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Decomposition of radical elements of a commutative residuated lattice

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1. Recently E. Schenkman [4] has pointed out the similarlity between the properties of ideals in a commutative ring and of normal subgroups of a group. In particular he obtained that *every radical*[1] A *of a group* G *such that* G/A *has finite principal series has a unique minimal decomposition as an intersection of primes*[2].

In the present note we shall define a radicial element of a commutative residuated *cm*-lattice[3] L, and obtain a decomposition theorem for radical elements of L, which is a lattice-formulaltion of the above result and of the minimal decomposition theorem[4] of radical ideals in (commutative) Noetherian rings.

2. Let L be a commutative residuated *cm*-lattice with a greatest element e, and suppose that $ab \leq a$ for any two elements a and b of L[5].

For example, the lattice of all normal subgroups of any group forms a commutative residuated *cm*-lattice with above properties, if we define a multiplication $A \cdot B$ of normal subgroups A and B as the subgroup generated by all commutators $xyx^{-1}y^{-1}$ $(x \in A,\ y \in B)$[6].

For any element a of L, we define inductively $a^{(1)}=a$, $a^{(\rho)}=a^{(\rho-1)} \cdot a^{(\rho-1)}$ for $\rho>1$[7]. Then we have

(1) $a \leq b$ implies $a^{(\rho)} \leq b^{(\rho)}$,

(2) $\rho \leq \sigma$ implies $a^{(\rho)} \geq a^{(\sigma)}$,

(3) $(a \cap b)^{(\rho)} \leq a^{(\rho)} \cap b^{(\rho)}$,

(4) $(a \cdot a)^{(\rho)} = a^{(\rho)} \cdot a^{(\rho)}$,

(5) $a^{(\rho)(\sigma)} = a^{(\sigma)(\rho)}$,

(6) $a^{(\rho\sigma)} \leq a^{(\rho)(\sigma)}$,

(7) $(a \cup b)^{(\rho\sigma)} \leq a^{(\rho)} \cup b^{(\sigma)}$.

(1), ⋯, (4) are immediate by induction on the whole number ρ.

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1), 2) Cf. [4, p. 376].

3) Cf. [1, p. 201]. The associative law for multiplication is not assumed.

4) Cf. [2, p. 202, Theorem 70].

5) The greatest element e is not necessarily a unity of L. If e is a unity then $ab \leq a$ for any two elements a and b of L.

6) Cf. [1, p. 204].

7) No confusion arises, even if we write $a^{\rho}=a^{(\rho)}$ for $\rho=1, 2$.

Proof of (5): Fix the whole number ρ. (5) is trivial for $\sigma=1$. Assume that $a^{(\rho)(\sigma-1)}=a^{(\sigma-1)(\rho)}$. Then $a^{(\rho)(\sigma)}=a^{(\rho)(\sigma-1)}\cdot a^{(\rho)(\sigma-1)}=a^{(\sigma-1)(\rho)}\cdot a^{(\sigma-1)(\rho)}=(a^{(\sigma-1)}\cdot a^{(\sigma-1)})^{(\rho)}$ $=a^{(\sigma)(\rho)}$. Similarly for ρ.

Proof of (6): Fix the whole number ρ. (6) is trivial for $\sigma=1$. We assume that $a^{(\rho(\sigma-1))}\leq a^{(\rho)(\sigma-1)}$. Then $a^{(\rho)(\sigma)}=a^{(\rho)(\sigma-1)}\cdot a^{(\rho)(\sigma-1)}\geq a^{(\rho(\sigma-1))}\cdot a^{(\rho(\sigma-1))}=a^{(\rho(\sigma-1)+1)}$ $\geq a^{(\rho(\sigma-1)+\rho)}=a^{(\rho\sigma)}$. Similarly for ρ.

Proof of (7): Since $(a\cup b)^{(\rho)}\leq a^{(\rho)}\cup b$, we have $(a\cup b)^{(\rho)(\sigma)}\leq(a^{(\rho)}\cup b)^{(\sigma)}\leq a^{(\rho)}\cup b^{(\sigma)}$. Hence, using (6), we obtain $(a\cup b)^{(\rho\sigma)}\leq a^{(\rho)}\cup b^{(\sigma)}$.

Let Γ_a be the set of all elements x which satisfies $a\leq x$ and $x^{(\rho)}\leq a$ for a suitable whole number $\rho=\rho(x)$.

DEFINITION. Sup $[\Gamma_a]$ *is called a radical of a, and denoted by* rad (a). *If* rad (a) $=a$, *then a is called a radical element of L.*

LEMMA 1. *In order that an element p is prime, it is necessary and sufficient that* (1°) *p is a radical element and* (2°) *p is a meet-irreducible element.*

Proof. Suppose that p is a prime element. Then it is easily verified that Γ_p consists of p only. Hence p is evidently a radical element. If $p=a\cap b$, then we have $ab\leq p$ because $ab\leq a\cap b$. Hence $a\leq p$ or $b\leq p$; and $a=p$ or $b=p$.

Conversely, if p has the properties (1°) and (2°), then $ab\leq p$ implies $(a\cup p)\cdot(b\cup p)=ab\cup ap\cup bp\cup p^2\leq p$. Since $((a\cup p)\cap(b\cup p))^2\leq(a\cup p)(b\cup p)\leq p\leq$ $(a\cup p)\cap(b\cup p)$, we have $(a\cup p)\cap(b\cup p)\leq rad(p)=p$. Hence $(a\cup p)\cap(b\cup p)=p$. This implies $a\cup p=p$ or $b\cup p=p$. That is, either $a\leq p$ or $b\leq p$.

LEMMA 2. *Let p be a prime element of L. Then $p:a$ is equal to e or equal to p according as $a\leq p$ or $a\not\leq p$.*

Proof. If $a\leq p$, then $ax\leq p$ for every element x of L. Particularly $ae\leq p$. Hence $p:a\geq e$, $p:a=e$. Since $(p:a)a\leq p$, we have $p:a\leq p$, if $a\not\leq p$. On the other hand $p\leq p:a$ is evident. Hence $p:a=p$.

LEMMA 3. *Suppose that a is a radical element of L. If $a=b\cap c$, then $a=b'\cap c'$ for any b' of Γ_b and c' of Γ_c.*

Proof. Since there exist whole numbers ρ and σ such that $b'^{(\rho)}\leq b$ and $c'^{(\sigma)}\leq c$, we have that $(b'\cap c')^{(\rho\sigma)}\leq b'^{(\rho\sigma)}\cap c'^{(\rho\sigma)}\leq b'^{(\rho)(\sigma)}\cap c'^{(\sigma)(\rho)}\leq b^{(\sigma)}\cap c^{(\rho)}\leq b\cap c=a$. It is evident that $a\leq b'\cap c'$. Hence we have that $b'\cap c'\leq rad(a)=a$, and $b'\cap c'=a$.

LEMMA 4. *Suppose that the ascending chain condition holds for the closed interval $[e, a]$. Then the radical of any element of $[e, a]$ is a radical element.*

Proof. Let c be any element of $[e, a]$. By the ascending chain condition for $[e, a]$, the lattice-ideal $J(\Gamma_c)$ generated by Γ_c of the sublattice $[e, a]$ forms a principal ideal: $J(\Gamma_c)=J(c^*)$ where $c^*=\sup[\Gamma_c]$. Since $c^*\in J(\Gamma_c)$, there exists a finite number of elements $u_1, \ldots, u_\lambda$ such that $c^*\leq u_1\cup\cdots\cup u_\lambda$, $u_i\in\Gamma_c$. Hence c^*

$=u_1\cup\cdots\cup u_\lambda$. Hence c^* is contained in Γ_c[8], i.e. $c^{*(\rho)}\leq c$ for some whole number ρ. Let x be any element of Γ_{c^*}. Then $x^{(\sigma)}\leq c^*\leq x$ for a suitable σ. Hence $x^{(\sigma\rho)}\leq x^{(\sigma)(\rho)}\leq c^{*(\rho)}\leq c\leq x$. Hence Γ_{c^*} is contained in Γ_c. We get therefore $c^*\leq sup[\Gamma_{c^*}]\leq sup[\Gamma_c]\leq sup[J(\Gamma_c)]=c^*$, $c^*=sup[\Gamma_{c^*}]=rad(c^*)$. This completes the proof.

DEFINITION. *If $a=a_1\cap\cdots\cap a_n$ where no a_ν can be omitted, then this decomposition is called minimal.*

THEOREM. *Let a be a radical element of L. If the ascending chain condition holds for the interval $[e, a]$, then a has a unique minimal decomposition of prime elements.*

Proof. It is easy to see that a can be decomposed as a meet of a finite number of meet-irreducible elements: $a=c_1\cap\cdots\cap c_n$. Then by Lemmas 3 and 4 we have

$$a=c_1^*\cap\cdots\cap c_n^*,\ c_i^*=rad(c_i).$$

If c_i^* is meet-reducible, then repeating the above arguments we obtain, after a finite number of steps, a meet-irreducible radical elements of a. That is, a can be decomposed into a finite number of prime elements: $a=p_1\cap\cdots\cap p_m$.

Now we suppose that a has two minimal decompositions of prime elements: $a=p_1\cap\cdots\cap p_m=p_1^*\cap\cdots\cap p_n^*$. Then either $m=n$ and the set of all p's coincides with the set of all p^*'s or else it is possible to pick a p_i or p_k^* which is not contained in the set of p^*'s or the set of p's respectively. For definiteness suppose it to be p_1. Then, using Lemma 2, we have[9] a: $p_1=(p_1: p_1)\cap(p_2: p_1)\cap\cdots\cap(p_m: p_1)$ $=e\cap p_2\cap\cdots\cap n_m=p_2\cap\cdots\cap p_m$. On the other hand a: $p_1=(p_1^*: p_1)\cap\cdots\cap(p_n^*: p_1)$ $=p_1^*\cap\cdots\cap p_n^*=a$. Hence $a=p_1\cap\cdots\cap p_m=p_2\cap\cdots\cap p_m$. This contradicts to the minimality of the decomposition. q. e. d.

REMARK 1. The prime elements determined by the Theorem is called the *prime elements of a*. Suppose that a is a radical element with prime elements $p_1, \dots, p_m$. If p is a prime element containing a then p contains one of the p_ν. For, since $(\cdots(p_1\cdot p_2)\cdots p_m)\leq p_1\cap p_2\cap\cdots\cap p_m\leq a\leq p$, there exists p_ν such that $p_\nu\leq p$.

REMARK 2. If $a:b=a$ then b is called *relatively prime to a*. Let a be a radical element with prime elements $p_1,\cdots,p_m$. Then in order that b is relatively prime to a, it is necessary and sufficient that b is contained in no p_ν. For, since a: $b=(p_1\cap\cdots\cap p_m)$: $b=(p_1: b)\cap\cdots\cap(p_m: b)$, we have $p_\nu:b=p_\nu$ if $a:b=a$. Hence by Lemma 2 we get $b\not\leq p_\nu$ ($\nu=1, \dots, m$). Conversely, if $b\not\leq p_\nu$, then $p_\nu: b=p_\nu$ ($\nu=1, \dots, m$). This implies $a:b=a$. Using the above results, we obtain the following:

Let a and b be two radical elements with minimal decompositions: $a=p_1\cap\cdots\cap p_m$, $b=p_1'\cap\cdots\cap p_n'$. *In order that b is relatively prime to a, it is necessary and*

8) Γ_c forms a sublattice.

9) Cf. [1, p. 202, Theorem 3].

sufficient that no p'_ν *is contained in a* p_μ.

REMARK 3. Suppose that L is associative and integral[10]. If any prime element is divisor-free, then it is verified that Lemma 2 in [3] holds for L, and we can prove the following:

Suppose that a *is a radical element such that the descending chain condition holds for* $[e, a]$. *If* $a=c_1\cap\cdots\cap c_\lambda$, *then* $a=rad(c_1)\cap\cdots\cap rad(c_\lambda)$, *where* $rad(c_\nu)$ *is a radical element of* L, $\nu=1, \ldots, \lambda$.

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10) Cf. [1, p. 202].