

## ON THE ISOMORPHISM CLASSES OF IWASAWA MODULES WITH $\lambda = 3$ AND $\mu = 0$

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(Received July 4, 2011, revised January 22, 2013)

### Abstract

For an odd prime number  $p$ , we classify the isomorphism classes of finitely generated torsion  $\Lambda = \mathbb{Z}_p[[T]]$ -modules with  $\lambda = 3$  and  $\mu = 0$ , which are free over  $\mathbb{Z}_p$ . We apply this classification to the Iwasawa module associated to the cyclotomic  $\mathbb{Z}_p$ -extension of an imaginary quadratic field.

### 1. Introduction

Let  $p$  be a fixed odd prime number and  $\Lambda = \mathbb{Z}_p[[T]]$  the ring of power series in one variable over  $\mathbb{Z}_p$ . In the classical Iwasawa theory, one studies Iwasawa modules up to pseudo-isomorphism. In this paper, we study Iwasawa modules up to  $\Lambda$ -isomorphism. Especially, our aim is to generalize Sumida's results (cf. [11], [12]).

For a distinguished polynomial  $f(T) \in \mathbb{Z}_p[T]$ , Sumida introduced the set

$$\mathcal{M}_{f(T)} = \left\{ [M]_{\mathbb{Q}_p} \mid \begin{array}{l} M \text{ is a finitely generated torsion } \Lambda\text{-module,} \\ \text{char}(M) = (f(T)) \text{ and } M \text{ is free over } \mathbb{Z}_p \end{array} \right\},$$

where  $[M]_{\mathbb{Q}_p}$  is the  $\Lambda$ -isomorphism class of  $M$  and  $\text{char}(M)$  is the characteristic ideal of  $M$ . Sumida showed that  $\mathcal{M}_{f(T)}$  is a finite set if and only if  $f(T)$  is a separable polynomial ([11], Theorem 2). Sumida and Koike determined  $\mathcal{M}_{f(T)}$  in the case  $\deg(f(T)) \leq 2$  ([7], Theorem 2.1 and [11], Proposition 10). In this paper, we determine the set  $\mathcal{M}_{f(T)}$  for

$$f(T) = (T - \alpha)(T - \beta)(T - \gamma),$$

where  $\alpha, \beta, \gamma$  are distinct elements of  $p\mathbb{Z}_p$  (Theorem 3.5). This is a generalization of Sumida's result [12]. (Precisely speaking, we work over  $\mathcal{O}[[T]]$  below where  $\mathcal{O}$  is the ring of integers of a finite extension of  $\mathbb{Q}_p$ .)

The motivation of this work lies in Iwasawa theory. We apply our theorem to the Iwasawa module associated to the cyclotomic  $\mathbb{Z}_p$ -extension of an imaginary quadratic field. Let  $k$  be an imaginary quadratic number field and  $k_\infty/k$  the cyclotomic  $\mathbb{Z}_p$ -extension of  $k$ . For each  $n \geq 0$ , we denote by  $k_n$  the unique intermediate field of

$k_\infty/k$  with  $[k_n : k] = p^n$ . Let  $A_n$  be the  $p$ -Sylow subgroup of the ideal class group of  $k_n$ . We put  $X = \varprojlim A_n$ , where the inverse limit is taken with respect to the relative norms. It is known that  $X$  is a finitely generated torsion  $\Lambda$ -module (cf. [5]). Moreover, it is known that  $X$  is a free  $\mathbb{Z}_p$ -module.

Therefore, we can apply our theorem to the Iwasawa module  $X$ . We apply our theorem in the case that  $p = 3$  and  $k = \mathbb{Q}(\sqrt{-d})$ . In this setting,  $f(T)$  can be approximately calculated by the  $p$ -adic  $L$ -functions (see Section 6).

The outline of this paper is as follows. Let  $E$  be a finite extension of  $\mathbb{Q}_p$  and  $\Lambda_E$  the ring of power series in one variable over the ring of integers of  $E$ . In Section 2, we introduce the set  $\mathcal{M}_{f(T)}^E$  which is the set of isomorphism classes of  $\Lambda_E$ -module satisfying some properties. In Section 3, we state our main theorem (Theorem 3.5). We define a certain equivalence relation  $\sim'$  on  $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathcal{O}_E$  and define  $Z' = (\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathcal{O}_E)/\sim'$ . We define  $Z$  to be a subset of  $Z'$  satisfying certain conditions. An element of  $Z'$  is written as  $(m, n, x)$ . We also define an equivalence relation  $\sim$  on  $Z$  and consider  $Z/\sim$ . An element of  $Z/\sim$  is written as  $[(m, n, x)]$ . Roughly speaking, Theorem 3.5 states that there is one to one correspondence between  $\mathcal{M}_{f(T)}^E$  and the equivalence classes of  $Z/\sim$ . Moreover, we prove Sumida's result ([12], Theorem 1) in Corollary 3.8, using our Theorem 3.5. In Section 4, we give a proof of Theorem 3.5. Section 5 is a preparation for Section 6. In this section, we study the structure of  $\Lambda$ -modules. In Section 6, we apply Theorem 3.5 to the Iwasawa module associated to the cyclotomic  $\mathbb{Z}_p$ -extension of an imaginary quadratic number field. We apply our theorem in the case that  $p = 3$  and  $k = \mathbb{Q}(\sqrt{-d})$  for all  $d$  such that  $1 < d < 10^5$  and  $d \not\equiv 2 \pmod{3}$ , that is to say  $p$  does not split in  $k$ . We determine the  $\Lambda$ -isomorphism class of the Iwasawa module associated to  $k$  in the case  $\lambda_p(k) = 3$ , where  $\lambda_p(k)$  is the Iwasawa  $\lambda$ -invariant. There are 1109 imaginary quadratic fields satisfying these properties. Among them, there are 1015 fields whose  $A_0$  are cyclic groups. We can determine  $[X]_{\mathbb{Q}_p}$  for these 1015 fields by Proposition 5.2 immediately. For remaining 94 fields whose  $A_0$  are not cyclic groups, there are 66 fields whose  $f(T)$  is reducible. We determine  $[X]_{\mathbb{Q}_p}$  for these 66 fields.

After I submitted this paper, I was informed from Sumida (Takahashi) of the thesis by C. Franks where he independently obtained the classification of  $\Lambda$ -modules. In Remark 3.6, I will explain the difference between our method and that in Franks.

## 2. Preliminaries

Let  $p$  be an odd prime number. Let  $E$  be a finite extension over the field  $\mathbb{Q}_p$  of  $p$ -adic numbers. Let  $\mathcal{O}_E$ ,  $\pi$ ,  $\text{ord}_E$  be the ring of integers in  $E$ , a prime element and the normalized additive valuation of  $E$  such that  $\text{ord}_E(\pi) = 1$ , respectively. We put  $\Lambda_E := \mathcal{O}_E[[T]]$  the ring of power series over  $\mathcal{O}_E$ .

Let  $M$  be a finitely generated torsion  $\Lambda_E$ -module. By the structure theorem of  $\Lambda_E$ -modules (cf. [13], Chapter 13), there is a  $\Lambda_E$ -homomorphism

$$\varphi: M \rightarrow \left( \bigoplus_i \Lambda_E / (\pi^{m_i}) \right) \oplus \left( \bigoplus_j \Lambda_E / (f_j(T)^{n_j}) \right)$$

with finite kernel and finite cokernel, where  $m_i, n_j$  are non-negative integers and  $f_j(T) \in \mathcal{O}_E[T]$  is a distinguished irreducible polynomial. We put

$$\text{char}(M) = \left( \prod_i \pi^{m_i} \prod_j f_j(T)^{n_j} \right)$$

which is an ideal in  $\Lambda_E$ . We define  $[M]_E$  to be the  $\Lambda_E$ -isomorphism class of  $M$ .

As in the introduction, for a distinguished polynomial  $f(T) \in \mathcal{O}_E[T]$ , we consider finitely generated torsion  $\Lambda_E$ -modules whose characteristic ideals are  $(f(T))$ , and define the set  $\mathcal{M}_{f(T)}^E$  by

$$(1) \quad \mathcal{M}_{f(T)}^E = \left\{ [M]_E \mid \begin{array}{l} M \text{ is a finitely generated torsion } \Lambda_E\text{-module,} \\ \text{char}(M) = (f(T)) \text{ and } M \text{ is free over } \mathcal{O}_E \end{array} \right\}.$$

Sumida showed that  $\mathcal{M}_{f(T)}^E$  is a finite set if and only if  $f(T)$  is separable [11]. Here, we say  $f(T)$  is separable when  $f(T)$  has no multiple roots in an algebraic closure of  $E$ . Sumida also determined  $\mathcal{M}_{f(T)}^E$  in the case  $f(T) = (T - \alpha)(T - \beta)(T - \gamma)$ , where  $\alpha, \beta, \gamma \in p\mathbb{Z}_p$  satisfy  $\alpha \not\equiv \beta, \beta \not\equiv \gamma, \gamma \not\equiv \alpha \pmod{p^2}$  (see [12], Theorem 1). We generalize this result to a general separable polynomial  $f(T)$  with degree 3.

Now we put

$$(2) \quad f(T) = (T - \alpha)(T - \beta)(T - \gamma),$$

where  $\alpha, \beta$  and  $\gamma$  are distinct elements of  $\pi\mathcal{O}_E$ . We determine all the elements of  $\mathcal{M}_{f(T)}^E$  in the next section.

Let  $[M]_E \in \mathcal{M}_{f(T)}^E$ . Since  $M$  has no non-trivial finite  $\Lambda_E$ -submodule, there exists an injective  $\Lambda_E$ -homomorphism

$$\varphi: M \hookrightarrow \Lambda_E / (T - \alpha) \oplus \Lambda_E / (T - \beta) \oplus \Lambda_E / (T - \gamma) =: \mathcal{E}$$

with finite cokernel. We write  $\mathcal{E}$  for the right hand side. The above fact implies that every class of  $\mathcal{M}_{f(T)}^E$  can be represented by a  $\Lambda_E$ -submodule of  $\mathcal{E}$ .

Now we fix a notation to express such submodules in  $\mathcal{E}$ . First, by using the canonical isomorphism  $\Lambda_E / (T - \alpha) \cong \mathcal{O}_E (f(T) \mapsto f(\alpha))$ , we define an isomorphism

$$\iota: \mathcal{E} = \Lambda_E / (T - \alpha) \oplus \Lambda_E / (T - \beta) \oplus \Lambda_E / (T - \gamma) \rightarrow \mathcal{O}_E^{\oplus 3}$$

by  $(f_1(T), f_2(T), f_3(T)) \mapsto (f_1(\alpha), f_2(\beta), f_3(\gamma))$ . We identify  $\mathcal{E}$  with  $\mathcal{O}_E^{\oplus 3}$  via  $\iota$ . Thus an element in  $\mathcal{E}$  is expressed as  $(a_1, a_2, a_3) \in \mathcal{O}_E^{\oplus 3}$ . Since the rank of  $M$  is equal to 3, we can write  $M$  in the form

$$M = \langle (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3) \rangle_{\mathcal{O}_E} \subset \mathcal{E},$$

where  $\langle * \rangle_{\mathcal{O}_E}$  is the  $\mathcal{O}_E$ -submodule generated by  $*$ . Further, we can express the action of  $T$  by

$$T(a_1, a_2, a_3) = (\alpha a_1, \beta a_2, \gamma a_3),$$

using this notation.

### 3. Main result

Let  $M$  be an  $\mathcal{O}_E$ -submodule of  $\mathcal{E}$  with  $\text{rank}(M) = 3$  of the form

$$M = \langle (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3) \rangle_{\mathcal{O}_E} \subset \mathcal{E}.$$

Let

$$\begin{aligned} s &= \min\{i \in \mathbb{Z}_{\geq 0} \mid \exists a, b \in \mathcal{O}_E \text{ s.t. } (\pi^i, a, b) \in M\}, \\ t &= \min\{i \in \mathbb{Z}_{\geq 0} \mid \exists c \in \mathcal{O}_E \text{ s.t. } (0, \pi^i, c) \in M\}, \\ u &= \min\{i \in \mathbb{Z}_{\geq 0} \mid (0, 0, \pi^i) \in M\}. \end{aligned}$$

Then we have

$$M = \langle (\pi^s, a, b), (0, \pi^t, c), (0, 0, \pi^u) \rangle_{\mathcal{O}_E}.$$

Suppose  $(a_1, a_2, a_3) \in M$ . Since  $\text{ord}_E(a_1) \geq s$ , there exists  $x \in \mathcal{O}_E$  such that  $a_1 = x\pi^s$ . So  $(a_1, a_2, a_3) - x(\pi^s, a, b) = (0, a_2 - xa, a_3 - xb) \in M$ . Since  $\text{ord}_E(a_2 - xa) \geq t$ , there exists  $y \in \mathcal{O}_E$  such that  $a_2 - xa = y\pi^t$ . Similarly by the same method as above, we get  $(0, 0, a_3 - xb - yc) \in M$ . Finally, there exists  $z \in \mathcal{O}_E$  such that  $a_3 - xb - yc = z\pi^u$ . Then we have  $(a_1, a_2, a_3) = x(\pi^s, a, b) + y(0, \pi^t, c) + z(0, 0, \pi^u)$ .

The following lemma is a necessary and sufficient condition for an  $\mathcal{O}_E$ -module  $M$  to be a  $\Lambda_E$ -submodule.

**Lemma 3.1.** *Let  $M = \langle (\pi^s, a, b), (0, \pi^t, c), (0, 0, \pi^u) \rangle_{\mathcal{O}_E}$ . Then the following two statements are equivalent:*

- (i) *The  $\mathcal{O}_E$ -module  $M$  is a  $\Lambda_E$ -submodule,*
- (ii) *Integers  $a, b, c, s, t$  and  $u$  satisfy*

$$\begin{cases} t \leq \text{ord}_E(\beta - \alpha) + \text{ord}_E(a), \\ u \leq \text{ord}_E\{(\gamma - \alpha)b - (\beta - \alpha)\pi^{-t}ac\}, \\ u \leq \text{ord}_E(\gamma - \beta) + \text{ord}_E(c). \end{cases}$$

Proof. We first suppose that  $M$  is a  $\Lambda_E$ -submodule. So  $M$  satisfies  $TM \subset M$  and we have

$$\begin{aligned} T(\pi^s, a, b) &= (\alpha\pi^s, \beta a, \gamma b) \\ &= \alpha(\pi^s, a, b) + (\beta - \alpha)\pi^{-t}a(0, \pi^t, c) \\ &\quad + \{(\gamma - \alpha)b - (\beta - \alpha)\pi^{-t}ac\}\pi^{-u}(0, 0, \pi^u), \\ T(0, \pi^t, c) &= (0, \beta\pi^t, \gamma c) \\ &= \beta(0, \pi^t, c) + (\gamma - \beta)c\pi^{-u}(0, 0, \pi^u). \end{aligned}$$

Since these coefficients belong to  $\mathcal{O}_E$ , we get (ii). Conversely, if an  $\mathcal{O}_E$ -module  $M$  satisfies (ii),  $M$  is naturally an  $\mathcal{O}_E[T]$ -module by the action as above. We show that  $M$  becomes a  $\Lambda_E$ -module. For a positive integer  $n$ , we put  $v_n = \sum_{k=0}^n d_k T^k \in \mathcal{O}_E[T]$  and  $v = \sum_{n=0}^{\infty} d_n T^n \in \mathcal{O}_E[[T]]$ . Then we have

$$\begin{aligned} v_n(\pi^s, a, b) &= \left( \pi^s \sum_{k=0}^n d_k \alpha^k, a \sum_{k=0}^n d_k \beta^k, b \sum_{k=0}^n d_k \gamma^k \right) \\ &= \sum_{k=0}^n d_k \alpha^k (\pi^s, a, b) + a \left( \sum_{k=0}^n d_k \beta^k - \sum_{k=0}^n d_k \alpha^k \right) \pi^{-t}(0, \pi^t, c) \\ &\quad + \left\{ b \left( \sum_{k=0}^n d_k \gamma^k - \sum_{k=0}^n d_k \alpha^k \right) - \left( \sum_{k=0}^n d_k \beta^k - \sum_{k=0}^n d_k \alpha^k \right) \pi^{-t}ac \right\} \pi^{-u}(0, 0, \pi^u). \end{aligned}$$

Because  $M$  is an  $\mathcal{O}_E[T]$ -module, we have  $v_n(\pi^s, a, b) \in M$ . Thus we obtain

$$a \left( \sum_{k=0}^n d_k \beta^k - \sum_{k=0}^n d_k \alpha^k \right) \pi^{-t} \in \mathcal{O}_E$$

and

$$\left\{ b \left( \sum_{k=0}^n d_k \gamma^k - \sum_{k=0}^n d_k \alpha^k \right) - \left( \sum_{k=0}^n d_k \beta^k - \sum_{k=0}^n d_k \alpha^k \right) \pi^{-t}ac \right\} \pi^{-u} \in \mathcal{O}_E.$$

Since  $d_k \alpha^k, d_k \beta^k, d_k \gamma^k \rightarrow 0$  ( $k \rightarrow \infty$ ),  $\sum_{k=0}^{\infty} d_k \alpha^k, \sum_{k=0}^{\infty} d_k \beta^k$  and  $\sum_{k=0}^{\infty} d_k \gamma^k$  converge in  $\mathcal{O}_E$ . Thus we have  $v(\pi^s, a, b) \in M$ . For  $(0, \pi^t, c)$  and  $(0, 0, \pi^u)$ , we can define the action of the elements of  $\Lambda_E$  by the same method as above.  $\square$

We use the following lemma to fix a representative of the  $\Lambda_E$ -isomorphism class of  $M$ .

**Lemma 3.2** (Lemma 1 in Sumida [12] ). *Let  $M = \langle (a_1, a_2, a_3), (b_1, b_2, b_3), (c_1, c_2, c_3) \rangle_{\mathcal{O}_E}$  be a  $\Lambda_E$ -submodule of  $\mathcal{E}$  and  $u_1, u_2, u_3 \in \mathcal{O}_E \setminus \{0\}$ . Then we have*

$$M \cong \langle (u_1a_1, u_2a_2, u_3a_3), (u_1b_1, u_2b_2, u_3b_3), (u_1c_1, u_2c_2, u_3c_3) \rangle_{\mathcal{O}_E}$$

as  $\Lambda_E$ -modules.

We take  $M$  to be a  $\Lambda_E$ -submodule of  $\mathcal{E}$  with finite index. Then we can write

$$M = \langle (\pi^s, a, b), (0, \pi^t, c), (0, 0, \pi^u) \rangle_{\mathcal{O}_E}$$

as we explained in the beginning of this section. By Lemma 3.2, there exist non-negative integers  $m, n$  and  $x \in \mathcal{O}_E$  such that there is an isomorphism

$$M \cong \langle (1, 1, 1), (0, \pi^m, x), (0, 0, \pi^n) \rangle_{\mathcal{O}_E}$$

as  $\Lambda_E$ -modules. In fact, by Lemma 3.2,  $M$  is isomorphic to  $M' = \langle (1, a, b), (0, \pi^t, c), (0, 0, \pi^u) \rangle_{\mathcal{O}_E}$ . In the case  $\text{ord}_E(a) \leq t$ , by Lemma 3.2,  $M$  is isomorphic to  $\langle (1, 1, b), (0, a^{-1}\pi^t, c), (0, 0, \pi^u) \rangle_{\mathcal{O}_E}$ . On the other hand, in the case  $\text{ord}_E(a) > t$ , since  $M' = \langle (1, a + \pi^t, b + c), (0, \pi^t, c), (0, 0, \pi^u) \rangle_{\mathcal{O}_E}$ , we can proceed by the same method as in the case  $\text{ord}_E(a) \leq t$ . Therefore  $M$  is isomorphic to  $M'' = \langle (1, 1, b), (0, a'\pi^t, c), (0, 0, \pi^u) \rangle_{\mathcal{O}_E}$  for some  $a' \in E$ . By applying the same method as above,  $M''$  is isomorphic to  $\langle (1, 1, 1), (0, \pi^m, x), (0, 0, \pi^n) \rangle_{\mathcal{O}_E}$  for some non-negative integers  $m, n$  and  $x \in \mathcal{O}_E$ .

We define  $M(m, n, x)$  by

$$M(m, n, x) := \langle (1, 1, 1), (0, \pi^m, x), (0, 0, \pi^n) \rangle_{\mathcal{O}_E} \subset \mathcal{E}.$$

**Proposition 3.3.** *Let  $f(T) \in \mathcal{O}_E[T]$  be a distinguished polynomial. Then we have*

$$\mathcal{M}_{f(T)}^E = \{[M(m, n, x)]_E \mid m, n, x \text{ satisfy } (*)\},$$

where  $[M(m, n, x)]_E$  is the  $\Lambda_E$ -isomorphism class of  $M(m, n, x)$  and  $(*)$  is as follows:

$$(*) \quad \begin{cases} (A) & 0 \leq m \leq \text{ord}_E(\beta - \alpha), \\ (B) & 0 \leq n \leq \text{ord}_E(\gamma - \beta) + \text{ord}_E(x), \\ (C) & n \leq \text{ord}_E\{(\gamma - \alpha) - (\beta - \alpha)\pi^{-m}x\}. \end{cases}$$

*Proof.* Let  $M$  be a  $\Lambda_E$ -module such that  $[M]_E \in \mathcal{M}_{f(T)}^E$ . Then we saw that  $[M]_E = [M(m, n, x)]_E$  for some  $m, n, x$  satisfying  $(*)$  by Lemma 3.1. We will show the converse. We suppose that  $m, n$  and  $x$  satisfy  $(*)$ . By Lemma 3.1,  $M(m, n, x)$  becomes a finitely generated  $\Lambda_E$ -module. Since  $f(T) = (T - \alpha)(T - \beta)(T - \gamma)$  annihilates  $M(m, n, x)$ ,  $M(m, n, x)$  is a torsion  $\Lambda_E$ -module. Moreover, by the definition of  $M(m, n, x)$ ,  $M(m, n, x)$  is a free  $\mathcal{O}_E$ -module. Finally we show that  $\text{char}(M(m, n, x)) = (f(T))$ . The  $\Lambda_E$ -module  $M(m, n, x)$

is a submodule of  $\mathcal{E}$  with finite index. In fact, since  $\text{rank}_{\mathcal{O}_E}(\mathcal{E}) = \text{rank}_{\mathcal{O}_E}(M(m, n, x)) = 3$ ,  $\mathcal{E}/M(m, n, x)$  is finite. This implies that  $\text{char}(M(m, n, x)) = \text{char}(\mathcal{E})$ . Thus we get  $[M(m, n, x)]_E \in \mathcal{M}_{f(T)}^E$ . □

REMARK 3.4. (i) If  $x \equiv x' \pmod{\pi^n}$ , we have  $M(m, n, x) = M(m, n, x')$  since  $(0, \pi^m, x) = (0, \pi^m, x') + a(0, 0, \pi^n)$  for some  $a \in \mathcal{O}_E$ . In particular, if  $\text{ord}_E(x) \geq n$ , we have  $M(m, n, x) = M(m, n, 0)$ . This means that we may assume that  $x = 0$  or  $\text{ord}_E(x) < n$ .

(ii) We have

$$\frac{(\gamma - \alpha)(\gamma - \beta)}{\pi^n} = \frac{(\gamma - \beta)x}{\pi^n} \cdot \frac{\beta - \alpha}{\pi^m} + (\gamma - \beta) \cdot \frac{(\gamma - \alpha) - (\beta - \alpha)\pi^{-m}x}{\pi^n}.$$

Therefore if (\*) holds, we get

$$0 \leq n \leq \text{ord}_E(\gamma - \alpha) + \text{ord}_E(\gamma - \beta).$$

Let  $M(m, n, x)$  and  $M(m', n', x') \in \mathcal{M}_{f(T)}^E$ . We will investigate a relation among  $m, m', n, n', x$  and  $x'$  when  $M(m, n, x)$  is isomorphic to  $M(m', n', x')$  as a  $\Lambda_E$ -module. We note that we may assume  $x = 0$  or  $\text{ord}_E(x) < n$  by Remark 3.4 (i).

First of all, we prepare some notation. For  $(m, n, x)$  and  $(m', n', x') \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathcal{O}_E$ , we define

$$(m, n, x) \sim' (m', n', x') \iff m = m', n = n' \text{ and } x \equiv x' \pmod{\pi^n \mathcal{O}_E}.$$

We put  $Z' := (\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathcal{O}_E) / \sim'$  and introduce a set

$$(3) \quad Z := \{\overline{(m, n, x)} \in Z' \mid m, n, x \text{ satisfy } (*)\},$$

where (\*) is the inequalities (A), (B) and (C) in Proposition 3.3 and  $\overline{(m, n, x)}$  is the equivalence class of  $(m, n, x)$ . The class  $\overline{(m, n, x)}$  is determined by  $m, n$  and  $x \pmod{\pi^n \mathcal{O}_E}$ . We note that the condition (\*) does not depend on the choice of a representative of  $(m, n, x)$ .

For an element for  $x \in \mathcal{O}_E$  and  $z = \bar{x} \in \mathcal{O}_E / \pi^n \mathcal{O}_E$ , we define  $\text{ord}_E(z) = \text{ord}_E(x \pmod{\pi^n})$  as follows:

$$\text{ord}_E(z) := \begin{cases} \text{ord}_E(x) & \text{if } \bar{x} \neq 0, \\ \infty & \text{if } \bar{x} = 0. \end{cases}$$

For  $\overline{(m, n, x)}$  and  $\overline{(m', n', x')} \in Z$ , let  $k = \text{ord}_E(x \pmod{\pi^n})$  and  $l = \text{ord}_E(x' - \pi^m)$ . We define  $\overline{(m, n, x)} \sim \overline{(m', n', x')}$  as follows.

(I) Suppose  $m \neq 0$ .

(a) When  $l + k \geq n$ , we define

$$\overline{(m, n, x)} \sim \overline{(m', n', x')} \iff m = m', n = n' \text{ and } \bar{x} = \bar{x}' \text{ in } \mathcal{O}_E / \pi^n \mathcal{O}_E.$$

(b) When  $l + k < n$ , we define

$$\begin{aligned} \overline{(m, n, x)} \sim \overline{(m', n', x')} &\iff m = m', n = n' \quad \text{and} \\ \bar{x} = \varepsilon \bar{x}' &\quad \text{in } \mathcal{O}_E/\pi^n \mathcal{O}_E \quad \text{for some } \varepsilon \in 1 + \pi^l \mathcal{O}_E. \end{aligned}$$

(II) Suppose  $m = 0$ . We define

$$\begin{aligned} \overline{(m, n, x)} \sim \overline{(m', n', x')} &\iff m = m' = 0, \quad n = n', \\ \text{ord}_E(x \bmod \pi^n) = \text{ord}_E(x' \bmod \pi^n) &\quad \text{and} \\ \overline{1 - x} = \varepsilon \overline{1 - x'} &\quad \text{in } \mathcal{O}_E/\pi^n \mathcal{O}_E \quad \text{for some } \varepsilon \in \mathcal{O}_E^\times. \end{aligned}$$

Here, for  $s \leq 0$ , we define  $1 + \pi^s \mathcal{O}_E = \mathcal{O}_E^\times$ . We can prove that  $\sim$  is an equivalence relation. The following is our main theorem. We will prove this theorem in the next section.

**Theorem 3.5.** *There is a bijection  $\Phi$ :*

$$\begin{array}{ccc} \mathcal{M}_{f(T)}^E & \longrightarrow & Z/\sim \\ \Psi & & \Psi \\ [M(m, n, x)]_E & \longmapsto & \overline{(m, n, x)}, \end{array}$$

where  $\mathcal{M}_{f(T)}^E$  is defined by (1) in Section 2,  $Z$  is defined by (3) after Remark 3.4, and  $\sim$  is the equivalence relation of  $Z$  defined above.  $[M(m, n, x)]_E$  is the class of  $M(m, n, x)$  defined by Proposition 3.3 and  $\overline{(m, n, x)}$  is the class of  $(m, n, x)$ .

REMARK 3.6. After we submitted this paper, we learned from Sumida the existence of the thesis by Chase Franks where he independently classified the isomorphism classes of  $\Lambda$ -modules with  $\lambda = 3$ . He also gave an algorithm to determine the  $\Lambda$ -isomorphism classes for any separable  $f(T)$  which has arbitrary degree [2]. His method is essentially the same as our paper, but there are some differences which we will explain here.

1. We give in this paper an explicit method to compute  $m$  and  $n$  using the action of  $T - \alpha, T - \beta$  etc. (cf. Lemma 4.1).

2. Our inequalities about orders of  $p$ -adic numbers ((5), (6), (7) in Section 4) are obtained from a different point of view from Franks'. He did not solve completely his equations which are essentially equivalent to our inequalities, but we solved our inequalities completely in the case  $\lambda = 3$ .

3. We explicitly give a subgroup  $H \subset \mathbb{Z}_p^\times$  such that  $M(m, n, x) \cong M(m', n', x')$  if and only if  $m = m', n = n'$  and  $x/x' \in H$  ( $H$  depends on  $\text{ord}_p(x)$ ). Also, we use the higher Fitting ideals (cf. Section 5 and 6). This is a different argument from Franks'.

4. As an application, we apply our classification to the Iwasawa module associated to the cyclotomic  $\mathbb{Z}_p$ -extension of an imaginary quadratic field (cf. Section 6). On the other hand, Franks determined the isomorphism class of the Pontryagin dual of the  $p$ -Selmer group of elliptic curves over the cyclotomic  $\mathbb{Z}_p$ -extension for  $\lambda = 2$ .

5. Franks' method has some merits. He gave an algorithm to decide whether two  $\Lambda$ -modules are isomorphic or not. This algorithm is to check whether some matrices he defined belong to  $GL_\lambda(\mathbb{Z}_p)$ . This algorithm works for arbitrary  $\lambda$  and separable  $f(T)$ .

REMARK 3.7. When  $\overline{(m, n, x)} \sim \overline{(m', n', x')}$  and  $l+k \leq n$ , we have  $l = \text{ord}_E(x' - \pi^m) = \text{ord}_E(x - \pi^m)$ .

Sumida determined all elements of  $\mathcal{M}_{f(T)}$  for  $f(T) = (T - \alpha)(T - \beta)(T - \gamma)$  and  $\text{ord}_p(\alpha - \beta) = \text{ord}_p(\beta - \gamma) = \text{ord}_p(\gamma - \alpha) = 1$  ([12], Theorem 1). We can also obtain the same result from our Theorem 3.5.

**Corollary 3.8.** (Sumida) *Let  $f(T)$  be the same as (2) in Section 2 and  $E = \mathbb{Q}_p$ . We assume  $\text{ord}_p(\alpha - \beta) = \text{ord}_p(\beta - \gamma) = \text{ord}_p(\gamma - \alpha) = 1$ . Then we have  $\#\mathcal{M}_{f(T)} = 7$  and*

$$\mathcal{M}_{f(T)}^{\mathbb{Q}_p} = \{(0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 2, up), (1, 1, 0), (0, 1, 2)\},$$

where  $u = (\gamma - \alpha)/(\beta - \alpha)$  and  $(m, n, x)$  means  $[M(m, n, x)]_{\mathbb{Q}_p}$ .

Proof. We prove this corollary using Theorem 3.5. By fixing integers  $m$  and  $n$ , we put

$$Z(m, n) = \{\text{the equivalence class of } \overline{(m, n, x)} \text{ in } Z/\sim \mid \overline{(m, n, x)} \in Z\}.$$

Then, by definition, we have

$$Z/\sim = \coprod_m \coprod_n Z(m, n).$$

We determine all the elements of  $Z(m, n)$  for each  $m$  and  $n$  in order to determine all the elements of  $\mathcal{M}_{f(T)}$ .

We first assume  $[\overline{(m, n, x)}] \in Z/\sim$ , where  $[\overline{(m, n, x)}]$  is the equivalence class of  $\overline{(m, n, x)}$ . Then by Proposition 3.3,  $M(m, n, x)$  is a  $\Lambda_E$ -module satisfying (A), (B) and (C). By the inequality (A), we have  $0 \leq m \leq 1$ . Now we investigate  $\coprod_n Z(m, n)$  for  $m = 0, 1$ .

(I) Suppose  $m = 0$ .

In this case, by the inequalities (B) and (C), we have  $0 \leq n \leq 1$ . When  $n \geq 2$ , we get  $\text{ord}_p(x) = 0$  by (C). This contradicts to (B). When  $n = 0$ , we have  $\overline{(0, 0, x)} = \overline{(0, 0, 0)}$ . Therefore we get  $Z(0, 0) = \{[\overline{(0, 0, 0)}]\}$ . When  $n = 1$ , we have  $Z(0, 1) =$

$\{[\overline{(0, 1, 0)}], [\overline{(0, 1, 1)}], [\overline{(0, 1, 2)}]\}$ . By the definition of the equivalence relation, we have  $\overline{(0, 1, x)} \sim \overline{(0, 1, x')}$  if and only if

$$\text{ord}_p(x \bmod p) = \text{ord}_p(x' \bmod p) \quad \text{and} \quad \overline{1-x} = \varepsilon \overline{(1-x')}$$

for some  $\varepsilon \in \mathbb{Z}_p^\times$ . By the definition of  $\text{ord}_p(x \bmod p)$ , we have

$$\text{ord}_p(x \bmod p) = \begin{cases} 0 & x \notin p\mathbb{Z}_p, \\ \infty & x \in p\mathbb{Z}_p. \end{cases}$$

We investigate the case  $\text{ord}_p(x \bmod p) = 0$ . Suppose  $x = 1$ . Then we have

$$\begin{aligned} [\overline{(0, 1, 1)}] &= \{ \overline{(0, 1, x)} \mid \overline{(0, 1, 1)} \sim \overline{(0, 1, x)} \} \\ &= \{ \overline{(0, 1, x)} \mid \text{ord}_p(x) = 0, \overline{0} = \varepsilon \overline{(1-x)} \text{ for some } \varepsilon \in \mathbb{Z}_p^\times \} \\ &= \{ \overline{(0, 1, x)} \mid x \equiv 1 \pmod p \} \\ &= \{ \overline{(0, 1, 1)} \}. \end{aligned}$$

Suppose  $x = 2$ . Then we have

$$\begin{aligned} [\overline{(0, 1, 2)}] &= \{ \overline{(0, 1, x)} \mid \text{ord}_p(x) = 0, \overline{-1} = \varepsilon \overline{(1-x)} \text{ for some } \varepsilon \in \mathbb{Z}_p^\times \} \\ &= \{ \overline{(0, 1, x)} \mid x \not\equiv 0, 1 \} \\ &= \{ \overline{(0, 1, 2)}, \dots, \overline{(0, 1, p-1)} \}. \end{aligned}$$

Therefore we get  $Z(0, 1) = \{[\overline{(0, 1, 0)}], [\overline{(0, 1, 1)}], [\overline{(0, 1, 2)}]\}$ .

(II) Suppose  $m = 1$ .

By Remark 3.4 (ii), we have  $0 \leq n \leq 2$ . When  $n = 0$ , we have  $Z(1, 0) = \{[\overline{(1, 0, 0)}]\}$ . When  $n = 1$ , we have  $Z(1, 1) = \{[\overline{(1, 1, 0)}]\}$ . In fact, we suppose  $[\overline{(1, 1, x)}] \in Z(1, 1)$ . Then we have  $\overline{x} = 0$  by (C). When  $n = 2$ , we have  $Z(1, 2) = \{[\overline{(1, 2, up)}]\}$ . Indeed, we suppose  $[\overline{(1, 2, x)}] \in Z(1, 2)$ . For some  $v \in \mathbb{Z}_p^\times$ , we have

$$\begin{aligned} x &= \left(1 - \frac{vp^2}{\gamma - \alpha}\right) \frac{\gamma - \alpha}{\beta - \alpha} p \\ &\equiv \frac{\gamma - \alpha}{\beta - \alpha} p \pmod{p^2}, \end{aligned}$$

by (C). Thus,

$$Z/\sim = \{[\overline{(0, 0, 0)}], [\overline{(0, 1, 0)}], [\overline{(1, 0, 0)}], [\overline{(0, 1, 1)}], [\overline{(1, 2, up)}], [\overline{(1, 1, 0)}], [\overline{(0, 1, 2)}]\}.$$

We complete the proof by Theorem 3.5. □

**Corollary 3.9.** *Let  $f(T)$  be the same as (2) in Section 2 and  $E = \mathbb{Q}_p$ . We assume  $\text{ord}_p(\alpha - \beta) = \text{ord}_p(\beta - \gamma) = \text{ord}_p(\gamma - \alpha) = 2$ . Then we have  $\# \mathcal{M}_{f(T)} = p + 18$  and*

$$\mathcal{M}_{f(T)}^{\mathbb{Q}_p} = \left\{ \begin{array}{l} (0, 0, 0), (0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), \\ (0, 2, 2), (0, 2, p), (0, 2, p + 1), (1, 0, 0), (1, 1, 0), \\ (1, 1, 1), (1, 2, 0), (1, 2, p), \dots, (1, 2, (p - 1)p), (1, 3, up), \\ (2, 0, 0), (2, 1, 0), (2, 2, 0), (2, 3, up^2), (2, 4, up^2) \end{array} \right\},$$

where  $u = (\gamma - \alpha)/(\beta - \alpha)$  and  $(m, n, x)$  means  $[M(m, n, x)]_{\mathbb{Q}_p}$ .

Proof. We use the same notation as Corollary 3.8. By definition, we have

$$Z/\sim = \coprod_m \coprod_n Z(m, n).$$

We determine all the elements of  $Z(m, n)$  for each  $m$  and  $n$  in order to determine all the elements of  $\mathcal{M}_{f(T)}^{\mathbb{Q}_p}$ .

We first assume  $[(m, n, x)] \in Z/\sim$ , where  $[(m, n, x)]$  is the equivalence class of  $(m, n, x)$ . Then  $M(m, n, x)$  becomes a  $\Lambda_E$ -module satisfying (A), (B) and (C). By the inequality (A), we have  $0 \leq m \leq 2$ . Now we investigate  $\coprod_n Z(m, n)$  for each  $m$ .

(I) Suppose  $m = 0$ .

In this case, by the inequalities (B) and (C), we have  $0 \leq n \leq 2$ . In fact, if  $\text{ord}_p(x) \geq 1$ , we get  $n \leq 2$  by (C) and if  $\text{ord}_p(x) = 0$ , we get  $n \leq 2$  by (B). When  $n = 0$ , we have  $(0, 0, x) = (0, 0, 0)$  and  $Z(0, 0) = \{[(0, 0, 0)]\}$ . When  $n = 1$ , we have  $Z(0, 1) = \{[(0, 1, 0)], [(0, 1, 1)], [(0, 1, 2)]\}$  by the same method as Corollary 3.8. When  $n = 2$ , we have

$$(4) \quad Z(0, 2) = \{[(0, 2, 0)], [(0, 2, 1)], [(0, 2, 2)], [(0, 2, p)], [(0, 2, p + 1)]\}.$$

In fact, we suppose  $[(0, 2, x)] \in Z(0, 2)$ , then we have  $\bar{x} = \bar{0}$  or  $\text{ord}_p(\bar{x}) \leq 1$ . We first investigate the case  $\text{ord}_p(x) = 0$ . Then,  $(0, 2, x) \sim (0, 2, x')$  if and only if

$$0 = \text{ord}_p(x) = \text{ord}_p(x') \quad \text{and} \quad \overline{1 - x} = \varepsilon \overline{(1 - x')} \quad \text{for some} \quad \varepsilon \in \mathbb{Z}_p^\times.$$

By the same method as above, we get

$$\begin{aligned} \overline{[(0, 2, 1)]} &= \overline{\{(0, 2, 1)\}}, \\ \overline{[(0, 2, 2)]} &= \overline{\{(0, 2, x) \mid \bar{x} \neq \bar{0}, \bar{1}\}}, \\ (0, 2, p + 1)t &= \overline{\{(0, 2, x) \mid \text{ord}_p(x) = 0, \overline{-p} = \varepsilon \overline{(1 - x)} \text{ for some } \varepsilon \in \mathbb{Z}_p^\times\}} \\ &= \overline{\{(0, 2, 1 + x_1 p) \mid 1 \leq x_1 < p\}}. \end{aligned}$$

Next, we investigate the case  $\text{ord}_p(x) = 1$ , let  $x = p$ . Then we have

$$\begin{aligned} \overline{[(0, 2, p)]} &= \overline{[(0, 2, x)] \mid \text{ord}_p(x) = 1, 1 - p = \varepsilon(1 - x) \text{ for some } \varepsilon \in \mathbb{Z}_p^\times} \\ &= \overline{[(0, 2, x_1 p)] \mid 1 \leq x_1 < p}. \end{aligned}$$

Thus we get (4).

(II) Suppose  $m = 1$ .

By the inequalities (B) and (C), we have  $0 \leq n \leq 3$ . If  $\text{ord}_p(x) \leq 1$ , we have  $n \leq 3$  by (B). If  $\text{ord}_p(x) > 1$ , we have  $n \leq 2$  by (C). When  $n = 0$ , we have  $Z(1, 0) = \{\overline{[(1, 0, 0)]}\}$ . When  $n = 1$ , we have  $Z(1, 1) = \{\overline{[(1, 1, 0)]}, \overline{[(1, 1, 1)]}\}$ . We suppose  $\overline{[(1, 1, x)]} \in Z(1, 1)$ . Then we have  $\bar{x} = 0$  or  $\text{ord}_p(\bar{x}) = 0$ . We suppose  $\text{ord}_p(\bar{x}) = 0$ . We have  $l = \text{ord}_p(x - p) = 0$ . This is the case (b). By the definition of the equivalence relation,  $\overline{(1, 1, x)} \sim \overline{(1, 1, x')}$  if and only if

$$\bar{x} = \varepsilon \bar{x}' \quad \text{for some } \varepsilon \in \mathbb{Z}_p^\times.$$

Here we note that  $l = \text{ord}_E(x' - p) = 0$ . Then we have

$$\begin{aligned} \overline{[(1, 1, x)]} &= \overline{[(1, 1, x')] \mid \bar{x} = \varepsilon \bar{x}' \text{ for some } \varepsilon \in \mathbb{Z}_p^\times} \\ &= \overline{[(1, 1, x')] \mid x' \neq 0}. \end{aligned}$$

Therefore we get  $Z(1, 1) = \{\overline{[(1, 1, 0)]}, \overline{[(1, 1, 1)]}\}$ . When  $n = 2$ , we have  $Z(1, 2) = \{\overline{[(1, 2, x)]} \mid x = 0, p, 2p, \dots, (p-1)p\}$ . In fact, we suppose  $\overline{[(1, 2, x)]} \in Z(1, 2)$ . By the inequality (C), we have

$$2 \leq \text{ord}_p\{(\gamma - \alpha) - (\beta - \alpha)p^{-1}x\}.$$

If  $\text{ord}_p(x) = 0$ , the order of the right hand side is 1. This is contradiction. Thus we may assume  $1 \leq \text{ord}_p(x)$ . If  $\text{ord}_p(x) \geq 2$ , we get  $\overline{[(1, 2, x)]} = \overline{[(1, 2, 0)]}$ . We suppose  $\text{ord}_p(x) = 1$ . Then  $\overline{(1, 2, x)} \sim \overline{(1, 2, x')}$  if and only if

$$\bar{x} = \bar{x}'.$$

Here we note that this is the case (a) since  $l = \text{ord}_p(x' - p) \geq 1$ . For each  $x = \varepsilon p$ , where  $1 \leq \varepsilon < p$ , we have

$$\overline{[(1, 2, x)]} = \overline{[(1, 2, x)]}.$$

Thus we get  $Z(1, 2) = \{\overline{[(1, 2, x)]} \mid x = 0, p, 2p, \dots, (p-1)p\}$ . When  $n = 3$ , we have  $Z(1, 3) = \{\overline{[(1, 3, up)]}\}$ . In fact, we suppose  $\overline{[(1, 3, x)]} \in Z(1, 3)$ . By the same method as in the case  $n = 2$ , we get  $\text{ord}_p(x) = 1$  and  $\overline{(1, 3, x)} \sim \overline{(1, 3, up)}$  if and only if

$$\bar{x} = \varepsilon \bar{up} \quad \text{for some } \varepsilon \in 1 + p\mathbb{Z}_p.$$

Here we note that this is the case (b) since  $l = \text{ord}_E(up - p) = 1$ . Moreover, by (C), we have

$$x = \left(1 - \frac{vp^3}{\gamma - \alpha}\right) \frac{\gamma - \alpha}{\beta - \alpha} p \quad \text{for some } v \in \mathbb{Z}_p^\times.$$

Since  $1 - vp^3/(\gamma - \alpha) \in 1 + p\mathbb{Z}_p$ , we have

$$[(1, 3, up)] = \{(\overline{1, 3, x}) \mid \bar{x} = \varepsilon \bar{u} \bar{p} \text{ for some } \varepsilon \in 1 + p\mathbb{Z}_p\},$$

where  $u = (\gamma - \alpha)/(\beta - \alpha)$ . Thus we get  $Z(1, 3) = \{[(1, 3, up)]\}$ .

(III) Suppose  $m = 2$ .

By the same method as (I) and (II), we get  $Z(2, 0) = \{[(2, 0, 0)]\}$ ,  $Z(2, 1) = \{[(2, 1, 0)]\}$ ,  $Z(2, 2) = \{[(2, 2, 0)]\}$ ,  $Z(2, 3) = \{[(2, 3, up^2)]\}$  and  $Z(2, 4) = \{[(2, 4, up^2)]\}$ . Thus we complete the proof.  $\square$

**4. Proof of Theorem 3.5**

For any  $\xi \in \Lambda_E$ , we define a map  $\Pi_\xi = \Pi_\xi^M: M \rightarrow M$  by  $\Pi_\xi(y) = \xi y$ .

**Lemma 4.1.** *Let  $q = \#(\mathcal{O}_E/(\pi))$  and  $M = M(m, n, x)$ . Then we have*

$$\begin{aligned} \#(\text{Ker}(\Pi_{(T-\alpha)}^N)/\text{Im}(\Pi_{(T-\beta)}^N)) &= q^{\{\text{ord}_E(\alpha-\beta)-m\}}, \\ \#(\text{Ker}(\Pi_{(T-\gamma)}^M)/\text{Im}(\Pi_{(T-\alpha)(T-\beta)}^M)) &= q^{\{\text{ord}_E(\gamma-\alpha)+\text{ord}_E(\gamma-\beta)-n\}}, \end{aligned}$$

where  $N = \text{Im}(\Pi_{(T-\gamma)})$ .

*Proof.* We first compute  $\text{Ker}(\Pi_{(T-\gamma)})$ . For  $y \in M = M(m, n, x)$ , there exist  $\lambda_1, \lambda_2, \lambda_3 \in \mathcal{O}_E$  such that

$$\begin{aligned} y &= \lambda_1(1, 1, 1) + \lambda_2(0, \pi^m, x) + \lambda_3(0, 0, \pi^n) \\ &= (\lambda_1, \lambda_1 + \lambda_2\pi^m, \lambda_1 + \lambda_2x + \lambda_3\pi^n). \end{aligned}$$

Thus we have  $\Pi_{(T-\gamma)}(y) = ((\alpha - \gamma)\lambda_1, (\beta - \gamma)(\lambda_1 + \lambda_2\pi^m), 0)$ . If  $y \in \text{Ker}(\Pi_{(T-\gamma)})$ , we get  $\lambda_1 = 0$  and  $\lambda_1 + \lambda_2\pi^m = 0$ , since  $\alpha, \beta$  and  $\gamma$  are distinct elements of  $\mathcal{O}_E$ . Therefore  $y = (0, 0, \lambda_3\pi^n)$  and  $\text{Ker}(\Pi_{(T-\gamma)}) = (0, 0, \pi^n\mathcal{O}_E)$ . On the other hand, by  $y = (\lambda_1, \lambda_1 + \lambda_2\pi^m, \lambda_1 + \lambda_2x + \lambda_3\pi^n)$ , we have

$$\begin{aligned} \Pi_{(T-\alpha)(T-\beta)}(y) &= \Pi_{(T-\alpha)}((\alpha - \beta)\lambda_1, 0, (\gamma - \beta)(\lambda_1 + \lambda_2x + \lambda_3\pi^n)) \\ &= (0, 0, (\gamma - \alpha)(\gamma - \beta)(\lambda_1 + \lambda_2x + \lambda_3\pi^n)). \end{aligned}$$

Thus we have  $\text{Im}(\Pi_{(T-\alpha)(T-\beta)}) = (0, 0, \pi^{\text{ord}_E(\gamma-\alpha)+\text{ord}_E(\gamma-\beta)}\mathcal{O}_E)$  and

$$\begin{aligned} \#(\text{Ker}(\Pi_{(T-\gamma)})/\text{Im}(\Pi_{(T-\alpha)(T-\beta)})) &= \#(\pi^n\mathcal{O}_E/\pi^{\text{ord}_E(\gamma-\alpha)+\text{ord}_E(\gamma-\beta)}\mathcal{O}_E) \\ &= q^{\{\text{ord}_E(\gamma-\alpha)+\text{ord}_E(\gamma-\beta)-n\}}. \end{aligned}$$

Next we put  $N = \text{Im}(\Pi_{(T-\gamma)})$ . We have

$$\begin{aligned} \text{Ker}(\Pi_{(T-\alpha)}^N) &= (\pi^{\text{ord}_E(\alpha-\gamma)+m} \mathcal{O}_E, 0, 0), \\ \text{Im}(\Pi_{(T-\beta)}^N) &= (\pi^{\text{ord}_E(\alpha-\gamma)+\text{ord}_E(\alpha-\beta)} \mathcal{O}_E, 0, 0). \end{aligned}$$

Therefore we get

$$\#(\text{Ker}(\Pi_{(T-\alpha)}^N)/\text{Im}(\Pi_{(T-\beta)}^N)) = q^{\{\text{ord}_E(\alpha-\beta)-m\}}. \quad \square$$

**Corollary 4.2.** *Let  $[M]_E, [M']_E \in \mathcal{M}_{f(T)}^E$  and  $M = M(m, n, x)$ ,  $M' = M(m', n', x')$ . If  $[M]_E = [M']_E$ , then we have  $m = m'$  and  $n = n'$ .*

Proof. Since  $M \cong M'$ , we have  $N = \text{Im}(\Pi_{(T-\gamma)}^M) \cong \text{Im}(\Pi_{(T-\gamma)}^{M'}) = N'$  and therefore

$$\text{Ker}(\Pi_{(T-\alpha)}^N)/\text{Im}(\Pi_{(T-\beta)}^N) \cong \text{Ker}(\Pi_{(T-\alpha)}^{N'})/\text{Im}(\Pi_{(T-\beta)}^{N'}).$$

This implies  $m = m'$  by Lemma 4.1. We get  $n = n'$  by the same method. □

The isomorphism

$$\iota: \mathcal{E} = \Lambda_E/(T-\alpha) \oplus \Lambda_E/(T-\beta) \oplus \Lambda_E/(T-\gamma) \rightarrow \mathcal{O}_E^{\oplus 3}$$

defined in Section 2, induces an isomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_E} E \xrightarrow{\sim} E^{\oplus 3}$$

such that  $(f_1(T), f_2(T), f_3(T)) \otimes y \mapsto (f_1(\alpha)y, f_2(\beta)y, f_3(\gamma)y)$ .

**Proposition 4.3.** *Let  $[M]_E, [M']_E \in \mathcal{M}_{f(T)}^E$ ,  $M = M(m, n, x)$ ,  $M' = M(m, n, x')$  and  $g: M \rightarrow M'$  be a  $\Lambda_E$ -isomorphism. We define an  $E$ -linear map  $F_A$  by the following commutative diagram*

$$\begin{array}{ccc} M & \xrightarrow{g} & M' \\ \varphi \otimes 1 \downarrow & & \downarrow \varphi' \otimes 1 \\ \mathcal{E} \otimes_{\mathcal{O}_E} E & \longrightarrow & \mathcal{E} \otimes_{\mathcal{O}_E} E \\ \iota \otimes 1 \downarrow & & \downarrow \iota \otimes 1 \\ E^{\oplus 3} & \xrightarrow{F_A} & E^{\oplus 3}. \end{array}$$

In the diagram,  $\varphi$  and  $\varphi'$  are natural inclusions (Section 2). When we take the standard basis of  $E^{\oplus 3}$ ,  $F_A$  corresponds to a diagonal matrix

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \text{ for some } a_1, a_2, a_3 \in \mathcal{O}_E^\times.$$

Proof. Consider the map  $\Pi_T: M \rightarrow M$ . Then  $\Pi_T$  induces a map  $F_B: E^{\oplus 3} \rightarrow E^{\oplus 3}$  and the following commutative diagram

$$\begin{array}{ccc}
 M & \xrightarrow{\Pi_T} & M \\
 \varphi \otimes 1 \downarrow & & \downarrow \varphi \otimes 1 \\
 \mathcal{E} \otimes_{\mathcal{O}_E} E & \longrightarrow & \mathcal{E} \otimes_{\mathcal{O}_E} E \\
 \iota \otimes 1 \downarrow & & \downarrow \iota \otimes 1 \\
 E^{\oplus 3} & \xrightarrow{F_B} & E^{\oplus 3}.
 \end{array}$$

Thus we get

$$(\ddagger) \quad F_B \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(x) = (\iota \otimes 1) \circ (\varphi \otimes 1)(Tx)$$

for  $x \in M$ . Let  $A$  be the matrix corresponding to  $F_A$ . By the diagram, we get

$$(\#) \quad F_A \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(Tx) = (\iota \otimes 1) \circ (\varphi' \otimes 1)(g(Tx)).$$

By  $(\ddagger)$  and the diagrams, the left hand side of  $(\#)$  is

$$F_A \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(Tx) = F_A \circ F_B \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(x).$$

The right hand side of  $(\#)$  is

$$\begin{aligned}
 (\iota \otimes 1) \circ (\varphi' \otimes 1)(Tg(x)) &= F_B \circ (\iota \otimes 1) \circ (\varphi' \otimes 1)(g(x)) \\
 &= F_B \circ F_A \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(x).
 \end{aligned}$$

Since this holds for any  $x \in M$ , we have  $F_A \circ F_B = F_B \circ F_A$ . Taking the standard basis of  $E^{\oplus 3}$ ,  $F_B$  corresponds to the matrix

$$B = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

Therefore we have

$$A \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} A.$$

Since  $\alpha, \beta$  and  $\gamma$  are distinct elements and we get

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \quad \text{with} \quad a_1, a_2, a_3 \in E.$$

Because  $g((1, 1, 1)) = (a_1, a_2, a_3) \in M'$ , we get  $a_1, a_2$  and  $a_3 \in \mathcal{O}_E$ . Furthermore, by the same argument for  $g^{-1}$ , we have  $a_1^{-1}, a_2^{-1}, a_3^{-1} \in \mathcal{O}_E$ . So we get  $a_1, a_2, a_3 \in \mathcal{O}_E^\times$ .  $\square$

By the commutativity of the diagram, we obtain the following corollary.

**Corollary 4.4.** *Suppose that  $M, F_A, \iota, \varphi$  and  $\varphi'$  are the same as Proposition 4.3. Then we have*

$$\langle (F_A \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(M)) \rangle_{\mathcal{O}_E} = \langle (\iota \otimes 1) \circ (\varphi' \otimes 1) \circ g(M) \rangle_{\mathcal{O}_E}.$$

**Proposition 4.5.** *Let  $[M]_E, [M']_E \in \mathcal{M}_{f(T)}^E$  and  $M = M(m, n, x), M' = M(m, n, x')$ . Then the following two statements are equivalent:*

- (i) *We have  $M \cong M'$  as  $\Lambda_E$ -modules,*
- (ii) *There exist  $a_1, a_2, a_3 \in \mathcal{O}_E^\times$  satisfying*

- (5)  $\text{ord}_E(a_2 - a_1) \geq m,$
- (6)  $\text{ord}_E(a_3x - a_2x') \geq n,$
- (7)  $\text{ord}_E\{a_3 - a_1 - (a_2 - a_1)\pi^{-m}x'\} \geq n.$

*Proof.* We first prove (i) implies (ii). If  $M$  is isomorphic to  $M'$  as a  $\Lambda_E$ -module, there exists a  $\Lambda_E$ -isomorphism  $g: M \xrightarrow{\sim} M'$ . By Proposition 4.3, there exists a diagonal matrix  $A$  which can be written as

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \text{ such that } a_1, a_2, a_3 \in \mathcal{O}_E^\times,$$

which corresponds to  $g$ . We have

$$\begin{aligned} F_A \circ (\iota \otimes 1) \circ (\varphi \otimes 1)(M) &= F_A(M(m, n, x)) \\ &= \langle (a_1, a_2, a_3), (0, a_2\pi^m, a_3x), (0, 0, a_3\pi^n) \rangle_{\mathcal{O}_E} \end{aligned}$$

and

$$\begin{aligned} (\iota \otimes 1) \circ (\varphi' \otimes 1) \circ g(M) &= (\iota \otimes 1) \circ (\varphi' \otimes 1)(M') \\ &= \langle (1, 1, 1), (0, \pi^m, x'), (0, 0, \pi^n) \rangle_{\mathcal{O}_E}. \end{aligned}$$

By Corollary 4.4, we get

$$\langle (a_1, a_2, a_3), (0, a_2\pi^m, a_3x), (0, 0, a_3\pi^n) \rangle_{\mathcal{O}_E} = \langle (1, 1, 1), (0, \pi^m, x'), (0, 0, \pi^n) \rangle_{\mathcal{O}_E}.$$

Because the left hand side is contained in the right hand side, we have

$$\begin{aligned} (a_1, a_2, a_3) &= a_1(1, 1, 1) + (a_2 - a_1)\pi^{-m}(0, \pi^m, x') \\ &\quad + \{a_3 - a_1 - (a_2 - a_1)\pi^{-m}x'\}\pi^{-n}(0, 0, \pi^n), \\ (0, a_2\pi^m, a_3x) &= a_2(0, \pi^m, x') + (a_3x - a_2x')\pi^{-n}(0, 0, \pi^n). \end{aligned}$$

Since these coefficients should belong to  $\mathcal{O}_E$ , we have (5), (6), (7). It is easy to prove that (ii) implies (i). □

We can simplify the inequalities (5), (6), (7). The following is easy to see.

**Lemma 4.6.** *The followings are equivalent:*

- (i) *There exist  $a_1, a_2, a_3 \in \mathcal{O}_E^\times$  satisfying (5), (6), (7),*
- (ii) *There exist  $a_1, a_2 \in \mathcal{O}_E^\times$  satisfying*

- (8)  $\text{ord}_E(a_2 - a_1) \geq m,$
- (9)  $\text{ord}_E(x - a_2x') \geq n,$
- (10)  $\text{ord}_E\{1 - a_1 - (a_2 - a_1)\pi^{-m}x'\} \geq n.$

**Corollary 4.7.** *Let  $[M]_E, [M']_E \in \mathcal{M}_{f(T)}^E$  and  $M = M(m, n, x), M' = M(m, n, x')$ . Assume  $\text{ord}_E(x) < n$ . If  $[M]_E = [M']_E$ , we have  $\text{ord}_E(x) = \text{ord}_E(x')$ .*

*Proof.* If  $\text{ord}_E(x) < \text{ord}_E(x')$ , by the inequality (6), we have  $n \leq \text{ord}_E(a_3x - a_2x') = \text{ord}_E(x)$ . This contradicts to the assumption  $\text{ord}_E(x) < n$ . If we assume  $\text{ord}_E(x) > \text{ord}_E(x')$ , we would get the same contradiction. Therefore we obtain  $\text{ord}_E(x) = \text{ord}_E(x')$ . □

To prove Theorem 3.5, we prepare a lemma and some propositions.

**Proposition 4.8.** *The following two statements are equivalent:*

- (i) *We have  $M(m, n, x) \cong M(m, n, 0)$  as  $\Lambda_E$ -modules,*
- (ii) *We have  $\overline{(m, n, x)} \sim \overline{(m, n, 0)}$ , where  $\sim$  is the equivalence relation defined in Section 3.*

*Proof.* We show that (i) implies (ii). If  $\text{ord}_E(x) < n$ , we have  $\text{ord}_E(x) = \text{ord}_E(0)$  by Corollary 4.7, which is a contradiction. So we have  $\text{ord}_E(x) \geq n$  and  $M(m, n, x) = M(m, n, 0)$ . Then  $\overline{(m, n, x)} = \overline{(m, n, 0)}$  by Remark 3.4 (i). □

Let  $M = M(m, n, x)$  and  $M' = M(m, n, x')$ . Now we suppose that  $x' \neq 0$  and the existence of  $a_1, a_2 \in \mathcal{O}_E^\times$  satisfying (8), (9) and (10). By Proposition 4.5 and Lemma 4.6,

$M$  is isomorphic to  $M'$ . From the inequalities (8) and (9), there are  $s, v \in \mathcal{O}_E$  such that  $a_2 - a_1 = \pi^m s$  and  $x - a_2 x' = \pi^n v$ . Thus we have

$$(11) \quad a_1 = \frac{x}{x'} - \frac{\pi^n}{x'} v - \pi^m s,$$

$$(12) \quad a_2 = \pi^m s + a_1 = \frac{x}{x'} - \frac{\pi^n}{x'} v.$$

By the inequality (10), we get

$$(13) \quad x'(x' - \pi^m)s - \pi^n v + \pi^n x'w = x' - x$$

for some  $w \in \mathcal{O}_E$ .

**Lemma 4.9.** *Let  $m, n \neq 0$  and  $\text{ord}_E(x) < n$ . The following two statements are equivalent:*

- (i) *There exist  $a_1, a_2 \in \mathcal{O}_E^\times$  satisfying (8), (9), (10),*
- (ii) *We have  $\text{ord}_E(x) = \text{ord}_E(x')$  and there exist  $s, v, w \in \mathcal{O}_E$  satisfying (13).*

*Proof.* We have already proved that (i) implies (ii). We will prove that (ii) implies (i). We put  $a_1$  and  $a_2$  by the equalities (11) and (12). Since  $m, n \neq 0$  and  $\text{ord}_E(x) = \text{ord}_E(x') < n$ , we have  $a_1, a_2 \in \mathcal{O}_E^\times$ . Then we have

$$a_2 - a_1 = \pi^m s, \quad x - a_2 x' = \pi^n v$$

and

$$1 - a_1 - (a_2 - a_1)\pi^{-m}x' = \pi^n w.$$

Therefore we get (8), (9) and (10). □

**Proposition 4.10.** *Let  $m, n \neq 0$  and  $\text{ord}_E(x) < n$ . Then the followings are equivalent:*

- (i) *We have  $\overline{M(m, n, x)} \cong \overline{M(m, n, x')}$  as  $\Lambda_E$ -modules,*
- (ii) *We have  $\overline{(m, n, x)} \sim \overline{(m, n, x')}$ .*

*Proof.* We first suppose that  $M(m, n, x)$  is isomorphic to  $M(m, n, x')$  as a  $\Lambda_E$ -module. Let  $k = \text{ord}_E(x)$  and  $l = \text{ord}_E(x' - \pi^m)$ . By Lemma 4.9, we have  $\text{ord}_E(x) = \text{ord}_E(x') = k$  and there exist  $s, v, w \in \mathcal{O}_E$  such that

$$x'(x' - \pi^m)s - \pi^n v + \pi^n x'w = x' - x.$$

We put  $\varepsilon = xx'^{-1} \in \mathcal{O}_E^\times$ . Dividing the above equality by  $x'$ , we have

$$(x' - \pi^m)s - \frac{\pi^n}{x'}v + \pi^n w = 1 - \varepsilon.$$

Thus we have

$$\begin{aligned} \text{ord}_E(1 - \varepsilon) &\geq \min \left\{ \text{ord}_E((x' - \pi^m)s), \text{ord}_E\left(-\frac{\pi^n}{x'}v\right), \text{ord}_E(\pi^n w) \right\} \\ &\geq \min\{l, n - k, n\} = \min\{l, n - k\}. \end{aligned}$$

In the case (a)  $l \geq n - k$ , we have  $\text{ord}_E(1 - \varepsilon) \geq n - k$ . Thus we get  $\bar{x} = \overline{\varepsilon x'} = \overline{x'}$  in  $\mathcal{O}_E/\pi^n\mathcal{O}_E$ . Therefore we have  $\overline{(m, n, x)} \sim \overline{(m, n, x')}$ . In the case (b)  $l < n - k$ , we have  $\text{ord}_E(1 - \varepsilon) \geq l$  and  $\bar{x} = \overline{\varepsilon x'}$  in  $\mathcal{O}_E/\pi^n\mathcal{O}_E$ . Therefore we get  $\overline{(m, n, x)} \sim \overline{(m, n, x')}$ . Conversely we assume that  $\overline{(m, n, x)} \sim \overline{(m, n, x')}$ . In the case (a), we have  $\bar{x} = \overline{x'}$  in  $\mathcal{O}_E/\pi^n\mathcal{O}_E$  and  $(x' - x)/\pi^n \in \mathcal{O}_E$ . Put  $s = w = 0$  and  $v = (x - x')/\pi^n \in \mathcal{O}_E$ . Then we get

$$x'(x' - \pi^m)s - \pi^n v + \pi^n x'w = x' - x.$$

By Lemma 4.9,  $M$  and  $M'$  are isomorphic as  $\Lambda_E$ -modules. In the case (b), We have  $\bar{x} = \overline{\varepsilon x'}$  in  $\mathcal{O}_E/\pi^n\mathcal{O}_E$  for some  $\varepsilon \in 1 + \pi^l\mathcal{O}_E$ . Since  $\text{ord}_E(1 - \varepsilon) \geq l$ , we have  $(1 - \varepsilon)/(x' - \pi^m) \in \mathcal{O}_E$ . Put  $v = w = 0$  and  $s = (1 - \varepsilon)/(x' - \pi^m) \in \mathcal{O}_E$ . Then we get

$$x'(x' - \pi^m)s - \pi^n v + \pi^n x'w = x' - \varepsilon x'.$$

By Lemma 4.9, we get  $M(m, n, x) = M(m, n, \varepsilon x') \cong M(m, n, x')$ . □

The following propositions treat the case  $m = 0$  and the case  $n = 0$ .

**Proposition 4.11.** *Suppose  $m = 0, n \neq 0$  and  $\text{ord}_E(x) < n$ . Then the followings are equivalent:*

- (i) *We have  $M(0, n, x) \cong M(0, n, x')$  as  $\Lambda_E$ -modules,*
- (ii) *We have  $\overline{(0, n, x)} \sim \overline{(0, n, x')}$ .*

*Proof.* Suppose that  $M(0, n, x)$  is isomorphic to  $M(0, n, x')$  as a  $\Lambda_E$ -module. By Proposition 4.5 and Lemma 4.6, there exist  $a_1, a_2 \in \mathcal{O}_E^\times$  satisfying (9) and (10). By the inequality (9), we have  $\bar{x} = a_2\overline{x'}$ . By the inequality (10), we have  $\overline{1 - a_2x'} = \overline{a_1(1 - x')}$ . Therefore we get

$$\text{ord}_E(x) = \text{ord}_E(x') \quad \text{and} \quad \overline{1 - x} = a_1\overline{(1 - x')}.$$

Thus we get  $\overline{(0, n, x)} \sim \overline{(0, n, x')}$ . Conversely we suppose that  $\overline{(0, n, x)} \sim \overline{(0, n, x')}$ . There exists  $a_1 \in \mathcal{O}_E^\times$  such that  $\overline{1 - x} = a_1\overline{(1 - x')}$ . Put  $a_2 = x/x'$ . Then we have (9) and (10). Indeed, we have  $1 - a_1 - (a_2 - a_1)\pi^{-m}\overline{x'} = 1 - a_1 - (a_2 - a_1)\overline{x'} = \overline{0}$ . By Proposition 4.5 and Lemma 4.6,  $M(0, n, x)$  and  $M(0, n, x')$  are isomorphic as  $\Lambda_E$ -modules. □

**Proposition 4.12.** *Suppose  $n = 0$ . The followings are equivalent:*

- (i) We have  $M(m, 0, x) \cong M(m, 0, x')$  as  $\Lambda_E$ -modules,
- (ii) We have  $\overline{(m, 0, x)} \sim \overline{(m', 0, x')}$ .

Proof. By Remark 3.4 (i), we have  $M(m, 0, x) = M(m, 0, x') = M(m, 0, 0)$  and  $\overline{(m, 0, x)} = \overline{(m, 0, x')} = \overline{(m, 0, 0)}$ . □

Now we can prove Theorem 3.5.

Proof of Theorem 3.5. For  $[M(m, n, x)]_E \in \mathcal{M}_{f(T)}^E$ , we may assume  $x = 0$  or  $\text{ord}_E(x) < n$  by Remark 3.4 (i). At first,  $\Phi$  is well-defined by Corollary 4.2 and Propositions 4.8, 4.10, 4.11 and 4.12. The surjectivity follows from Proposition 3.3 and Remark 3.4. On the other hand,  $\Phi$  is injective by Propositions 4.8, 4.10, 4.11 and 4.12. □

### 5. Complementary properties

In this section, we show some propositions in order to determine the Iwasawa module associated to an imaginary quadratic field in the next section.

For a non-negative integer  $n$ , we put  $\omega_n = \omega_n(T) = (1 + T)^{p^n} - 1$ .

**Proposition 5.1.** *For a distinguished polynomial  $f(T) \in \mathbb{Z}_p[T]$ , let  $E$  be the splitting field of  $f(T)$  over  $\mathbb{Q}_p$ . Then the natural map*

$$\Psi: \mathcal{M}_{f(T)}^{\mathbb{Q}_p} \rightarrow \mathcal{M}_{f(T)}^E \quad ([M] \mapsto [M \otimes_{\Lambda} \Lambda_E]_E)$$

is injective.

Proof. We suppose that  $M \otimes_{\Lambda} \Lambda_E \cong M' \otimes_{\Lambda} \Lambda_E$  for  $[M]$  and  $[M'] \in \mathcal{M}_{f(T)}^{\mathbb{Q}_p}$ . Since  $M \otimes_{\Lambda} \Lambda_E \cong M^n$  as  $\Lambda$ -modules, we get  $M^n \cong M'^n$  as  $\Lambda$ -modules, where  $n$  is the degree of the extension  $E/\mathbb{Q}_p$ .

We assume that  $M \not\cong M'$  as  $\Lambda$ -modules. Since  $M$  is a finitely generated  $\Lambda$ -module,  $M$  is a profinite module and we have  $M = \varprojlim M/\mathfrak{m}^n M$  where  $\mathfrak{m} = (\pi, T)$ . Since  $M \not\cong M'$ , there exists a positive integer  $l$  such that  $M/\mathfrak{m}^l M \not\cong M'/\mathfrak{m}^l M'$  ([11], Proposition 5). Because both  $M/\mathfrak{m}^l M$  and  $M'/\mathfrak{m}^l M'$  are of finite length, we can decompose these modules into indecomposable modules

$$M/\mathfrak{m}^l M = \bigoplus_i N_i^{\oplus e_i}, \quad M'/\mathfrak{m}^l M' = \bigoplus_i N_i^{\oplus e'_i},$$

where  $N_i$ 's are indecomposable modules,  $N_i \not\cong N_j$  ( $i \neq j$ ) and  $e_i, e'_i$  are non-negative integers. By Krull–Remak–Schmidt’s theorem, there exists  $i$  such that  $e_i \neq e'_i$ . Furthermore we have

$$(M/\mathfrak{m}^l M)^n = \bigoplus_i N_i^{\oplus ne_i}, \quad (M'/\mathfrak{m}^l M')^n = \bigoplus_i N_i^{\oplus ne'_i}.$$

Thus we get  $ne_i \neq ne'_i$  for some  $i$ . By Krull–Remak–Schmidt’s theorem, we have  $(M/\mathfrak{m}^l M)^n \not\cong (M'/\mathfrak{m}^l M')^n$ . This implies  $M^n \not\cong M'^n$ . This contradicts to our assumption.  $\square$

Let  $f(T) \in \mathbb{Z}_p[T]$  be a distinguished polynomial,  $E$  the splitting field of  $f(T)$  and we put

$$f(T) = (T - \alpha)(T - \beta)(T - \gamma),$$

where  $\alpha, \beta$  and  $\gamma \in \pi\mathcal{O}_E$  as in Section 2.

**Proposition 5.2.** *Let  $E$  and  $f(T)$  be as above and  $[M]_E \in \mathcal{M}_{f(T)}^E$ . If  $M$  is a cyclic  $\Lambda_E$ -module, then we have*

$$M \cong M(\text{ord}_E(\beta - \alpha), \text{ord}_E(\gamma - \alpha) + \text{ord}_E(\gamma - \beta), u\pi^{\text{ord}_E(\beta - \alpha)})$$

as  $\Lambda_E$ -modules, where  $u = (\gamma - \alpha)/(\beta - \alpha)$ .

*Proof.* Let  $M \cong M(m, n, x) \subset \mathcal{E}$ . Suppose that  $M$  is cyclic and put

$$M = \langle (a, b, c) \rangle_{\Lambda_E} \subset \mathcal{E}$$

for some  $a, b, c \in \mathcal{O}_E$ . Because  $(1, 1, 1) \in \langle (a, b, c) \rangle_{\Lambda_E}$ , we have  $(1, 1, 1) = h(T)(a, b, c) = (h(\alpha)a, h(\beta)b, h(\gamma)c)$  for some  $h(T) \in \Lambda_E$ . Therefore we get  $a, b, c \in \mathcal{O}_E^\times$ . Since  $(0, \pi^m, x)$  and  $(0, 0, \pi^n) \in \langle (a, b, c) \rangle_{\Lambda_E}$ , we have

$$\begin{aligned} (0, \pi^m, x) &= q(T)(a, b, c) = (q(\alpha)a, q(\beta)b, q(\gamma)c), \\ (0, 0, \pi^n) &= r(T)(a, b, c) = (r(\alpha)a, r(\beta)b, r(\gamma)c) \end{aligned}$$

for some  $q(T)$  and  $r(T) \in \Lambda_E$ . Since  $(T - \alpha) \mid q(T)$  and  $(T - \alpha)(T - \beta) \mid r(T)$ , we get  $m = \text{ord}_E(q(\beta)) \geq \text{ord}_E(\beta - \alpha)$  and  $n = \text{ord}_E(r(\gamma)) \geq \text{ord}_E(\gamma - \alpha) + \text{ord}_E(\gamma - \beta)$ . On the other hand, by Proposition 3.3 and Remark 3.4, we have  $m \leq \text{ord}_E(\beta - \alpha)$  and  $n \leq \text{ord}_E(\gamma - \alpha) + \text{ord}_E(\gamma - \beta)$ . Therefore we obtain  $m = \text{ord}_E(\beta - \alpha)$  and  $n = \text{ord}_E(\gamma - \alpha) + \text{ord}_E(\gamma - \beta)$ . Furthermore,

$$\begin{aligned} (T - \alpha)(1, 1, 1) &= (0, \beta - \alpha, \gamma - \alpha) \\ &= (\beta - \alpha)\pi^{-m}(0, \pi^m, x) + \{\gamma - \alpha - (\beta - \alpha)\pi^{-m}x\}\pi^{-n}(0, 0, \pi^n). \end{aligned}$$

Because  $\text{ord}_E\{\gamma - \alpha - (\beta - \alpha)\pi^{-m}x\} \geq n$ , we have  $x = (\gamma - \alpha)/(\beta - \alpha)\pi^m(1 - \pi^n v/(\gamma - \alpha))$  for some  $v \in \mathcal{O}_E$ . By Remark 3.4 (i), we get

$$M(m, n, x) = M(\text{ord}_E(\beta - \alpha), \text{ord}_E(\gamma - \alpha) + \text{ord}_E(\gamma - \beta), u\pi^{\text{ord}_E(\beta - \alpha)}). \quad \square$$

**Proposition 5.3.** *Let  $f(T)$  be as above. Assume  $\text{ord}_E(\alpha - \beta) = \text{ord}_E(\beta - \gamma) = \text{ord}_E(\gamma - \alpha) = 1$  and  $\text{ord}_E(\alpha) \geq \text{ord}_E(\beta) \geq \text{ord}_E(\gamma)$ . Then, we have*

$$\mathcal{M}_{f(T)}^E = \{(0, 0, 0), (0, 1, 0), (1, 0, 0), (0, 1, 1), (1, 2, u\pi), (1, 1, 0), (0, 1, 2)\},$$

where  $u = (\gamma - \alpha)/(\beta - \alpha)$  and  $(m, n, x)$  means  $[M(m, n, x)]_E$ . The following is the table of the structure of  $\mathcal{O}_E$ -modules  $M/\omega_0 M$  for  $\Lambda_E$ -modules  $M$ .

$M$	$M/\omega_0 M$
$M(0, 0, 0)$	$\mathcal{O}_E/(\alpha) \oplus \mathcal{O}_E/(\beta) \oplus \mathcal{O}_E/(\gamma)$
$M(0, 1, 0)$	$\mathcal{O}_E/(\beta) \oplus \mathcal{O}_E/(\alpha\gamma)$
$M(0, 1, 1)$	$\mathcal{O}_E/(\alpha) \oplus \mathcal{O}_E/(\beta\gamma)$
$M(0, 1, 2)$	$\mathcal{O}_E/(\beta) \oplus \mathcal{O}_E/(\alpha\gamma)$
$M(1, 0, 0)$	$\mathcal{O}_E/(\gamma) \oplus \mathcal{O}_E/(\alpha\beta)$
$M(1, 1, 0)$	$\mathcal{O}_E/(\gamma) \oplus \mathcal{O}_E/(\alpha\beta)$
$M(1, 2, u\pi)$	$\mathcal{O}_E/(\alpha\beta\gamma)$

Proof. The former is Corollary 3.8. We show the latter. Let  $[M]_E \in \mathcal{M}_{f(T)}^E$ . There exist  $m, n$  and  $x$  such that

$$M = \langle (1, 1, 1), (0, \pi^m, x), (0, 0, \pi^n) \rangle_{\mathcal{O}_E}$$

and we have

$$\omega_0 M = \langle (\alpha, \beta, \gamma), (0, \beta\pi^m, \gamma x), (0, 0, \gamma\pi^n) \rangle_{\mathcal{O}_E}.$$

Since  $\mathcal{O}_E$  is a principal ideal domain, we can use the structure theorem over the principal ideal domain. We consider the map  $\Pi_{\omega_0}: M \rightarrow M$  and take  $(1, 1, 1), (0, \pi^m, x)$  and  $(0, 0, \pi^n)$  as a basis of  $M$ . Then we have

$$(14) \quad \begin{aligned} T(1, 1, 1) &= \alpha(1, 1, 1) + (\beta - \alpha)\pi^{-m}(0, \pi^m, x) \\ &\quad + \{\gamma - \alpha - (\beta - \alpha)\pi^{-m}x\}\pi^{-n}(0, 0, \pi^n), \end{aligned}$$

$$(15) \quad \begin{aligned} T(0, \pi^m, x) &= (0, \beta\pi^m, \gamma x) \\ &= \beta(0, \pi^m, x) + (\gamma - \beta)x\pi^{-n}(0, 0, \pi^n). \end{aligned}$$

By the equalities (14) and (15), the matrix corresponding to  $\Pi_{\omega_0}$  is

$$\begin{pmatrix} \alpha & 0 & 0 \\ (\beta - \alpha)\pi^{-m} & \beta & 0 \\ \{\gamma - \alpha - (\beta - \alpha)\pi^{-m}x\}\pi^{-n} & (\gamma - \beta)x\pi^{-n} & \gamma \end{pmatrix}.$$

In order to verify the table, we have only to transform this matrix by elementary row

and column operations. For example, the case  $M = M(0, 1, 0)$ , we get the matrix

$$\begin{pmatrix} \alpha & 0 & 0 \\ \beta - \alpha & \beta & 0 \\ (\gamma - \alpha)\pi^{-1} & 0 & \gamma \end{pmatrix}.$$

By the elementary row and column operations, we have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \alpha\gamma \end{pmatrix}.$$

So we get  $M/\omega_0M \cong \mathcal{O}_E/(\beta) \oplus \mathcal{O}_E/(\alpha\gamma)$ . The rest of the table can be checked by the same method. □

**Proposition 5.4.** *Let  $f(T) = (T - \alpha)g(T)$ , where  $\alpha \in p\mathbb{Z}_p$  and  $g(T) \in \mathbb{Z}_p[T]$  is a distinguished irreducible polynomial of degree 2. Let  $E$  be the splitting field of  $g(T)$  over  $\mathbb{Q}_p$ . If  $[M(m, n, x)]_E \in \text{Image}(\Psi: \mathcal{M}_{f(T)}^{\mathbb{Q}_p} \rightarrow \mathcal{M}_{f(T)}^E([M] \mapsto [M \otimes_{\Lambda} \Lambda_E]_E))$ , we have*

$$\text{ord}_E(x) = m.$$

*Proof.* Let  $[M] \in \mathcal{M}_{f(T)}^{\mathbb{Q}_p}$  and  $M \otimes_{\Lambda} \Lambda_E \cong M(m, n, x) \subset \mathcal{E}$ . There is a natural injective map

$$M \rightarrow \Lambda/(f(T)) \rightarrow \Lambda/(T - \alpha) \oplus \Lambda/(g(T))$$

([13], Lemma 13.8). By this injective map, we have

$$M = \langle (a_1, b_1T + c_1), (a_2, b_2T + c_2), (a_3, b_3T + c_3) \rangle_{\mathbb{Z}_p} \subset \Lambda/(T - \alpha) \oplus \Lambda/(g(T))$$

for some  $a_i, b_i$  and  $c_i \in \mathbb{Z}_p$ . Because  $M \otimes_{\Lambda} \Lambda_E = \langle (a_1, b_1T + c_1), (a_2, b_2T + c_2), (a_3, b_3T + c_3) \rangle_{\mathcal{O}_E}$ , by the same argument before Lemma 3.1, we can write

$$M \otimes_{\Lambda} \Lambda_E = \langle (a'_1, b'_1T + c'_1), (0, b'_2T + c'_2), (0, c'_3) \rangle_{\mathcal{O}_E}$$

for some  $a'_i, b'_i$  and  $c'_i \in \mathbb{Z}_p$ . Furthermore there is an injective map ([13], Lemma 13.8)

$$\Lambda_E/(T - \alpha) \oplus \Lambda_E/(g(T)) \rightarrow \mathcal{E}, \quad (s(t), u(t)) \mapsto (s(\alpha), u(\beta), u(\gamma)),$$

where  $\beta$  and  $\gamma$  are the roots of  $g(T)$  in  $E$ . By this map,  $M \otimes_{\Lambda} \Lambda_E$  is isomorphic to the module

$$M' = \langle (a'_1, b'_1\beta + c'_1, b'_1\gamma + c'_1), (0, b'_2\beta + c'_2, b'_2\gamma + c'_2), (0, c'_3, c'_3) \rangle_{\mathcal{O}_E} \subset \mathcal{E}.$$

Since  $\beta$  and  $\gamma$  are conjugate, we have  $\text{ord}_E(b'_1\beta + c'_1) = \text{ord}_E(b'_1\gamma + c'_1)$  and  $\text{ord}_E(b'_2\beta + c'_2) = \text{ord}_E(b'_2\gamma + c'_2)$ . By the same arguments after Lemma 3.2, we get

$$M' \cong \langle (1, 1, 1), (0, \pi^m, x), (0, 0, \pi^n) \rangle_{\mathcal{O}_E}$$

for some  $m, n, x$  which satisfy  $m = \text{ord}_E(x)$ . Indeed, we may assume  $\text{ord}_E(b'_2\beta + c'_2) \leq \text{ord}_E(c'_3)$ . By Lemma 3.2, we have

$$M' \cong \langle (1, b'_1\beta + c'_1, b'_1\gamma + c'_1), (0, b'_2\beta + c'_2, b'_2\gamma + c'_2), (0, c'_3, c'_3) \rangle_{\mathcal{O}_E}.$$

In the case  $\text{ord}_E(b'_1\beta + c'_1) \leq \text{ord}_E(b'_2\beta + c'_2)$ , we have

$$M' \cong \left\langle (1, 1, b'_1\gamma + c'_1), \left(0, \frac{b'_2\beta + c'_2}{b'_1\beta + c'_1}, b'_2\gamma + c'_2\right), \left(0, \frac{c'_3}{b'_1\beta + c'_1}, c'_3\right) \right\rangle_{\mathcal{O}_E}.$$

Since  $\text{ord}_E(b'_1\gamma + c'_1) \leq \text{ord}_E(b'_2\gamma + c'_2) \leq \text{ord}_E(c'_3)$ , we get

$$\begin{aligned} M' &\cong \left\langle (1, 1, 1), \left(0, \frac{b'_2\beta + c'_2}{b'_1\beta + c'_1}, \frac{b'_2\gamma + c'_2}{b'_1\gamma + c'_1}\right), \left(0, \frac{c'_3}{b'_1\beta + c'_1}, \frac{c'_3}{b'_1\gamma + c'_1}\right) \right\rangle_{\mathcal{O}_E} \\ &= \left\langle (1, 1, 1), \left(0, \frac{b'_2\beta + c'_2}{b'_1\beta + c'_1}, \frac{b'_2\gamma + c'_2}{b'_1\gamma + c'_1}\right), \left(0, 0, \frac{c'_3}{b'_1\gamma + c'_1} - \frac{c'_3}{b'_2\beta + c'_2} \cdot \frac{b'_2\gamma + c'_2}{b'_1\gamma + c'_1}\right) \right\rangle_{\mathcal{O}_E}. \end{aligned}$$

Thus we get

$$\begin{aligned} m &= \text{ord}_E\left(\frac{b'_2\beta + c'_2}{b'_1\beta + c'_1}\right), \quad n = \text{ord}_E\left(\frac{c'_3}{b'_1\gamma + c'_1} - \frac{c'_3}{b'_2\beta + c'_2} \cdot \frac{b'_2\gamma + c'_2}{b'_1\gamma + c'_1}\right), \\ x &= \pi^{-m} \cdot \frac{b'_1\beta + c'_1}{b'_2\beta + c'_2} \cdot \frac{b'_2\gamma + c'_2}{b'_1\gamma + c'_1}. \end{aligned}$$

Therefore we obtain  $m = \text{ord}_E(x)$ . On the other hand, in the case  $\text{ord}_E(b'_1\beta + c'_1) > \text{ord}_E(b'_2\beta + c'_2)$ , we have

$$M' = \langle a'_1, (b'_1 - b'_2)\beta + (c'_1 - c'_2), (b'_1 - b'_2)\gamma + (c'_1 - c'_2), (0, b'_2\beta + c'_2, b'_2\gamma + c'_2), (0, c'_3, c'_3) \rangle_{\mathcal{O}_E}.$$

Because  $\text{ord}_E(b'_1\beta + c'_1 - (b'_2\beta + c'_2)) = \text{ord}_E(b'_2\beta + c'_2)$ , we get the same conclusion as in the case  $\text{ord}_E(b'_1\beta + c'_1) \leq \text{ord}_E(b'_2\beta + c'_2)$ . □

**Proposition 5.5.** *Let  $f(T) = (T - \alpha)g(T)$ , where  $\alpha \in p\mathbb{Z}_p$  and  $g(T) \in \mathbb{Z}_p[T]$  is an irreducible polynomial of degree 2. Let  $E$  be the splitting field of  $g(T)$  over  $\mathbb{Q}_p$ . We assume  $\text{ord}_E(\alpha - \beta) = \text{ord}_E(\beta - \gamma) = \text{ord}_E(\gamma - \alpha) = 1$ ,*

$$M/\omega_0 M \cong \mathbb{Z}/p^i\mathbb{Z} \oplus \mathbb{Z}/p^j\mathbb{Z} \quad (i, j \in \mathbb{Z}_{\geq 1})$$

and  $E/\mathbb{Q}_p$  is a totally ramified extension. Then we have

$$\Psi(M) = M \otimes_{\Lambda} \Lambda_E \cong M(0, 1, 1) \cong \Lambda_E/(T - \alpha) \oplus \Lambda_E/(T - \beta)(T - \gamma).$$

**Proof.** Since  $M/\omega_0 M \cong \mathbb{Z}/p^i \mathbb{Z} \oplus \mathbb{Z}/p^j \mathbb{Z}$ , we have  $M/\omega_0 M \otimes_{\Lambda} \Lambda_E \cong \mathcal{O}_E/(\pi^{2i}) \oplus \mathcal{O}_E/(\pi^{2j})$ . Since  $E/\mathbb{Q}_p$  is a totally ramified extension,  $\text{ord}_E(\alpha) = 2 \text{ord}_p(\alpha) \geq 2$ . Thus we get  $\text{ord}_E(\beta) = \text{ord}_E(\gamma) = 1$ . Because  $\text{ord}_E(\pi^{2i}) = 2i$  and  $\text{ord}_E(\pi^{2j}) = 2j$  are even, we get

$$M \otimes_{\Lambda} \Lambda_E \cong M(0, 1, 1)$$

by the table of the Proposition 5.3. The isomorphism  $M(0, 1, 1) \cong \Lambda_E/(T - \alpha) \oplus \Lambda_E/(T - \beta)(T - \gamma)$  is Lemma 3 in Sumida [12].  $\square$

**Corollary 5.6.** *Let  $f(T), g(T)$  and  $E$  be as in Propositions 5.5 and  $[M]_{\mathbb{Q}_p} \in \mathcal{M}_{f(T)}^{\mathbb{Q}_p}$ . We assume the same conditions of Proposition 5.5 and we put  $g(T) = T^2 + c_1 T + c_0$ . Then*

(a) *Suppose  $p \geq 5$ . For  $n \geq 0$ , we have*

$$\#(M/\omega_n M \otimes \Lambda_E) = p^{\text{ord}_E(\omega_n(\alpha)\omega_n(\beta)\omega_n(\gamma))} = p^{6n+2+\text{ord}_E(\alpha)}.$$

(b) *Suppose  $p = 3$ . For  $n \geq 1$ , we have*

$$\#(M/\omega_n M \otimes \Lambda_E) = \begin{cases} p^{\text{ord}_E(\omega_n(\alpha)\omega_n(\beta)\omega_n(\gamma))} = p^{6n+\text{ord}_E(\alpha)+4 \text{ord}_3(c_0-3)-2} \\ \text{if } \text{ord}_3(c_0 - 3) \leq \text{ord}_3(c_1 - 3), \\ p^{\text{ord}_E(\omega_n(\alpha)\omega_n(\beta)\omega_n(\gamma))} = p^{6n+\text{ord}_E(\alpha)+4 \text{ord}_3(c_0-3)-2} \\ \text{if } \text{ord}_3(c_0 - 3) > \text{ord}_3(c_1 - 3). \end{cases}$$

**Proof.** Put  $N = \langle (1, 1, 1), (0, 1, 1), (0, 0, \pi) \rangle_{\mathcal{O}_E} \subset \mathcal{E}$ . By Proposition 5.5, we have  $M \otimes_{\Lambda} \Lambda_E \cong N$  as  $\Lambda_E$ -modules. Thus we have

$$M/\omega_n M \otimes \Lambda_E \cong (M \otimes_{\Lambda} \Lambda_E)/\omega_n(M \otimes_{\Lambda} \Lambda_E) \cong N/\omega_n N$$

as  $\Lambda_E/\omega_n \Lambda_E$ -modules. By the same method as Proposition 5.3, we consider the map  $\Pi_{\omega_n} : N \rightarrow N$  and take  $(1, 0, 0), (0, 1, 1)$  and  $(0, 0, \pi)$  as a basis of  $N$ . The matrix corresponding to  $\Pi_{\omega_n}$  is

$$\begin{pmatrix} \omega_n(\alpha) & 0 & 0 \\ 0 & \omega_n(\beta) & 0 \\ 0 & (\omega_n(\beta) - \omega_n(\gamma))\pi^{-1} & \omega_n(\gamma) \end{pmatrix}.$$

We first consider the case (a). We have  $\text{ord}_E(\omega_n(\beta) - \omega_n(\gamma)) = \text{ord}_E(\beta - \gamma) + n \text{ord}_E(3) = 2n + 1$  (cf. [7], Lemma 2.5). Furthermore, we have  $\text{ord}_E(\omega_n(\alpha)) = 2n + \text{ord}_E(\alpha)$ , and we get  $\text{ord}_E\{(\omega_n(\beta) - \omega_n(\gamma))\pi^{-1}\} = 2n < \text{ord}_E(\omega_n(\beta))$  since  $\text{ord}_E(\omega_n(\beta)) = \text{ord}_E(\omega_n(\gamma)) = 2n + 1$ . Thus we can transform the above matrix into

$$\begin{pmatrix} \pi^{2n+\text{ord}_E(\alpha)} & 0 & 0 \\ 0 & \pi^{2n} & 0 \\ 0 & 0 & \pi^{2n+2} \end{pmatrix}.$$

This implies  $N/\omega_n N \cong \mathcal{O}_E/(\pi^{2n+\text{ord}_E(\alpha)}) \oplus \mathcal{O}_E/(\pi^{2n}) \oplus \mathcal{O}_E/(\pi^{2n+2})$ .

Next, we prove the case (b). For  $n \geq 1$ , we have

$$\text{ord}_E(\omega_n(\beta)) = \begin{cases} 2 \text{ord}_3(c_0 - 3) + 2n - 1 & \text{if } \text{ord}_3(c_0 - 3) \leq \text{ord}_3(c_1 - 3), \\ 2 \text{ord}_3(c_1 - 3) + 2n & \text{if } \text{ord}_3(c_0 - 3) > \text{ord}_3(c_1 - 3). \end{cases}$$

On the other hand, for  $n \geq 1$ , we have

$$\text{ord}_E(\omega_n(\beta) - \omega_n(\gamma)) \begin{cases} = 2 \text{ord}_3(c_0 - 3) + 2n - 1 & \text{if } \text{ord}_3(c_0 - 3) \leq \text{ord}_3(c_1 - 3), \\ > 2 \text{ord}_3(c_1 - 3) + 2n & \text{if } \text{ord}_3(c_0 - 3) > \text{ord}_3(c_1 - 3) \end{cases}$$

(cf. [7], Lemma 2.5). The rest can be proved by the same method as the case (a).  $\square$

In order to determine the structure of  $X$ , we will use the higher Fitting ideals. For a commutative ring  $R$  and a finitely presented  $R$ -module  $M$ , we consider the following exact sequence

$$R^m \xrightarrow{f} R^n \rightarrow M \rightarrow 0,$$

where  $m$  and  $n$  are positive integers. For an integer  $i \geq 0$  such that  $0 \leq i < n$ , the  $i$ -th Fitting ideal of  $M$  is defined to be the ideal of  $R$  generated by all  $(n - i) \times (n - i)$  minors of the matrix corresponding to  $f$ . This definition does not depend on the choice of the above exact sequence (see [9]).

**Proposition 5.7.** *Let  $f(T) = (T - \alpha)g(T)$ , where  $\alpha \in p\mathbb{Z}_p$  and  $g(T) \in \mathbb{Z}_p[T]$  is an irreducible polynomial of degree 2. Let  $E$  be the splitting field of  $g(T)$  over  $\mathbb{Q}_p$ . Let  $[M]_E \in \mathcal{M}_{f(T)}^E$  and  $M = M(m, n, x)$ .*

(1) *Assume  $m = 0$  and  $(\gamma - \beta)x\pi^{-n} \in \mathcal{O}_E^\times$ . Then we have*

$$\text{Fitt}_{1,\Delta}(M) = \begin{cases} (T - \alpha, (\alpha - \beta)(\alpha - \gamma)) & \text{if } x = 1, \\ (T - \alpha, (\alpha - \beta)(\alpha - \gamma)(1 - x)\pi^{-n}) & \text{if } x \neq 1. \end{cases}$$

(2) *Assume  $n = 0$  and  $(\beta - \alpha)\pi^{-m} \in \mathcal{O}_E^\times$ . Then we have*

$$\text{Fitt}_{1,\Delta}(M) = (T - \gamma, (\alpha - \gamma)(\beta - \gamma)).$$

(3)

$$\text{Fitt}_{1,\Delta}((T - \alpha)M) = \begin{cases} (T - \beta, (\beta - \gamma)\pi^{-n}) & \text{if } n \leq \text{ord}_E(\pi^m - x), \\ \left(T - \beta, \frac{\gamma - \beta}{\pi^m - x}\right) & \text{if } n > \text{ord}_E(\pi^m - x). \end{cases}$$

Proof. By the action of  $T$ , we have

$$\begin{aligned} T(1, 1, 1) &= (\alpha, \beta, \gamma) \\ &= \alpha(1, 1, 1) + (\beta - \alpha)\pi^{-m}(0, \pi^m, x) \\ &\quad + \{\gamma - \alpha - (\beta - \alpha)\pi^{-m}x\}\pi^{-n}(0, 0, \pi^n), \\ T(0, \pi^m, x) &= (0, \beta\pi^m, \gamma x) \\ &= \beta(0, \pi^m, x) + (\gamma - \beta)x\pi^{-n}(0, 0, \pi^n), \\ T(0, 0, \pi^n) &= \gamma(0, 0, \pi^n). \end{aligned}$$

Then we get the following matrix

$$\begin{pmatrix} T - \alpha & -(\beta - \alpha)\pi^{-m} & -\{(\gamma - \alpha) - (\beta - \alpha)\pi^{-m}x\}\pi^{-n} \\ 0 & T - \beta & -(\gamma - \beta)x\pi^{-n} \\ 0 & 0 & T - \gamma \end{pmatrix}.$$

We first show (1). Under the assumption of (1), the matrix is

$$\begin{pmatrix} T - \alpha & -\beta + \alpha & -\{(\gamma - \alpha) - (\beta - \alpha)x\}\pi^{-n} \\ 0 & T - \beta & -(\gamma - \beta)x\pi^{-n} \\ 0 & 0 & T - \gamma \end{pmatrix}.$$

By elementary row and column operations, we can transform the above matrix into

$$\begin{pmatrix} T - \alpha & (\alpha - \gamma)(1 - x)\pi^{-n}(T - \beta) & 0 \\ 0 & (T - \beta)(T - \gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Therefore we get

$$\begin{aligned} \text{Fitt}_{1,\Delta}(M) &= (T - \alpha, (\alpha - \beta)(\alpha - \gamma), (\alpha - \beta)(\alpha - \beta)(1 - x)\pi^{-n}) \\ &= \begin{cases} (T - \alpha, (\alpha - \beta)(\alpha - \gamma)) & \text{if } x = 1, \\ (T - \alpha, (\alpha - \beta)(\alpha - \gamma)(1 - x)\pi^{-n}) & \text{if } x \neq 1. \end{cases} \end{aligned}$$

Next we show (2). Under the assumption of (2), the matrix is

$$\begin{pmatrix} T - \alpha & -(\beta - \alpha)\pi^{-m} & -(\gamma - \alpha) + (\beta - \alpha)\pi^{-m}x \\ 0 & T - \beta & -(\gamma - \beta)x \\ 0 & 0 & T - \gamma \end{pmatrix}.$$

By elementary row and column operations, we can transform the above matrix into

$$\begin{pmatrix} T - \alpha & 1 & 0 \\ 0 & T - \beta & 0 \\ 0 & 0 & T - \gamma \end{pmatrix}.$$

Therefore we get

$$\begin{aligned} \text{Fitt}_{1,\Delta}(M) &= ((T - \alpha)(T - \beta), (T - \beta)(T - \gamma), (T - \alpha)(T - \gamma), (T - \gamma)) \\ &= (T - \gamma, (\alpha - \gamma)(\beta - \gamma)). \end{aligned}$$

Finally we show (3). We note that

$$\begin{aligned} (T - \alpha)M &= \langle (0, \beta - \alpha, \gamma - \alpha), (0, (\beta - \alpha)\pi^m, (\gamma - \alpha)x), (0, 0, (\gamma - \alpha)\pi^n) \rangle_{\mathcal{O}_E} \\ &= \begin{cases} \langle (0, \beta - \alpha, \gamma - \alpha), (0, 0, (\gamma - \alpha)\pi^n) \rangle_{\mathcal{O}_E} & \text{if } n \leq \text{ord}_E(\pi^m - x), \\ \langle (0, \beta - \alpha, \gamma - \alpha), (0, 0, (\gamma - \alpha)(\pi^m - x)) \rangle_{\mathcal{O}_E} & \text{if } n > \text{ord}_E(\pi^m - x). \end{cases} \end{aligned}$$

In the case  $n \leq \text{ord}_E(\pi^m - x)$ , by the action of  $T$ , we have

$$\begin{aligned} T(0, \beta - \alpha, \gamma - \alpha) &= (0, \beta(\beta - \alpha), \gamma(\gamma - \alpha)) \\ &= \beta(0, \beta - \alpha, \gamma - \alpha) + (\gamma - \beta)\pi^{-n}(0, 0, (\gamma - \alpha)\pi^n), \\ T(0, 0, (\gamma - \alpha)\pi^n) &= \gamma(0, 0, (\gamma - \alpha)\pi^n). \end{aligned}$$

Thus we get the following matrix

$$\begin{pmatrix} T - \beta & -(\gamma - \beta)\pi^{-n} \\ 0 & T - \gamma \end{pmatrix}.$$

Therefore we get

$$\begin{aligned} \text{Fitt}_{1,\Delta}((T - \alpha)M) &= (T - \beta, T - \gamma, (\gamma - \beta)\pi^{-n}) \\ &= (T - \beta, (\gamma - \beta)\pi^{-n}). \end{aligned}$$

In the case  $n > \text{ord}_E(\pi^m - x)$ , by the same method as above, we get the following matrix

$$\begin{pmatrix} T - \beta & -\frac{\gamma - \beta}{\pi^m - x} \\ 0 & T - \gamma \end{pmatrix}.$$

Therefore we get

$$\begin{aligned} \text{Fitt}_{1,\Delta}((T - \alpha)M) &= \left( T - \beta, T - \gamma, \frac{\gamma - \beta}{\pi^m - x} \right) \\ &= \left( T - \beta, \frac{\gamma - \beta}{\pi^m - x} \right). \quad \square \end{aligned}$$

## 6. Numerical examples

In this section, we introduce some numerical examples which were computed using Pari-Gp.

Let  $p = 3$  and  $k = \mathbb{Q}(\sqrt{-d})$  where  $d$  is a positive square-free integer. For simplicity, let  $d \not\equiv 2 \pmod{3}$ . Our assumption  $d \not\equiv 2 \pmod{3}$  implies that  $p = 3$  is inert or ramified in  $k$ . This assumption is also needed to get the isomorphism (16) below. In this section, we determine the  $\Lambda$ -isomorphism class of the Iwasawa module associated to  $k = \mathbb{Q}(\sqrt{-d})$  in the range  $1 < d < 10^5$  with  $\lambda_p(k) = 3$ , where  $\lambda_p(k)$  is the Iwasawa  $\lambda$ -invariant with respect to the cyclotomic  $\mathbb{Z}_p$ -extension. There are 1109 imaginary quadratic fields satisfying these properties.

Let  $k_\infty/k$  be the cyclotomic  $\mathbb{Z}_p$ -extension of  $k$ . For each  $n \geq 0$ , we denote by  $k_n$  the intermediate field of  $k_\infty/k$  such that  $k_n$  is the unique cyclic extension over  $k$  of degree  $p^n$ . Let  $A_n$  be the  $p$ -Sylow subgroup of the ideal class group of  $k_n$ . We put  $X = \varprojlim A_n$ , where the inverse limit is taken with respect to the relative norms. Then  $X$  becomes a  $\mathbb{Z}_p[[\text{Gal}(k_\infty/k)]]$ -module. Since there is a ring isomorphism between  $\Lambda = \mathbb{Z}_p[[T]]$  and  $\mathbb{Z}_p[[\text{Gal}(k_\infty/k)]]$  which depends on the choice of a topological generator of  $\text{Gal}(k_\infty/k)$ ,  $X$  becomes a finitely generated torsion  $\Lambda$ -module. Let  $f(T)$  be the distinguished polynomial which generates  $\text{char}(X)$ . It is known that  $X$  is a free  $\mathbb{Z}_p$ -module so  $[X]_{\mathbb{Q}_p} \in \mathcal{M}_{f(T)}^{\mathbb{Q}_p}$  and we can apply Theorem 3.5 to the Iwasawa module  $X$ .

We can calculate the polynomial  $f(T) \pmod{p^n}$  for small  $n$  numerically. Let  $\chi$  be the Dirichlet character associated to  $k$ ,  $\omega$  be the Teichmüller character and  $f_0$  be the least common multiple of  $p$  and conductor of  $\chi$ . By the Iwasawa main conjecture, there exists a power series  $g_{\chi^{-1}\omega}(T) \in \Lambda$  such that

$$\text{char}(X) = (g_{\chi^{-1}\omega}(T)).$$

Here,  $g_{\chi^{-1}\omega}(T)$  is the  $p$ -adic  $L$ -function constructed by Iwasawa. We can approximate  $g_{\chi^{-1}\omega}(T)$  such as

$$g_{\chi^{-1}\omega}(T) \equiv -\frac{1}{2f_0p^n} \sum_{0 < a < f_0p^n, (a, f_0p^n)=1} a\chi\omega^{-1}(a)(1+T)^{i_n(a)} \pmod{\omega_n},$$

where  $i_n(a)$  is the unique integer such that  $a\omega^{-1}(a) \equiv (1+p)^{i_n(a)} \pmod{p^{n+1}}$  and  $0 \leq i_n(a) < p^n$ . By Weierstrass preparation theorem ([13], Theorem 7.3), there exists  $u_{\chi^{-1}\omega} \in \Lambda^\times$  such that  $g_{\chi^{-1}\omega}(T) = f(T)u_{\chi^{-1}\omega}(T)$ . Thus we can get  $f(T)$  approximately ([13], Proposition 7.2). For the detail about computation of  $g_{\chi^{-1}\omega}(T)$ , see [1] and [4]. We computed  $f(T)$  by Mizusawa's program Iwapoly.ub ([8], Research, Programing, Approximate Computation of Iwasawa Polynomials by UBASIC), and referred Fukuda's table for the  $\lambda$ -invariants of imaginary quadratic fields [3].

Now we classify the Iwasawa module  $X$ . There are two cases

- (I)  $A_0$  is a cyclic group,
- (II)  $A_0$  is not a cyclic group.

In order to determine the structure of  $X$ , we use the following fact. In our case, exactly one prime is ramified in  $k_\infty/k$  and it is totally ramified. So there are

$\Lambda$ -isomorphisms

$$(16) \quad X/\omega_n X \cong A_n$$

for any non-negative integers ([13], Proposition 13.22).

We determine the  $\Lambda$ -isomorphism class of  $X$  by the information on the structures of  $A_n$  for some  $n \geq 0$ .

There are 1015 fields whose  $A_0$  are cyclic groups among 1109 fields. First of all, we determine the isomorphism classes in the case (I). In this case,  $X$  becomes a  $\Lambda_E$ -cyclic module by Nakayama’s Lemma. Thus we can use Proposition 5.2 to get

$$M \cong M(\text{ord}_E(\beta - \alpha), \text{ord}_E(\gamma - \alpha) + \text{ord}_E(\gamma - \beta), u\pi^{\text{ord}_E(\beta - \alpha)}).$$

In the above range of  $d$ , no  $f(T)$  splits completely in  $\mathbb{Q}_p[T]$ , so we have to consider the minimal splitting field  $E$  of  $f(T)$ , which is quadratic over  $\mathbb{Q}_p$ .

EXAMPLE 6.1. Let  $k = \mathbb{Q}(\sqrt{-886})$ . Then we have  $A_0 \cong \mathbb{Z}/9\mathbb{Z}$  (cf. [10]). By using Mizusawa’s program [8], we have

$$f(T) \equiv (T - 195)(T^2 + 291T + 429) \pmod{3^6}.$$

By Hensel’s lemma, there exist  $\alpha \in \mathbb{Z}_p$  and  $g(T) \in \mathbb{Z}_p[T]$  such that

$$f(T) = (T - \alpha)g(T),$$

where  $\alpha \equiv 195 \pmod{3^5}$  and  $g(T) \equiv T^2 + 48T + 186 \pmod{3^5}$ . Since  $g(T)$  is an Eisenstein polynomial,  $E/\mathbb{Q}_p$  is a totally ramified extension. Let  $E$  be the minimal splitting field of  $g(T)$  and  $g(T) = (T - \beta)(T - \gamma)$ , where  $\beta, \gamma \in E$ . Because  $\beta\gamma \equiv 186 \pmod{3^5}$ , we get  $\text{ord}_E(\beta) = \text{ord}_E(\gamma) = 1$  and  $\text{ord}_E(\alpha - \gamma) = \text{ord}_E(\alpha - \beta) = 1$ . Since  $(\beta - \gamma)^2 = (\beta + \gamma)^2 - 4\beta\gamma \equiv 1560 \pmod{3^5}$ , we have  $\text{ord}_E(\beta - \gamma) = 1$ . By Proposition 5.1 and 5.2, we get  $X \otimes_{\Lambda} \Lambda_E \cong M(1, 2, u\pi)$ , where  $u = (\gamma - \alpha)/(\beta - \alpha)$ .

Next, we determine the isomorphism classes in the case (II). There are 94 fields whose  $A_0$  are not cyclic groups. There are 66 fields whose  $A_0$  are not cyclic groups and whose  $f(T)$  is reducible. We will determine  $[X]_{\mathbb{Q}_p}$  for these 66 fields. We can determine the  $\Lambda$ -isomorphism class of  $X$  for 60 fields by Proposition 5.5. The following example is a case that we can determine the  $\Lambda$ -isomorphism class of  $X$  by Proposition 5.5.

EXAMPLE 6.2. Let  $k = \mathbb{Q}(\sqrt{-6583})$ . In this case, we have  $A_0 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$  (cf. [10]). We have

$$f(T) \equiv (T - 96)(T^2 + 96T + 696) \pmod{3^6}.$$

By Hensel's lemma, there exist  $\alpha \in \mathbb{Z}_p$  and  $g(T) \in \mathbb{Z}_p[T]$  such that

$$f(T) = (T - \alpha)g(T),$$

where  $\alpha \equiv 96 \pmod{3^5}$  and  $g(T) \equiv T^2 + 96T + 210 \pmod{3^5}$ . Let  $E$  be the minimal splitting field of  $g(T)$  and  $g(T) = (T - \beta)(T - \gamma)$ , where  $\beta, \gamma \in E$ . Then,  $E/\mathbb{Q}_p$  is a totally ramified extension and we get  $\text{ord}_E(\alpha - \beta) = \text{ord}_E(\beta - \gamma) = \text{ord}_E(\gamma - \alpha) = 1$ ,  $\text{ord}_E(\alpha) = 2$  and  $\text{ord}_E(\beta) = \text{ord}_E(\gamma) = 1$ . Therefore we get  $X \otimes_{\Lambda} \Lambda_E \cong M(0, 1, 1)$  by Proposition 5.5.

There are remaining 6 fields which we cannot determine the structure of  $X$  by Proposition 5.5. For these fields, we have to investigate the action of the group  $\Gamma_1 = \text{Gal}(k_1/k)$ . Explicitly, the remaining 6 fields are  $\mathbb{Q}(\sqrt{-9574})$ ,  $\mathbb{Q}(\sqrt{-30994})$ ,  $\mathbb{Q}(\sqrt{-41631})$ ,  $\mathbb{Q}(\sqrt{-64671})$ ,  $\mathbb{Q}(\sqrt{-82774})$ ,  $\mathbb{Q}(\sqrt{-92515})$ .

EXAMPLE 6.3. Let  $k = \mathbb{Q}(\sqrt{-9574})$ . In this case, we have  $A_0 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$  (cf. [10]) and  $A_1 \cong \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/27\mathbb{Z}$ . We have

$$f(T) \equiv (T - 192)(T^2 + 1173T + 1422) \pmod{3^7}.$$

By Hensel's Lemma, there exist  $\alpha \in \mathbb{Z}_p$  and  $g(T) \in \mathbb{Z}_p[T]$  such that

$$f(T) = (T - \alpha)g(T),$$

where  $\alpha \equiv 192 \pmod{3^5}$  and  $g(T) \equiv T^2 + 201T + 207 \pmod{3^5}$ . Let  $E$  be the splitting field of  $g(T)$  and  $g(T) = (T - \beta)(T - \gamma)$ , where  $\beta, \gamma \in E$ . Because the discriminant of  $g(T)$  is  $3^2 \cdot 4397 \pmod{3^7}$  and 4397 is a quadratic nonresidue,  $E/\mathbb{Q}_p$  is an unramified extension. Since the discriminant of  $f(T)$  is  $2^8 \cdot 3^6 \cdot 43 \cdot 89 \cdot 1039 \pmod{3^7}$ , we get  $\text{ord}_E(\alpha - \beta) = \text{ord}_E(\beta - \gamma) = \text{ord}_E(\gamma - \alpha) = 1$  and  $\text{ord}_E(\alpha) = \text{ord}_E(\beta) = \text{ord}_E(\gamma) = 1$ . By checking the structures of  $A_0$  and  $A_1$  as  $\mathcal{O}_E$ -modules, we get

$$X \otimes_{\Lambda} \Lambda_E \cong M(0, 1, 1), M(0, 1, 2), M(1, 0, 0) \text{ or } M(1, 1, 0).$$

Now we investigate the structure of  $A_1$  as a  $\Gamma_1$ -module. We have an isomorphism  $A_1 \cong \mathbb{Z}/27\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ . Furthermore, Pari-Gp gives explicit generators which give this isomorphism. Let  $\mathfrak{a}_1$ ,  $\mathfrak{a}_2$  and  $\mathfrak{a}_3$  be the generators which was computed. (We do not write down  $\mathfrak{a}_1$ ,  $\mathfrak{a}_2$  and  $\mathfrak{a}_3$  because they are complicated.) Let  $\sigma$  be the generator of  $\Gamma_1$ , which was computed by Pari-Gp. We compute,

$$(\sigma - 1)\mathfrak{a}_1 = 3\mathfrak{a}_2 - \mathfrak{a}_3,$$

$$(\sigma - 1)\mathfrak{a}_2 = 6\mathfrak{a}_2,$$

$$(\sigma - 1)\mathfrak{a}_3 = 18\mathfrak{a}_1 + 6\mathfrak{a}_2.$$

There is a topological generator  $\tilde{\sigma} \in \text{Gal}(k_\infty/k)$  such that  $\tilde{\sigma}$  is an extension of  $\sigma$ . By this topological generator, we have an isomorphism

$$\mathbb{Z}_p[[\text{Gal}(k_\infty/k)]] \cong \Lambda = \mathbb{Z}_p[[T]] \quad \text{such that} \quad \tilde{\sigma} \leftrightarrow 1 + T.$$

We regard  $X$  as a  $\Lambda$ -module by this isomorphism. We note that  $f(T)$  depends on the choice of  $\tilde{\sigma}$ , but we can easily check that  $\mathcal{M}_{f(T)}^E$  does not depend on the choice of  $\tilde{\sigma}$ . Because  $\mathbb{Z}_p[[\Gamma_1]] \cong \Lambda/\omega_1\Lambda$ , we get

$$\begin{aligned} \overline{T}a_1 &= 3a_2 - a_3, \\ \overline{T}a_2 &= 6a_2, \\ \overline{T}a_3 &= 18a_1 + 6a_2, \end{aligned}$$

where  $\overline{T} = T \bmod \omega_1$ . Now we have

$$\begin{aligned} \overline{(T^2 + 18)}a_1 + \overline{6}a_2 &= 0, \\ \overline{(T - 6)}a_2 &= 0, \\ \overline{3T}a_1 &= 0, \\ \overline{27}a_1 &= 0, \\ \overline{9}a_2 &= 0. \end{aligned}$$

Therefore we can calculate the 1-st Fitting ideal of  $A_1 \otimes \mathcal{O}_E$ :

$$\text{Fitt}_{1, \Lambda_E/\omega_1\Lambda_E}(A_1 \otimes \mathcal{O}_E) = (T, 3) \quad \text{mod } \omega_1,$$

where  $\text{Fitt}_{1, \Lambda_E/\omega_1\Lambda_E}(A_1 \otimes \mathcal{O}_E)$  is the 1-st Fitting ideal of  $A_1 \otimes \mathcal{O}_E$  as a  $\Lambda_E/\omega_1\Lambda_E$ -module. On the other hand, by Proposition 5.7 (1) and (2), for  $M(0, 1, 2)$ ,  $M(1, 0, 0)$ ,  $M(0, 1, 1)$ , we have

$$\text{Fitt}_{1, \Lambda_E/\omega_1\Lambda_E}(M/\omega_1M) = \begin{cases} (T, 3) \bmod \omega_1 & \text{if } M = M(0, 1, 2), \\ (T - \gamma, 9) \bmod \omega_1 & \text{if } M = M(1, 0, 0), \\ (T - \alpha, 9) \bmod \omega_1 & \text{if } M = M(0, 1, 1). \end{cases}$$

Therefore we have

$$X \otimes_\Lambda \Lambda_E \cong M(0, 1, 2) \quad \text{or} \quad M(1, 1, 0).$$

We investigate the module  $(T - \alpha)(M/\omega_1M)$ . By Proposition 5.7 (3), for  $M(0, 1, 2)$ ,  $M(1, 1, 0)$  we get

$$\text{Fitt}_{1, \Lambda_E/\omega_1\Lambda_E}((T - \alpha)(M/\omega_1M)) = \begin{cases} (T, 3) \bmod \omega_1 & \text{if } M = M(0, 1, 2), \\ \Lambda_E/\omega_1\Lambda_E & \text{if } M = M(1, 1, 0). \end{cases}$$

We can compute the following from the above data

$$\text{Fitt}_{1, \Lambda_E/\omega_1\Lambda_E}(\overline{(T - \alpha)}A_1 \otimes \mathcal{O}_E) = (T, 3) \quad \text{mod } \omega_1.$$

Therefore, we get  $X \otimes_{\Lambda} \Lambda_E \cong M(0, 1, 2)$ .

By the same method as above, we can determine the isomorphism classes of  $X$  of  $\mathbb{Q}(\sqrt{-30994})$ ,  $\mathbb{Q}(\sqrt{-82774})$  and  $\mathbb{Q}(\sqrt{-92515})$ . For the 3 fields, we can show that  $X \otimes_{\Lambda} \Lambda_E \cong M(0, 1, 2)$ .

Finally we determine the structure of  $X$  for remaining 2 fields  $\mathbb{Q}(\sqrt{-41631})$  and  $\mathbb{Q}(\sqrt{-64671})$ .

EXAMPLE 6.4. Let  $k = \mathbb{Q}(\sqrt{-41631})$ . In this case, we have  $A_0 \cong \mathbb{Z}/3^3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$  (cf. [10]) and  $A_1 \cong \mathbb{Z}/3^4\mathbb{Z} \oplus \mathbb{Z}/3^2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$  computing by Pari-Gp. We have

$$f(T) \equiv (T - 42)(T^2 - 279T + 594) \pmod{3^7}.$$

By Hensel's lemma, there exist  $\alpha \in \mathbb{Z}_p$  and  $g(T) \in \mathbb{Z}_p[T]$  such that

$$f(T) = (T - \alpha)g(T),$$

where  $\alpha \equiv 42 \pmod{3^5}$  and  $g(T) \equiv T^2 + 36T + 108 \pmod{3^5}$ . Let  $E$  be the minimal splitting field of  $g(T)$  and  $g(T) = (T - \beta)(T - \gamma)$ , where  $\beta, \gamma \in E$ . Then  $E/\mathbb{Q}_p$  is a totally ramified extension with  $\text{ord}_E(\alpha - \beta) = \text{ord}_E(\gamma - \alpha) = 2$ ,  $\text{ord}_E(\beta - \gamma) = 3$ ,  $\text{ord}_E(\alpha) = 2$ , and  $\text{ord}_E(\beta) = \text{ord}_E(\gamma) = 3$ . Let  $\pi$  be a prime element of  $E$ . In this case, the elements  $M(m, n, x) \in \mathcal{M}_{f(T)}^E$  which satisfy the conclusion of Proposition 5.4 are

$$\left\{ \begin{array}{l} (0, 0, 0), (0, 1, 1), (0, 1, 2), (0, 2, 1), (0, 2, 2), (0, 2, 1 + \pi), (0, 3, 1), \\ (0, 3, 1 + \pi), (0, 3, 1 + \pi^2), (1, 0, 0), (1, 1, 0), (1, 1, 1), (1, 2, \pi), \\ (1, 2, 2\pi), (1, 3, \pi), (1, 3, \pi + \pi^2), (1, 3, \pi + 2\pi^2), (1, 4, u\pi), \\ (2, 0, 0), (2, 1, 0), (2, 2, 0), (2, 3, u\pi^2), (2, 4, u\pi^2), (2, 5, u\pi^2) \end{array} \right\},$$

where  $u = (\gamma - \alpha)/(\beta - \alpha)$ . By checking the structures of  $A_0$  and  $A_1$  as  $\mathcal{O}_E$ -modules, we get

$$X \otimes_{\Lambda} \Lambda_E \cong M(0, 3, 1), M(0, 3, 1 + \pi), M(0, 3, 1 + \pi^2), \\ M(1, 3, \pi + \pi^2), M(1, 3, \pi + 2\pi^2) \quad \text{or} \quad M(2, 3, u\pi^2).$$

We have an isomorphism  $A_1 \cong \mathbb{Z}/81\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ . Let  $\mathfrak{a}_1, \mathfrak{a}_2$  and  $\mathfrak{a}_3$  be the generators which were computed by Pari-Gp. By Pari-Gp we have:

$$(\sigma - 1)\mathfrak{a}_1 = 54\mathfrak{a}_1 + 6\mathfrak{a}_2 + \mathfrak{a}_3, \\ (\sigma - 1)\mathfrak{a}_2 = 54\mathfrak{a}_1, \\ (\sigma - 1)\mathfrak{a}_3 = 54\mathfrak{a}_1 + 3\mathfrak{a}_2,$$

for a certain generator  $\sigma$  of  $\Gamma_1$ . By the same method as  $k = \mathbb{Q}(\sqrt{-9574})$ , we fix a topological generator  $\tilde{\sigma} \in \text{Gal}(k_{\infty}/k)$  such that  $\tilde{\sigma}$  is an extension of  $\sigma$ . Because

$\mathbb{Z}_p[[\Gamma_1]] \cong \Lambda/\omega_1\Lambda$ , we have

$$\begin{aligned} \overline{(T^2 - 54T - 54)}\mathbf{a}_1 - \overline{3}\mathbf{a}_2 &= 0, \\ \overline{54}\mathbf{a}_1 - \overline{T}\mathbf{a}_2 &= 0, \\ \overline{3T}\mathbf{a}_1 &= 0, \\ \overline{81}\mathbf{a}_1 &= 0, \\ \overline{9}\mathbf{a}_2 &= 0, \end{aligned}$$

where  $\overline{T} = T \bmod \omega_1$ . Therefore we get the 1-st Fitting ideal of  $A_1 \otimes \mathcal{O}_E$ ;

$$\text{Fitt}_{1, \Lambda_E/\omega_1\Lambda_E}(A_1 \otimes \mathcal{O}_E) = (T, 3) \bmod \omega_1.$$

On the other hand, by Proposition 5.7 (1) and (2), we have

$$\text{Fitt}_{1, \Lambda_E/\omega_1\Lambda_E}(M/\omega_1M) = \begin{cases} (T - \alpha, 9) \bmod \omega_1 & \text{if } M = M(0, 3, 1), \\ (T, 3) \bmod \omega_1 & \text{if } M = M(0, 3, 1 + \pi), \\ (T - \alpha, \pi^3) \bmod \omega_1 & \text{if } M = M(0, 3, 1 + \pi^2), \end{cases}$$

for  $M(0, 3, 1)$ ,  $M(0, 3, 1 + \pi)$  and  $M(0, 3, 1 + \pi^2)$ . Therefore we have

$$X \otimes_{\Lambda} \Lambda_E \cong M(0, 3, 1 + \pi), M(1, 3, \pi + \pi^2), M(1, 3, \pi + 2\pi^2) \text{ or } M(2, 3, u\pi^2).$$

As in the case  $k = \mathbb{Q}(\sqrt{-9574})$ , we investigate the structure of  $(T - \alpha)(M/\omega_1M)$ . By Proposition 5.7 (3), we get

$$\text{Fitt}_{1, \Lambda_E/\omega_1\Lambda_E}((T - \alpha)(M/\omega_1M)) = \begin{cases} (T, 3) \bmod \omega_1 & \text{if } M = M(0, 3, 1 + \pi), \\ \Lambda_E/\omega_1\Lambda_E & \text{if } M = M(1, 3, \pi + \pi^2), \\ (T, \pi) \bmod \omega_1 & \text{if } M = M(1, 3, \pi + 2\pi^2), \\ \Lambda_E/\omega_1\Lambda_E & \text{if } M = M(2, 3, u\pi^2). \end{cases}$$

We can compute from the above data

$$\text{Fitt}_{1, \Lambda_E/\omega_1\Lambda_E}(\overline{(T - \alpha)}A_1 \otimes \mathcal{O}_E) = (T, 3) \bmod \omega_1.$$

Therefore we get  $X \otimes_{\Lambda} \Lambda_E \cong M(0, 3, 1 + \pi)$ .

We can determine the structure of  $\mathbb{Q}(\sqrt{-64671})$  by the same method as above. For  $\mathbb{Q}(\sqrt{-64671})$ , we can show that  $X \otimes_{\Lambda} \Lambda_E \cong M(0, 3, 1 + \pi)$ .

The following is the table of the  $X \otimes_{\Lambda} \Lambda_E$  for the fields such that  $A_0$  is not cyclic and  $f(T)$  is reducible. Here,  $m, n, x$  represent  $X \otimes \Lambda_E \cong M(m, n, x)$ , and ram. /unram. means that  $E/\mathbb{Q}_3$  is ramified /unramified extension, respectively. We marked (\*) on the remaining 6 fields for which we determined the structures in Example 6.3 and 6.4.

Table 1.

	$d$	$\text{ord}_E(\alpha - \beta)$	$\text{ord}_E(\beta - \gamma)$	$\text{ord}_E(\gamma - \alpha)$	$E/\mathbb{Q}_3$	$m$	$n$	$x$	$A_0$
	6583	1	1	1	ram.	0	1	1	(3, 3)
	8751	1	1	1	ram.	0	1	1	(3, 3)
	9069	1	1	1	ram.	0	1	1	(3, 3)
(*)	9574	1	1	1	unram.	0	1	2	(3 <sup>2</sup> , 3)
	12118	1	1	1	ram.	0	1	1	(3, 3)
	16627	1	1	1	ram.	0	1	1	(3, 3)
	21018	1	1	1	ram.	0	1	1	(3, 3)
	23178	1	1	1	ram.	0	1	1	(3, 3)
	24109	1	1	1	ram.	0	1	1	(3, 3)
	25122	1	1	1	ram.	0	1	1	(3, 3)
	29569	1	1	1	ram.	0	1	1	(3, 3)
	29778	1	1	1	ram.	0	1	1	(3, 3)
	29994	1	1	1	ram.	0	1	1	(3, 3)
(*)	30994	1	1	1	unram.	0	1	2	(3 <sup>2</sup> , 3)
	31999	1	1	1	ram.	0	1	1	(3, 3)
	34507	1	1	1	ram.	0	1	1	(3, 3)
	34867	1	1	1	ram.	0	1	1	(3, 3)
	35539	1	1	1	ram.	0	1	1	(3, 3)
	37213	1	1	1	ram.	0	1	1	(3, 3)
	37237	1	1	1	ram.	0	1	1	(3, 3)
	38226	1	1	1	ram.	0	1	1	(3, 3)
	38553	1	1	1	ram.	0	1	1	(3, 3)
	38926	1	1	1	ram.	0	1	1	(3, 3)
	40299	1	1	1	ram.	0	1	1	(3, 3)
	41583	1	1	1	ram.	0	1	1	(3, 3)
(*)	41631	2	3	2	ram.	0	3	$1 + \pi$	(3 <sup>3</sup> , 3)
	41671	1	1	1	ram.	0	1	1	(3, 3)
	45210	1	1	1	ram.	0	1	1	(3, 3)
	45753	1	1	1	ram.	0	1	1	(3, 3)
	45942	1	1	1	ram.	0	1	1	(3, 3)
	46198	1	1	1	ram.	0	1	1	(3, 3)
	47199	1	1	1	ram.	0	1	1	(3 <sup>2</sup> , 3)
	48667	1	1	1	ram.	0	1	1	(3, 3)

Table 2.

$d$	$\text{ord}_E(\alpha - \beta)$	$\text{ord}_E(\beta - \gamma)$	$\text{ord}_E(\gamma - \alpha)$	$E/\mathbb{Q}_3$	$m$	$n$	$x$	$A_0$
49074	1	1	1	ram.	0	1	1	(3, 3)
51142	1	1	1	ram.	0	1	1	(3, 3)
52858	1	1	1	ram.	0	1	1	(3, 3)
53839	1	1	1	ram.	0	1	1	(3, 3)
53862	1	1	1	ram.	0	1	1	(3, 3)
54319	1	1	1	ram.	0	1	1	(3, 3)
54853	1	1	1	ram.	0	1	1	(3, 3)
56773	1	1	1	ram.	0	1	1	(3, 3)
59478	1	1	1	ram.	0	1	1	(3, 3)
59578	1	1	1	ram.	0	1	1	(3, 3)
60099	1	1	1	ram.	0	1	1	(3, 3)
(*) 64671	2	3	2	ram.	0	3	$1 + \pi$	(3 <sup>2</sup> , 3)
68314	1	1	1	ram.	0	1	1	(3, 3)
72591	1	1	1	ram.	0	1	1	(3, 3)
75273	1	1	1	ram.	0	1	1	(3, 3)
75354	1	1	1	ram.	0	1	1	(3 <sup>2</sup> , 3)
75790	1	1	1	ram.	0	1	1	(3, 3)
75841	1	1	1	ram.	0	1	1	(3, 3)
78181	1	1	1	ram.	0	1	1	(3 <sup>2</sup> , 3)
80233	1	1	1	ram.	0	1	1	(3, 3)
80242	1	1	1	ram.	0	1	1	(3 <sup>2</sup> , 3)
80746	1	1	1	ram.	0	1	1	(3, 3)
(*) 82774	1	1	1	unram.	0	1	2	(3 <sup>2</sup> , 3)
87727	1	1	1	ram.	0	1	1	(3, 3)
87979	1	1	1	ram.	0	1	1	(3 <sup>2</sup> , 3)
88134	1	1	1	ram.	0	1	1	(3 <sup>2</sup> , 3)
88242	1	1	1	ram.	0	1	1	(3, 3)
(*) 92515	1	1	1	unram.	0	1	2	(3 <sup>2</sup> , 3)
94998	1	1	1	ram.	0	1	1	(3, 3)
95691	1	1	1	ram.	0	1	1	(3, 3)
97555	1	1	1	ram.	0	1	1	(3, 3)
98277	1	1	1	ram.	0	1	1	(3, 3)
98929	1	1	1	ram.	0	1	1	(3, 3)

ACKNOWLEDGMENTS. A part of this work including Theorem 3.5 was done in my master's thesis. I am deeply grateful to Yoshitaka Hachimori for giving me valuable comments on this paper, especially Proposition 4.3. I am also grateful to Professor Masato Kurihara for valuable suggestions on the statement of Theorem 3.5 and the use of Fitting ideals for Iwasawa modules in Section 6.

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