 2.4 we get $u \equiv 0$ in $\mathbb{R}^2 \times [0,T]$. This last fact is a contradiction with $u \neq 0$.

CASE (II): $u = 0$.

We denote

$$\nu_k = \|u_k\|_{L^2([0,T];L^2(B_R))}. \tag{3.13}$$

By defining $v_k = u_k/\nu_k$ we obtain,

$$\|v_k\|_{L^2([0,T];L^2(B_R))} = 1, \quad \forall k \in \mathbb{N}. \tag{3.14}$$

Next, we derive an uniform bound for the initial data $v_k(0)$ in $L^2(\mathbb{R}^2)$. Indeed, by using (H2) we deduce

$$\int_0^T \int_{\mathbb{R}^2} |u_k|^2 \, dx \, dt \le \frac{1}{\alpha_0} \int_0^T \int_{\mathbb{R}^2} a(x)|u_k|^2 \, dx \, dt + \int_0^T \int_{B_R} |u_k|^2 \, dx \, dt. \tag{3.15}$$

So, from equality (1.4) and (3.15) one has

$$\begin{aligned} \int_{\mathbb{R}^2} |u_k(0)|^2 \, dx &\le \frac{1}{T} \int_0^T \int_{\mathbb{R}^2} |u_k|^2 \, dx \, dt + 2 \int_0^T \int_{\mathbb{R}^2} a(x)|u_k|^2 \, dx \, dt \\ &\le \left(2 + \frac{1}{\alpha_0 T}\right) \int_0^T \int_{\mathbb{R}^2} a(x)|u_k|^2 \, dx \, dt + \frac{1}{T} \int_0^T \int_{B_R} |u_k|^2 \, dx \, dt. \end{aligned}$$

Finally, by using last inequality we get

$$\|v_k(0)\|^2_{L^2(\mathbb{R}^2)} \le \left(2 + \frac{1}{\alpha_0 T}\right) \int_0^T \int_{\mathbb{R}^2} a(x)|v_k|^2 \, dx \, dt + \frac{1}{T}, \tag{3.16}$$

which establishes a bound for the initial data $v_k(0)$ in L^2-level. In addition, since $T \gg 1$ we obtain from (3.5) and (3.16) that there are $\mu_0 > 0$ and $k_0 \in \mathbb{N}$ such that

$$\|v_k(0)\|_{L^2} \le \mu_0, \quad k \ge k_0.$$

Therefore, from Theorem 2.1 we get that v_k satisfies the equation,

$$i v_{t,k} + \Delta v_k + \lambda \nu_k^2 |v_k|^2 v_k + i a(x) v_k = 0, \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^2 \times [0,T]). \tag{3.17}$$

On the other hand, by using (3.5), (3.14) and the fact that $a(x) \ge \alpha_0 > 0$ for $|x| \ge R$, we deduce

$$\lim_{k \to +\infty} \int_0^T \int_{\mathbb{R}^2 \setminus B_R} |v_k(x,t)|^2 \, dx \, dt = 0. \tag{3.18}$$

Thus,

$$v_k \to 0 \quad \text{in} \quad L^2([0, T]; L^2(\mathbb{R}^2 \setminus B_R)). \tag{3.19}$$

So, we get a function v which verifies $v_k \rightharpoonup \tilde{v}$ in $L^2([0, T]; L^2(\mathbb{R}^2))$, where

$$\tilde{v} = \begin{cases} v, & \text{a.e. in} \quad B_R \times [0, T], \\ 0, & \text{a.e. in} \quad \mathbb{R}^2 \setminus B_R \times [0, T]. \end{cases}$$

Next, since $\nu_k = \|u_k\|_{L^2([0,T];L^2(B_R))} \to 0$, as $k \to +\infty$, and $|v_k|^2 v_k$ is bounded in $L^{4/3}(\mathbb{R}^2 \times [0, T])$, we can use similar arguments as in first case. These facts enable us to pass to the limit in equation (3.17) to obtain that v solves the linear equation

$$\begin{cases} i v_t + \Delta v = 0, & \text{in} \quad \mathcal{D}'(\mathbb{R}^2 \times [0, T]), \\ v = 0, & \text{a.e. in} \quad \mathbb{R}^2 \setminus B_R \times [0, T]. \end{cases} \tag{3.20}$$

Therefore, from Holmgreen's theorem we conclude that $v \equiv 0$ in $B_R \times [0, T]$. Our proof is not complete in the sense that we still not have a contradiction argument as in first case. In fact, it is not clear that $v_k \to 0$ strongly in $L^2([0, T]; L^2(B_R))$ as $k \to +\infty$ to get a contradiction with (3.14). For this purpose a compact embedding as determined in Theorem 2.2 makes necessary in this case. In [4] the arguments in this point are not clear since we conclude directly the required strong convergence above. Therefore, from Lemma 2.2 applied to v_k, Aubin–Lions' lemma and having in mind that $v \equiv 0$ one has

$$\lim_{k \to +\infty} \int_0^T \int_{B_R} |v_k(x, t)|^2 \, dx \, dt = 0. \tag{3.21}$$

In addition, note that from (3.14) we have

$$\lim_{k \to \infty} \int_0^T \int_{B_R} |v_k(x, t)|^2 \, dx \, dt = \lim_{k \to \infty} \|v_k\|^2_{L^2([0,T];L^2(B_R))} = 1, \tag{3.22}$$

which establishes a contradiction. The proof is now completed. □

Proof of Theorem 3.1. Indeed, from (1.4) and (3.3) we deduce that

$$\int_0^T E_0(t) \, dt \leq \frac{\alpha_0^{-1}}{2} E_0(0) + c \int_0^T \int_{\mathbb{R}^2} a(x) |u(x, t)|^2 \, dx \, dt, \quad \text{for all} \quad T \gg 1. \tag{3.23}$$

Next, by using identity of the energy in L^2-level, namely:

$$E_0(t) - E_0(0) = -2 \int_0^t \int_{\mathbb{R}^2} a(x) |u(x, t)|^2 \, dx \, dt, \quad \text{for all} \quad t \geq 0, \tag{3.24}$$

we infer that $E(t)$ is non-increasing, and, furthermore that

$$2\int_0^t \int_{\mathbb{R}^2} a(x)|u(x,s)|^2\,dx\,ds = E_0(0) - E_0(t) \quad \text{for all} \quad t \geq 0. \tag{3.25}$$

Thus, combining (3.23) and (3.25) we have,

$$\begin{aligned} E_0(T) \leq \frac{\alpha_0^{-1}}{2T}\left[E_0(T) + 2\int_0^T \int_{\mathbb{R}^2} a(x)|u(x,t)|^2\,dx\,dt\right] \\ + \frac{c}{T}\int_0^T \int_{\mathbb{R}^2} a(x)|u(x,t)|^2\,dx\,dt \end{aligned} \tag{3.26}$$

which implies that for $T \gg 1$,

$$\left(T - \frac{\alpha_0^{-1}}{2}\right)E_0(T) \leq (\alpha_0^{-1} + c)\int_0^T a(x)|u(x,t)|^2\,dx\,dt. \tag{3.27}$$

For $T \gg 1$, the last inequality yields

$$E_0(T) \leq c\int_0^T a(x)|u(x,t)|^2\,dx\,dt. \tag{3.28}$$

Finally, combining (3.25) and (3.28) we obtain

$$E_0(T) \leq c\left[\frac{E_0(0) - E_0(T)}{2}\right], \tag{3.29}$$

that is,

$$E_0(T) \leq \gamma E_0(0), \quad \text{where} \quad \gamma = \frac{c/2}{1 + c/2}. \tag{3.30}$$

Next, since we have global solutions in $L^2(\mathbb{R}^2)$, let us define $v(x,t) = u(x,t+T)$. We see that v is a solution related to the Schrödinger equation in (1.3) which belongs to $C([T,2T], L^2(\mathbb{R}^2))$. In addition, the new initial data is now $v(x,0) = u(x,T) \in L^2(\mathbb{R}^2)$. So we have from Lemma 3.2 applied to v that

$$\begin{aligned} E_0(2T) &= E_{0,v}(T) \\ &\leq c\int_0^T \int_{B_R} a(x)|v(x,t)|^2\,dx\,dt = c\int_0^T \int_{B_R} a(x)|u(x,t+T)|^2\,dx\,dt \\ &= c\int_T^{2T} \int_{B_R} a(x)|u(x,s)|^2\,dx\,ds = c\left[\frac{E_0(T) - E_0(2T)}{2}\right], \end{aligned}$$

where $E_{0,v}$ is the energy in $L^2(\mathbb{R}^2)$ associated with v. So, from the above we get

$$E_0(2T) \leq \gamma E_0(T) \leq \gamma^2 E_0(0).$$

Repeating this argument, one has $E_0(nT) \leq \gamma^n E_0(0)$, for all $n \in \mathbb{N}$. This last fact allow us to deduce the exponential decay. In fact, letting $t = nT + r$, where $0 \leq r < T$, we get

$$E_0(t) \leq E_0(nT) \leq \gamma^n E_0(0) = \gamma^{(t/T - r/T)} E_0(0).$$

Since $0 < \gamma < 1$, we can choose $\omega = -\ln(\gamma)/T > 0$ and $\alpha = \gamma^{-r/T} E_0(0)$ to obtain $E_0(t) \leq \alpha e^{-\omega t}$, $t \gg 1$.

The proof is now completed. $\square$

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