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ON OPERATORS WHICH ARE POWER SIMILAR TO HYPONORMAL OPERATORS

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Abstract

In this paper, we study power similarity of operators. In particular, we show that if $T \in PS(H)$ (defined below) for some hyponormal operator H, then T is subscalar. From this result, we obtain that such an operator with rich spectrum has a nontrivial invariant subspace. Moreover, we consider invariant and hyperinvariant subspaces for $T \in PS(H)$.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. As usual, we write $\sigma(T)$, $\sigma_l(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, $\sigma_{re}(T)$, and $\sigma_{le}(T)$ for the spectrum, the left spectrum, the point spectrum, the approximate point spectrum, the right essential spectrum, and the left essential spectrum of T, respectively.

A closed subspace $\mathcal{M}$ of $\mathcal{H}$ is called an *invariant subspace* for an operator $T \in \mathcal{L}(\mathcal{H})$ if $T\mathcal{M} \subset \mathcal{M}$. We say that $\mathcal{M} \subset \mathcal{H}$ is a *hyperinvariant subspace* for $T \in \mathcal{L}(\mathcal{H})$ if $\mathcal{M}$ is an invariant subspace for every $S \in \mathcal{L}(\mathcal{H})$ commuting with T.

An operator X in $\mathcal{L}(\mathcal{H})$ is a *quasiaffinity* if it has trivial kernel and dense range. An operator T in $\mathcal{L}(\mathcal{H})$ is said to be a *quasiaffine transform* of operator S in $\mathcal{L}(\mathcal{H})$ if there is a quasiaffinity X in $\mathcal{L}(\mathcal{H})$ such that $XT = SX$, and this relation of S and T is denoted by $T \prec S$. If both $T \prec S$ and $S \prec T$, then we say that S and T are *quasisimilar*.

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *p-hyponormal* if $(TT^*)^p \leq (T^*T)^p$, where $0 < p < \infty$. In particular, 1-hyponormal operators and 1/2-hyponormal operators are called *hyponormal* operators and *semi-hyponormal* operators, respectively. It is well known that

$$\text{hyponormal} \Rightarrow p\text{-hyponormal} \quad (0 < p < 1).$$

An arbitrary operator $T \in \mathcal{L}(\mathcal{H})$ has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{1/2}$ and U is the appropriate partial isometry satisfying $\ker(U) = \ker(|T|) =$

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$\ker(T)$ and $\ker(U^*) = \ker(T^*)$. Associated with T is a related operator $|T|^{1/2}U|T|^{1/2}$, called the *Aluthge transform* of T, and denoted throughout this paper by $\hat{T}$. For an operator $T \in \mathcal{L}(\mathcal{H})$, the sequence $\{\hat{T}^{(n)}\}$ of Aluthge iterates of T is defined by $\hat{T}^{(0)} = T$ and $\hat{T}^{(n+1)} = \widehat{\hat{T}^{(n)}}$ for every positive integer n (see [2], [9], and [10]). We note from [3] that if T is p-hyponormal, then $\hat{T}$ is $(p + 1/2)$-hyponormal.

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *scalar* of order m if it possesses a spectral distribution of order m, i.e., if there is a continuous unital morphism of topological algebras

$$\Phi\colon C_0^m(\mathbb{C}) \to \mathcal{L}(\mathcal{H})$$

such that $\Phi(z) = T$, where z stands for the identical function on $\mathbb{C}$ and $C_0^m(\mathbb{C})$ for the space of all compactly supported functions continuously differentiable of order m, $0 \le m \le \infty$. An operator is said to be *subscalar* of order m if it is similar to the restriction of a scalar operator of order m to an invariant subspace.

DEFINITION 1.1. Let $R \in \mathcal{L}(\mathcal{H})$ be given. We say that an operator $T \in \mathcal{L}(\mathcal{H})$ is *power similar* to R if there exists a positive integer n such that T^n is similar to R^n. In this case, we use the notation $T \overset{ps}{\sim} R$.

It is easy to check that the relation $\overset{ps}{\sim}$ is an equivalence relation. Indeed, if $T_1 \overset{ps}{\sim} T_2$ and $T_2 \overset{ps}{\sim} T_3$, then there exist positive integers n, m and invertible operators X, Y such that $XT_1^n = T_2^n X$ and $YT_2^m = T_3^m Y$. Let s be the least common multiplier of n and m. Then $s = nr = mt$ for some integers r, t. Hence $YXT_1^s = YXT_1^{nr} = YT_2^{nr}X = YT_2^{mt}X = T_3^{mt}YX = T_3^s YX$, i.e., $T_1 \overset{ps}{\sim} T_3$.

For a fixed operator $R \in \mathcal{L}(\mathcal{H})$, define the following subset of $\mathcal{L}(\mathcal{H})$:

$$PS_n(R) = \{T \in \mathcal{L}(\mathcal{H})\colon T^n \text{ is similar to } R^n\}$$

where n is a positive integer. We observe that the following relations hold:

$$PS_1(R) \subset PS_n(R) \subset PS_{n^2}(R) \subset PS_{n^3}(R) \subset \cdots$$

for each positive integer n. Set

$$PS(R) := \bigcup_{n=1}^{\infty} PS_n(R) = \{T \in \mathcal{L}(\mathcal{H})\colon T \overset{ps}{\sim} R\}.$$

We remark that there exists a non-hyponormal operator power similar to a hyponormal operator. For example, let $H \in \mathcal{L}(\mathcal{H})$ be a hyponormal operator and let $N \in \mathcal{L}(\mathcal{H})$ be a nilpotent operator of order $m > 1$. Since zero operators are the only nilpotent hyponormal operators, the direct sum $T := H \oplus N$ is not hyponormal, but $T \in$

$PS_n(H \oplus 0)$ for any integer $n \geq m$. Let's consider another example. Assume that $\{\alpha_k\}_{k=0}^{\infty}$ and $\{\beta_k\}_{k=0}^{\infty}$ are bounded sequences of positive real numbers, and let A and B be the weighted shifts in $\mathcal{L}(\mathcal{H})$ with weights $\{\alpha_k\}$ and $\{\beta_k\}$, respectively, that is, $Ae_k = \alpha_k e_{k+1}$ and $Be_k = \beta_k e_{k+1}$ for all $k \geq 0$, where $\{e_k\}_{k=0}^{\infty}$ is an orthonormal basis for $\mathcal{H}$. Suppose that $\{\alpha_k\}_{k=0}^{\infty}$ is an increasing sequence such that $\alpha_k\alpha_{k+1} = \beta_k\beta_{k+1}$ holds for each $k \geq 0$. Then A is hyponormal. In addition, we get that

$$\frac{\alpha_0\alpha_1 \cdots \alpha_{2k}}{\beta_0\beta_1 \cdots \beta_{2k}} = \frac{\alpha_0}{\beta_0} \quad \text{and} \quad \frac{\alpha_0\alpha_1 \cdots \alpha_{2k+1}}{\beta_0\beta_1 \cdots \beta_{2k+1}} = 1$$

for all nonnegative integers k. This implies that A is similar to B from [8], and so $B \in PS_1(A)$. In this case, we can choose a non-increasing weight sequence $\{\beta_k\}$ for B, which ensures that B is not hyponormal; in particular, if we select the beginning weight β_0 satisfying that $\beta_0^2 > \alpha_0\alpha_1$, then $\beta_0 > \beta_1$ and so B is a non-hyponormal operator power similar to the hyponormal operator A. Furthermore, Example 3.17 also gives $B \in PS_4(A)$ where A and B are the weighted shifts with weights $\{1/3, 1/2, 1, 1, 1, \dots\}$ and $\{1/6, 1, 1/2, 2, 1/2, 2, \dots\}$, respectively; here, we observe that A is hyponormal, but B is not.

In this paper, we study power similarity of operators. In particular, we show that if $T \in PS(H)$ for some hyponormal operator H, then T is subscalar. From this result, we obtain that such an operator with rich spectrum has a nontrivial invariant subspace. Moreover, we consider invariant and hyperinvariant subspaces for $T \in PS(H)$.

2. Preliminaries

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have the *single-valued extension property*, abbreviated SVEP, if for every open subset G of $\mathbb{C}$ and any analytic function $f: G \to \mathcal{H}$ such that $(T - z)f(z) \equiv 0$ on G, it results $f(z) \equiv 0$ on G. For an operator $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$, *the resolvent set* $\rho_T(x)$ of T at x is defined to consist of z_0 in $\mathbb{C}$ such that there exists an analytic function $f(z)$ on a neighborhood of z_0, with values in $\mathcal{H}$, which verifies $(T - z)f(z) \equiv x$. We denote *the local spectrum* of T at x by $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$, and by using local spectra, we define *the local spectral subspace* of T by $\mathcal{H}_T(F) = \{x \in \mathcal{H}: \sigma_T(x) \subset F\}$, where F is a subset of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Dunford's property* (C) if $\mathcal{H}_T(F)$ is closed for each closed subset F of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Bishop's property* (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n: G \to \mathcal{H}$ of $\mathcal{H}$-valued analytic functions such that $(T - z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, then $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. It is well known [13] that

$$\text{Bishop's property } (\beta) \Rightarrow \text{Dunford's property } (C) \Rightarrow \text{SVEP}.$$

For an operator $T \in \mathcal{L}(\mathcal{H})$ and a subset F of $\mathbb{C}$, we define the *glocal spectral subspace* $\widetilde{\mathcal{H}}_T(F)$ to consist of all $x \in \mathcal{H}$ such that there is an analytic function $f: \mathbb{C} \setminus F \to \mathcal{H}$ for which $(T-z)f(z) \equiv x$ on $\mathbb{C} \setminus F$. Clearly, if T has the single-valued extension property, then $\mathcal{H}_T(F) = \widetilde{\mathcal{H}}_T(F)$ for any subset F of $\mathbb{C}$. We say that an operator $T \in \mathcal{L}(\mathcal{H})$ has *property* (δ) if we have the decomposition $\mathcal{H} = \widetilde{\mathcal{H}}_T(\overline{U}) + \widetilde{\mathcal{H}}_T(\overline{V})$ for any open cover $\{U, V\}$ of $\mathbb{C}$.

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *upper semi-Fredholm* if T has closed range and $\dim\ker(T) < \infty$, and T is called *lower semi-Fredholm* if T has closed range and $\dim(\mathcal{H}/ran(T)) < \infty$. When T is either upper semi-Fredholm or lower semi-Fredholm, it is called *semi-Fredholm*. The *index of a semi-Fredholm operator* $T \in \mathcal{L}(\mathcal{H})$, denoted $\text{index}(T)$, is given by $\text{index}(T) = \dim\ker(T) - \dim(\mathcal{H}/ran(T))$ and this value is an integer or $\pm\infty$. Also an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *Fredholm* if it is both upper and lower semi-Fredholm. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *Weyl* if it is Fredholm of index zero. For an operator $T \in \mathcal{L}(H)$, if we can choose the smallest positive integer m such that $\ker(T^m) = \ker(T^{m+1})$, then m is called *the ascent* of T and T is said to have *finite ascent*. Moreover, if there is the smallest positive integer n satisfying $ran(T^n) = ran(T^{n+1})$, then n is called *the descent* of T and T is said to have *finite descent*. We say that $T \in \mathcal{L}(\mathcal{H})$ is *Browder* if it is Fredholm of finite ascent and finite descent. We define the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} \colon T - \lambda \text{ is not Weyl}\}$$

and

$$\sigma_b(T) = \{\lambda \in \mathbb{C} \colon T - \lambda \text{ is not Browder}\}.$$

It is evident that

$$\sigma_e(T) \subset \sigma_w(T) \subset \sigma_b(T).$$

We say that *Weyl's theorem holds* for T if

$$\sigma(T) \setminus \pi_{00}(T) = \sigma_w(T), \quad \text{or equivalently,} \quad \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$$

where $\pi_{00}(T) := \{\lambda \in \text{iso}\,\sigma(T) \colon 0 < \dim\ker(T-\lambda) < \infty\}$ and $\text{iso}\,\sigma(T)$ denotes the set of all isolated points of $\sigma(T)$. We say that *Browder's theorem holds* for $T \in \mathcal{L}(\mathcal{H})$ if $\sigma_b(T) = \sigma_w(T)$.

Let z be the coordinate function in the complex plane $\mathbb{C}$ and $d\mu(z)$ the planar Lebesgue measure. Consider a bounded (connected) open subset U of $\mathbb{C}$. We shall denote by $L^2(U, \mathcal{H})$ the Hilbert space of measurable functions $f \colon U \to \mathcal{H}$ such that

$$\|f\|_{2,U} = \left(\int_U \|f(z)\|^2 \, d\mu(z)\right)^{1/2} < \infty.$$

The space of functions $f \in L^2(U, \mathcal{H})$ which are analytic functions in U is denoted by

$$A^2(U, \mathcal{H}) = L^2(U, \mathcal{H}) \cap \mathcal{O}(U, \mathcal{H})$$

where $\mathcal{O}(U, \mathcal{H})$ denotes the Fréchet space of $\mathcal{H}$-valued analytic functions on U with respect to uniform topology. The space $A^2(U, \mathcal{H})$ is called *the Bergman space* for U, and it is a Hilbert space.

Now let us define a special Sobolev type space. Let U be again a bounded open subset of $\mathbb{C}$ and m be a fixed non-negative integer. The vector-valued Sobolev space $W^m(U, \mathcal{H})$ with respect to $\bar\partial$ and of order m will be the space of those functions $f \in L^2(U, \mathcal{H})$ whose derivatives $\bar\partial f, \dots, \bar\partial^m f$ in the sense of distributions still belong to $L^2(U, \mathcal{H})$. Endowed with the norm

$$\| f\|^2_{W^m} = \sum_{i=0}^{m} \|\bar\partial^i f\|^2_{2,U},$$

$W^m(U, \mathcal{H})$ becomes a Hilbert space contained continuously in $L^2(U, \mathcal{H})$. Note that the linear operator M of multiplication by z on $W^m(U, \mathcal{H})$ is continuous and it has a spectral distribution $\Phi_M : C_0^m(\mathbb{C}) \to \mathcal{L}(W^m(U, \mathcal{H}))$ of order m defined by the following relation:

$$\Phi_M(\varphi) f = \varphi f \quad \text{for} \quad \varphi \in C_0^m(\mathbb{C}) \quad \text{and} \quad f \in W^m(U, \mathcal{H}).$$

Therefore, M is a scalar operator of order m.

3. Main results

In this section, we first prove that if $T \in PS(H)$ for some hyponormal operator $H \in \mathcal{L}(\mathcal{H})$, then T has scalar extensions.

Theorem 3.1. *If $T \in PS_n(H)$ for some hyponormal operator $H \in \mathcal{L}(\mathcal{H})$ and some positive integer $n > 1$, then T is subscalar of order $2n$. Hence, if $T \in PS(H)$ for some hyponormal operator $H \in \mathcal{L}(\mathcal{H})$, then T is subscalar.*

Proof. Suppose that $T \in PS_n(H)$ for some hyponormal operator $H \in \mathcal{L}(\mathcal{H})$ and some positive integer $n > 1$. For any open disk D in $\mathbb{C}$ containing $\sigma(T)$, define the map $V : \mathcal{H} \to H(D)$ by

$$Vh = \widetilde{1 \otimes h} \quad (\equiv 1 \otimes h + \overline{(T - z)W^{2n}(D, \mathcal{H})})$$

where $H(D) := W^{2n}(D, \mathcal{H})/\overline{(T - z)W^{2n}(D, \mathcal{H})}$ and $1 \otimes h$ denotes the constant function sending any $z \in D$ to h. Let $X \in \mathcal{L}(\mathcal{H})$ be an invertible operator such that $T^n = X^{-1}H^n X$, and let $h_k \in \mathcal{H}$ and $f_k \in W^{2n}(D, \mathcal{H})$ be sequences such that

$$\lim_{k\to\infty} \|(T - z) f_k + 1 \otimes h_k\|_{W^{2n}} = 0. \tag{1}$$

By the definition of the norm of the Sobolev space and (1), we have that

$$\lim_{k\to\infty} \|(T-z)\bar{\partial}^i f_k\|_{2,D} = 0$$

for $i = 1, 2, \dots, 2n$, which implies that

$$\lim_{k\to\infty} \|(T^n - z^n)\bar{\partial}^i f_k\|_{2,D} = 0$$

for $i = 1, 2, \dots, 2n$. Since $T^n = X^{-1}H^nX$, we ensure that

$$\lim_{k\to\infty} \|(H^n - z^n)X\bar{\partial}^i f_k\|_{2,D} = \lim_{k\to\infty} \|(H-z)Q(H,z)X\bar{\partial}^i f_k\|_{2,D} = 0 \tag{2}$$

for $i = 1, 2, \dots, 2n$ where $Q(\lambda, z) = \lambda^{n-1} + z\lambda^{n-2} + \cdots + z^{n-1}$. By the fundamental theorem of algebra,

$$Q(\lambda, z) = (\lambda - p_1 z)\cdots(\lambda - p_{n-1}z)$$

where $p_1 z, \dots, p_{n-1}z$ list the zeros of $Q(\lambda, z)$ by multiplicities. Set $p_n = 1$. Since each p_j is nonzero, we obtain from (2) that

$$\lim_{k\to\infty} \left\| \prod_{j=1}^{n} \left(\frac{1}{p_j} H - z \right) X\bar{\partial}^i f_k \right\|_{2,D} = 0 \tag{3}$$

for $i = 1, 2, \dots, 2n$.

Claim. *It holds for $r = 1, 2, \dots, n$ that*

$$\lim_{k\to\infty} \left\| \prod_{j=r}^{n} \left(\frac{1}{p_j} H - z \right) X\bar{\partial}^i f_k \right\|_{2,D_r} = 0$$

for $i = 1, 2, \dots, 2(n-r)+2$, where $D_1 = D$ and each D_r is an open disk containing $\sigma(T)$ with $\overline{D_{r+1}} \subset D_r$ for $r = 1, 2, \dots, n-1$.

To prove the claim, we will apply the induction on r. If $r = 1$, then the claim holds clearly by (3). Suppose that the claim is true for some $r = t < n$, that is,

$$\lim_{k\to\infty} \left\| \left(\frac{1}{p_t} H - z \right) \prod_{j=t+1}^{n} \left(\frac{1}{p_j} H - z \right) X\bar{\partial}^i f_k \right\|_{2,D_t} = 0$$

for $i = 1, 2, \dots, 2(n-t)+2$. Since $(1/p_t)H$ is hyponormal, we obtain from [15, Proposition 2.1] that

$$\lim_{k\to\infty} \left\| (I-P) \prod_{j=t+1}^{n} \left(\frac{1}{p_j} H - z \right) X\bar{\partial}^i f_k \right\|_{2,D_t} = 0 \tag{4}$$

for $i = 1,2,\dots,2(n-t-1)+2$, where P denotes the orthogonal projection of $L^2(D_t,\mathcal{H})$ onto $A^2(D_t, \mathcal{H})$. Hence

$$\lim_{k\to\infty}\left\|\left(\frac{1}{p_t}H - z\right)P\prod_{j=t+1}^{n}\left(\frac{1}{p_j}H - z\right)X\bar{\partial}^i f_k\right\|_{2,D_t} = 0$$

for $i = 1,2,\dots,2(n-t-1)+2$. Since $(1/p_t)H$ is hyponormal, it has Bishop's property (β) and so

$$\lim_{k\to\infty}\left\|P\prod_{j=t+1}^{n}\left(\frac{1}{p_j}H - z\right)X\bar{\partial}^i f_k\right\|_{2,D_{t+1}} = 0 \tag{5}$$

for $i = 1, 2, \dots, 2(n-t-1)+2$. From (4) and (5) we get that

$$\lim_{k\to\infty}\left\|\prod_{j=t+1}^{n}\left(\frac{1}{p_j}H - z\right)X\bar{\partial}^i f_k\right\|_{2,D_{t+1}} = 0$$

for $i = 1, 2, \dots, 2(n-t-1)+2$, which completes the proof of our claim.

From the claim with $r = n$, we have

$$\lim_{k\to\infty}\|(H-z)X\bar{\partial}^i f_k\|_{2,D_n} = 0$$

for $i = 1, 2$. Since H is hyponormal, it follows from [15, Proposition 2.1] that

$$\lim_{k\to\infty}\|X(I-P)f_k\|_{2,D_n} = \lim_{k\to\infty}\|(I-P)Xf_k\|_{2,D_n} = 0 \tag{6}$$

where P denotes the orthogonal projection of $L^2(D_n, \mathcal{H})$ onto $A^2(D_n, \mathcal{H})$. Since X is invertible, it holds that

$$\lim_{k\to\infty}\|(I-P)f_k\|_{2,D_n} = 0. \tag{7}$$

From (1) and (7), we see that

$$\lim_{k\to\infty}\|(T-z)Pf_k + (1\otimes h_k)\|_{2,D_n} = 0.$$

Let Γ be a curve in D_n surrounding $\sigma(T)$. Then

$$\lim_{k\to\infty}\|Pf_k(z) + (T-z)^{-1}(1\otimes h_k)\| = 0$$

uniformly for $z \in \Gamma$, which yields that

$$\lim_{k\to\infty}\left\|\frac{1}{2\pi i}\int_{\Gamma} Pf_k(z)\,dz + h_k\right\| = 0$$

by Riesz–Dunford functional calculus. Since $(1/(2\pi i))\int_\Gamma Pf_k(z)\,dz = 0$ by Cauchy's theorem, we have $\lim_{k\to\infty}\|h_k\| = 0$, which means that the map V is one-to-one and has closed range.

The class of a vector f or an operator A on $H(D)$ will be denoted by $\widetilde{f}$, respectively $\widetilde{A}$. Let M be the multiplication by z on $W^{2n}(D,\mathcal{H})$. As noted at the end of section two, M is a scalar operator of order $2n$ and has a spectral distribution Φ_M. Since $\overline{(T-z)W^{2n}(D,\mathcal{H})}$ is invariant under $\Phi_M(\varphi)$ for every $\varphi \in C_0^{2n}(\mathbb{C})$, $\widetilde{M}$ is a scalar operator of order $2n$ with spectral distribution $\widetilde{\Phi_M}$. Since

$$VTh = \widetilde{1\otimes Th} = \widetilde{z\otimes h} = \widetilde{M}(\widetilde{1\otimes h}) = \widetilde{M}Vh$$

for every $h \in \mathcal{H}$, we get the identity $VT = \widetilde{M}V$. In particular, $ran(V)$ is invariant for $\widetilde{M}$. Furthermore, $ran(V)$ is closed by the argument above, and hence $ran(V)$ is a closed invariant subspace of the scalar operator $\widetilde{M}$. Since T is similar to the restriction $\widetilde{M}|_{ran(V)}$ and $\widetilde{M}$ is scalar of order $2n$, the operator T is subscalar of order $2n$. □

Corollary 3.2. *Assume that $T \in PS(H)$ for some hyponormal operator $H \in \mathcal{L}(\mathcal{H})$. If $\sigma(T)$ has nonempty interior, then T has a nontrivial invariant subspace.*

Proof. The proof follows from Theorem 3.1 and [6]. □

Corollary 3.3. *If $T \in PS(H)$ for some hyponormal operator $H \in \mathcal{L}(\mathcal{H})$, then the following statements hold.*

(a) *T has the single-valued extension property, Dunford's property (C), and Bishop's property (β).*

(b) *If Q is a quasinilpotent operator commuting with T, then $T+Q$ has the single-valued extension property.*

(c) *If f is any function analytic on a neighborhood of $\sigma(T)$, then both Weyl's and Browder's theorems hold for $f(T)$ and $\sigma_w(f(T)) = \sigma_b(f(T)) = f(\sigma_w(T)) = f(\sigma_b(T))$.*

(d) *$\sigma(f(T)) - \pi_{00}(f(T)) = f(\sigma(T) - \pi_{00}(T))$ for every analytic function f on a neighborhood of $\sigma(T)$.*

Proof. (a) From section two, it suffices to prove that T has Bishop's property (β). We note that Bishop's property (β) is transmitted from an operator to its restrictions to closed invariant subspaces and every scalar operator has Bishop's property (β) (see [15]). Since T is subscalar by Theorem 3.1, we complete the proof.

(b) Since T is subscalar from Theorem 3.1, the proof follows from (a) and [5].

(c) Let f be any function analytic on a neighborhood of $\sigma(T)$. Since T is subscalar from Theorem 3.1, so is $f(T)$ and thus Weyl's theorem holds for $f(T)$ from [1]. Moreover, since $f(T)$ has the single-valued extension property by [13], Browder's theorem holds for $f(T)$ and the given equalities are satisfied from [1, Corollary 3.72].

(d) Since both T and $f(T)$ satisfy Weyl's theorem by (c), it follows that $f(\sigma_w(T)) = f(\sigma(T) - \pi_{00}(T))$ and $\sigma_w(f(T)) = \sigma(f(T)) - \pi_{00}(f(T))$. Since the identity $\sigma_w(f(T)) = f(\sigma_w(T))$ holds from (c), we complete the proof. □

Corollary 3.4. *Let $T \in PS(H)$ for some hyponormal operator $H \in \mathcal{L}(\mathcal{H})$. Then the operator matrix $\begin{pmatrix} 0 & T \\ I & 0 \end{pmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$ has Bishop's property (β).*

Proof. Set $A = \begin{pmatrix} a0 & T \\ I & 0 \end{pmatrix}$. Since $A^2 = T \oplus T$ and T has Bishop's property (β) from Corollary 3.3, we obtain that A^2 has Bishop's property (β), and so does A by [13]. □

Corollary 3.5. *Let $T_1 \in PS(H_1)$ and $T_2 \in PS(H_2)$ for some hyponormal operators $H_1, H_2 \in \mathcal{L}(\mathcal{H})$. If T_1 and T_2 are quasisimilar, then $\sigma(T_1) = \sigma(T_2)$ and $\sigma_e(T_1) = \sigma_e(T_2)$.*

Proof. Since T_1 and T_2 have Bishop's property (β) by Corollary 3.3, the proof follows from [16]. □

If $T \in \mathcal{L}(\mathcal{H})$ and $x \in \mathcal{H}$, then $\{T^n x\}_{n=0}^{\infty}$ is called *the orbit* of x under T, and is denoted by $\mathcal{O}(x, T)$. If $\mathcal{O}(x, T)$ is dense in $\mathcal{H}$, then x is called *a hypercyclic vector* for T. If there exists a hypercyclic vector $x \in \mathcal{H}$, an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *hypercyclic*. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *hypertransitive* if every nonzero vector in $\mathcal{H}$ is hypercyclic for T. Denote the set of all nonhypertransitive operators in $\mathcal{L}(\mathcal{H})$ by (NHT). The hypertransitive operator problem is the open question whether $(NHT) = \mathcal{L}(\mathcal{H})$.

Proposition 3.6. *If $T \in PS(H)$ for some hyponormal operator $H \in \mathcal{L}(\mathcal{H})$, then T is nonhypertransitive. In particular, if T is invertible, then T and T^{-1} have a common nontrivial invariant closed subset.*

Proof. Since H is not hypercyclic, any power of H is not hypercyclic by [4]. Since T^n is similar to H^n for some positive integer n, we obtain that T^n is not hypercyclic, and neither is T by [4]. Therefore T is nonhypertransitive. In addition, the second result follows from the first statement and [11]. □

Corollary 3.7. *Let $T \in PS(H)$ for some hyponormal operator $H \in \mathcal{L}(\mathcal{H})$. If $\sigma_T(x) \cap \mathbb{D} \neq \emptyset$ and $\sigma_T(x) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$ for every nonzero $x \in \mathcal{H}$, where $\mathbb{D}$ stands for the open unit disk in $\mathbb{C}$, then T^* is hypercyclic.*

Proof. Suppose that $\sigma_T(x) \cap \mathbb{D} \neq \emptyset$ and $\sigma_T(x) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$ for all nonzero $x \in \mathcal{H}$. Then we get that $\mathcal{H}_T(\mathbb{C} \setminus \mathbb{D}) = \{0\}$ and $\mathcal{H}_T(\overline{\mathbb{D}}) = \{0\}$. Since T has Bishop's property (β) by Corollary 3.3, T^* has property (δ). Thus, by [13, Proposition 2.5.14], we can infer that both $\mathcal{H}_{T^*}(\mathbb{D})$ and $\mathcal{H}_{T^*}(\mathbb{C} \setminus \overline{\mathbb{D}})$ are dense in $\mathcal{H}$. By using [7, Theorem 3.2], T^* is hypercyclic. □

In the following proposition, we give some spectral properties under power similarity to a hyponormal operator. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *quasitriangular* if there is a sequence $\{P_k\}$ of finite rank orthogonal projections on $\mathcal{H}$ converging strongly to the identity operator I on $\mathcal{H}$ such that $\lim_{k\to\infty}\|(I-P_k)TP_k\| = 0$. When both T and T^* are quasitriangular, we say that *biquasitriangular*.

Proposition 3.8. *If $T \in PS(H)$ for some hyponormal operator $H \in \mathcal{L}(\mathcal{H})$, then the following statements hold.*
(a) $\sigma_{ap}(T)^* \subset \sigma_{ap}(T^*) = \sigma_l(T^*) = \sigma(T^*)$.
(b) *T is invertible if and only if T is right invertible.*
(c) *Suppose that T is not a scalar multiple of the identity operator on $\mathcal{H}$. If T has no nontrivial invariant subspace, then T is biquasitriangular.*
(d) *T has finite ascent.*

Proof. (a) Since T has the single-valued extension property from Corollary 3.3, we have $\sigma(T^*) = \sigma_{ap}(T^*)$ (see [1] or [13]). Hence it holds that

$$\sigma_{ap}(T)^* \subset \sigma(T)^* = \sigma(T^*) = \sigma_{ap}(T^*) = \sigma_l(T^*).$$

(b) The proof follows from (a); indeed, $\sigma_r(T) = \sigma_l(T^*)^* = \sigma(T^*)^* = \sigma(T)$.

(c) Since T has no nontrivial invariant subspace, then $\sigma_p(T^*) = \emptyset$. Thus T^* has the single-valued extension property. Since both T and T^* have the single-valued extension property, we conclude from [12] that T is biquasitriangular.

(d) If $T \in PS(H)$, then $T^n = X^{-1}H^nX$ for some positive integer n. It suffices to show the inclusion $\ker(T^{n+1}) \subset \ker(T^n)$. If $x \in \ker(T^{n+1})$, then $T^{2n}x = 0$ and $H^{2n}Xx = 0$ since $T^{2n} = X^{-1}H^{2n}X$. By the hyponormality of H, it holds that $\ker(H) = \ker(H^2)$, which implies that $H^nXx = 0$ and so $T^nx = 0$. Thus $\ker(T^{n+1}) \subset \ker(T^n)$. □

Corollary 3.9. *If $T \in PS(H)$ for some hyponormal operator $H \in \mathcal{L}(\mathcal{H})$, then* $\ker(T) \cap ran(T^n) = \{0\}$ *for some positive integer n.*

Proof. If $T \in PS(H)$ for some hyponormal operator $H \in \mathcal{L}(\mathcal{H})$, then we obtain from Proposition 3.8 that $\ker(T^n) = \ker(T^{n+1})$ for some positive integer n. If $y \in \ker(T) \cap ran(T^n)$, then $Ty = 0$ and $y = T^nx$ for some $x \in \mathcal{H}$. This implies that $T^{n+1}x = Ty = 0$. Since $x \in \ker(T^{n+1}) = \ker(T^n)$, we have $y = T^nx = 0$. Hence $\ker(T) \cap ran(T^n) = \{0\}$. □

In the following proposition, we show that the translation invariant property does not hold in $PS_n(H)$, in general.

Proposition 3.10. *Let $T, H \in \mathcal{L}(\mathcal{H})$. Then $T \in PS_1(H)$ if and only if there exists a positive integer n such that $T - \lambda \in PS_n(H - \lambda)$ for all $\lambda \in \mathbb{C}$.*

Proof. If there is a positive integer n such that $T-\lambda \in PS_n(H-\lambda)$ for all $\lambda \in \mathbb{C}$, then we can choose an invertible operator $X \in \mathcal{L}(\mathcal{H})$ with $(T-\lambda)^n = X^{-1}(H-\lambda)^n X$ for all $\lambda \in \mathbb{C}$, which implies that

$$\sum_{k=0}^{n}(-1)^{n-k}\lambda^{n-k}T^k = X^{-1}\left(\sum_{k=0}^{n}(-1)^{n-k}\lambda^{n-k}H^k\right)X$$

for all $\lambda \in \mathbb{C}$. Since both sides are $(-1)^n\lambda^n$ when $k=0$, we obtain the following equation:

$$\sum_{k=1}^{n}(-1)^{n-k}\lambda^{n-k}T^k = X^{-1}\left(\sum_{k=1}^{n}(-1)^{n-k}\lambda^{n-k}H^k\right)X$$

for all $\lambda \in \mathbb{C}$. Dividing both sides by λ^{n-1} when $\lambda \neq 0$, we get that

$$\begin{aligned}&\sum_{k=2}^{n}(-1)^{n-k}\lambda^{1-k}T^k + (-1)^{n-1}T\\ &= X^{-1}\left(\sum_{k=2}^{n}(-1)^{n-k}\lambda^{1-k}H^k\right)X + X^{-1}((-1)^{n-1}H)X\end{aligned}$$

for all nonzero $\lambda \in \mathbb{C}$. Set $\lambda = re^{i\theta}$ with $r>0$ and real θ. Then

$$\begin{aligned}&\sum_{k=2}^{n}(-1)^{n-k}\frac{e^{i(1-k)\theta}}{r^{k-1}}T^k + (-1)^{n-1}T\\ &= X^{-1}\left(\sum_{k=2}^{n}(-1)^{n-k}\frac{e^{i(1-k)\theta}}{r^{k-1}}H^k\right)X + X^{-1}((-1)^{n-1}H)X\end{aligned}$$

for all $r>0$ and all real θ. Letting $r \to \infty$, we have $T = X^{-1}HX$. Hence $T \in PS_1(H)$.

Conversely, if $T \in PS_1(H)$, then $T-\lambda \in PS_1(H-\lambda)$ for all $\lambda \in \mathbb{C}$, which completes the proof. □

We say that $T \in \mathcal{L}(\mathcal{H})$ has *Dunford's boundedness condition* (B) if T has the single-valued extension property and there exists a constant $K>0$ such that $\|x\| \leq K\|x+y\|$ whenever $\sigma_T(x)\cap\sigma_T(y)=\emptyset$, where K is independent of x and y.

Proposition 3.11. *Let $T \in PS(H)$ for some hyponormal operator $H \in \mathcal{L}(\mathcal{H})$. If T has the property that $\sigma_T(P_F(x)) \subset \sigma_T(x)$ for all $x \in \mathcal{H}$ and each closed set F in $\mathbb{C}$ where P_F denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_T(F)$, then it has Dunford's boundedness condition (B).*

Proof. Since T has Dunford's property (C) by Corollary 3.3, $\mathcal{H}_T(F)$ is closed. Let $x_1, x_2 \in \mathcal{H}$ be such that $\sigma_T(x_1)\cap\sigma_T(x_2)=\emptyset$. Set $F_j = \sigma_T(x_j)$ for $j=1,2$.

By the hypothesis, we have $\sigma_T(P_{F_2}x_1) \subset \sigma_T(x_1) = F_1$. Moreover, it is obvious that $\sigma_T(P_{F_2}x_1) \subset F_2$. Hence

$$\sigma_T(P_{F_2}x_1) \subset F_1 \cap F_2 = \sigma_T(x_1) \cap \sigma_T(x_2) = \emptyset.$$

Since T has the single-valued extension property from Corollary 3.3, we get that $P_{F_2}x_1 = 0$. This means that $x_1 \perp \mathcal{H}_T(F_2)$. But since $\sigma_T(x_2) = F_2$, it holds that $x_2 \in \mathcal{H}_T(F_2)$ and so $\langle x_1, x_2 \rangle = 0$. This implies that

$$\|x_1 + x_2\| = (\|x_1\|^2 + \|x_2\|^2)^{1/2} \geq \|x_1\|,$$

which completes our proof. $\square$

Lemma 3.12. *Let $T \in PS(H)$ for some hyponormal operator $H \in \mathcal{L}(\mathcal{H})$ with $T \neq \lambda I$ for any $\lambda \in \mathbb{C}$. If there exists $x \in \mathcal{H} \setminus \{0\}$ such that $\sigma_T(x) \subsetneqq \sigma(T)$, then T has a nontrivial hyperinvariant subspace.*

Proof. If there exists a nonzero vector $x \in \mathcal{H}$ such that $\sigma_T(x) \subsetneqq \sigma(T)$, set

$$\mathcal{M} := \mathcal{H}_T(\sigma_T(x)), \quad \text{i.e.,} \quad \mathcal{M} = \{y \in \mathcal{H} \colon \sigma_T(y) \subset \sigma_T(x)\}.$$

Since T has Dunford's property (C) by Corollary 3.3, $\mathcal{M}$ is a T-hyperinvariant subspace from [13]. Since $x \in \mathcal{M}$, we get that $\mathcal{M} \neq \{0\}$. Suppose that $\mathcal{M} = \mathcal{H}$. Since T has the single-valued extension property, it follows that

$$\sigma(T) = \bigcup\{\sigma_T(y) \colon y \in \mathcal{H}\} \subset \sigma_T(x) \subsetneqq \sigma(T).$$

But this is a contradiction, and hence $\mathcal{M}$ is a nontrivial T-hyperinvariant subspace. $\square$

Theorem 3.13. *Let $T \in PS(H)$ for some hyponormal operator $H \in \mathcal{L}(\mathcal{H})$ with $T \neq \lambda I$ for any $\lambda \in \mathbb{C}$. If there exists $x \in \mathcal{H} \setminus \{0\}$ such that $\|T^n x\| \leq Cr^n$ for all positive integers n, where $C > 0$ and $0 < r < r(T)$ are constants, then T has a nontrivial hyperinvariant subspace.*

Proof. Put $f(z) := -\sum_{n=0}^{\infty} z^{-(n+1)} T^n x$, which is analytic for $|z| > r$; in fact, $\omega = z^{-1}$ for $|z| > r$, then $f(\omega) = -\sum_{n=0}^{\infty} \omega^{n+1} T^n x$ for $0 < |\omega| < 1/r$. Since the hypothesis implies that $\limsup_{n \to \infty} \|T^n x\|^{1/n} \leq r$, the radius of convergence for the power series $\sum_{n=0}^{\infty} \omega^{n+1} T^n x$ is at least $1/r$. Setting $f(0) := 0$, we get that $f(\omega)$ is analytic for $|\omega| < 1/r$, i.e., $f(z)$ is analytic for $|z| > r$. Since

$$(T - z) f(z) = -\sum_{n=0}^{\infty} z^{-(n+1)} T^{n+1} x + \sum_{n=0}^{\infty} z^{-n} T^n x = x$$

for all $z \in \mathbb{C}$ with $|z| > r$, we have $\rho_T(x) \supset \{z \in \mathbb{C} : |z| > r\}$, i.e.,

$$\sigma_T(x) \subset \{z \in \mathbb{C} : |z| \leq r\}.$$

Since $r < r(T)$, it holds that $\sigma_T(x) \subsetneqq \sigma(T)$. Thus, we conclude from Lemma 3.12 that T has a nontrivial hyperinvariant subspace. $\square$

Finally, we consider a special case of power similarity.

Proposition 3.14. *Let $T \in \mathcal{L}(\mathcal{H})$. Suppose that $R \in \mathcal{L}(\mathcal{H})$ is an operator satisfying the following conditions*:
(a) $T^n = R^n$,
(b) $T^{n-2}R = R^{n-1}$, $R^{n-2}T = T^{n-1}$, *and*
(c) $T^{n-1} + R^{n-1} \neq 0$
for some positive integer $n \geq 2$. If T has a nontrivial hyperinvariant subspace, then R has a nontrivial invariant subspace.

Proof. Suppose that R has no nontrivial invariant subspace. Then T and R have no common nontrivial invariant subspace. Define $A = T^{n-1} + R^{n-1}$ for some positive integer $n \geq 2$. Then we have $AT = (T^{n-1} + R^{n-1})T = T^n + R^{n-1}T$ and $RA = R(T^{n-1} + R^{n-1}) = RT^{n-1} + R^n$. Since $R^{n-1}T = RR^{n-2}T = RT^{n-1}$, we get that $AT = RA$. Similarly, $AR = TA$ holds. By [14, Lemma], $A = 0$ or A is a quasiaffinity. However, A is nonzero by (c), and so it should be a quasiaffinity. This implies that T and R are quasisimilar. Since T has nontrivial a hyperinvariant subspace by hypothesis, [17, Theorem 6.19] implies that R has a nontrivial hyperinvariant subspace. So we have a contradiction. Hence R has a nontrivial invariant subspace. $\square$

As some applications of Proposition 3.14, we get the following corollaries.

Corollary 3.15. *Under the same hypotheses as in* Proposition 3.14, *if T is a normal operator that is not a scalar multiple of the identity operator on $\mathcal{H}$ or T is nonzero and is not a quasiaffinity, then R has a nontrivial invariant subspace.*

Proof. If T satisfies the first condition, then T has a nontrivial hyperinvariant subspace by [17, Corollary 1.17]. If T is nonzero and is not a quasiaffinity, then $\sigma_p(T) \cup \sigma_p(T^*) \neq \emptyset$, and so T has a nontrivial hyperinvariant subspace. Hence, in both cases, R has a nontrivial invariant subspace from Proposition 3.14. $\square$

Corollary 3.16. *Let $A \in PS_2(B)$ for some $B \in \mathcal{L}(\mathcal{H})$, i.e., there exists an invertible operator X such that $A^2 = X^{-1}B^2X$, and $XAX^{-1} + B \neq 0$. If B has a nontrivial hyperinvariant subspace, then A has a nontrivial invariant subspace.*

Proof. Since $B^2 = XA^2X^{-1} = (XAX^{-1})^2$, taking $R = XAX^{-1}$ and $T = B$ in Proposition 3.14, we obtain that A has a nontrivial invariant subspace. □

We observe that even if T is hyponormal in Proposition 3.14, it is not necessary that R is hyponormal from the following examples.

EXAMPLE 3.17. Let A and B be weighted shifts defined by $Ae_k = \alpha_k e_{k+1}$ and $Be_k = \beta_k e_{k+1}$ with positive weight sequences $\{\alpha_k\}_{k=0}^{\infty}$ and $\{\beta_k\}_{k=0}^{\infty}$. Note that A and B satisfy the conditions in Proposition 3.14 if and only if

$$\begin{cases} \alpha_k\alpha_{k+1}\cdots\alpha_{k+n-1} = \beta_k\beta_{k+1}\cdots\beta_{k+n-1}, \\ \alpha_{k+1}\alpha_{k+2}\cdots\alpha_{k+n-2} = \beta_{k+1}\beta_{k+2}\cdots\beta_{k+n-2} \end{cases} \tag{8}$$

for all nonnegative integers k. In particular, we note that if A and B satisfy the conditions in Proposition 3.14 for $n = 3$, then they must be the same by (8).

Let $\{\alpha_k\}_{k=0}^{\infty} = \{1/3, 1/2, 1, 1, 1, \dots\}$ and $\{\beta_k\}_{k=0}^{\infty} = \{1/6, 1, 1/2, 2, 1/2, 2, \dots\}$. Then equation (8) holds for $n = 4$. Hence, we obtain that A and B satisfy all conditions in Proposition 3.14 for $n = 4$. Since $\{\alpha_k\}_{k=0}^{\infty}$ is increasing but $\{\beta_k\}_{k=0}^{\infty}$ is not, we conclude that A is hyponormal, while B is not.

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