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THE MODULI SPACE OF TRANSVERSE CALABI–YAU STRUCTURES ON FOLIATED MANIFOLDS

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Abstract

In this paper, we develop a moduli theory of transverse structures given by calibrations on foliated manifolds, including transverse Calabi–Yau structures. We show that the moduli space of the transverse structures is a smooth manifold of finite dimension under a cohomological assumption. We also prove a local Torelli type theorem. If the foliation is taut, we can construct a Riemannian metric on the set of transverse Riemannian structures. This metric induces a distance on the moduli space of the transverse structures given by a calibration. As an application, we show the moduli space of transverse Calabi–Yau structures is a Hausdorff and smooth manifold of finite dimension.

1. Introduction

Kodaira and Spencer introduced the deformation theory of compact complex manifolds [12]. They showed that there exists a deformation of complex structures parameterized by a smooth finite dimensional space which is versal, under a cohomological assumption. Kuranishi proved a general theorem on the existence of a versal deformation space for any given complex structure, where the versal deformation space (Kuranishi space) is given by an analytic space which is not necessarily smooth [13]. Bogomolov, Tian and Todorov proved that the Kuranishi space of Calabi–Yau structures is smooth by using the Kodaira–Spencer–Kuranishi theory [1], [14] and [16]. Goto provided a deformation theory of Calabi–Yau, hyperkähler, G_2 and Spin(7) structures by a method which is different from the deformation theory of complex manifolds [10]. He considered these structures as systems of closed differential forms (called *calibrations*), and showed that deformation spaces are smooth and moduli spaces become smooth manifolds under a cohomological condition.

In the geometry of holomorphic foliations, the theory of deformations was initiated by Kodaira and Spencer. Duchamp–Kalka [4] and Gómez-Mont [9] showed a weak version of Kuranishi's theorem for deformations of transversely holomorphic foliations on compact manifolds. Girbau, Haefliger and Sundararaman constructed the Kuranishi space of deformations of transversely holomorphic foliations on compact manifolds [8]. In a

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previous paper [18], we provided a deformation theory of transverse geometric structures other than transversely holomorphic structures. We considered transverse geometric structures defined in terms of closed forms and called such closed forms *transverse calibrations*. The transverse calibrations include transverse Calabi–Yau, hyperkähler, G_2 and Spin(7) structures as examples. By modifying Goto's deformation theory, we obtained the deformation theory of transverse calibrations. We fixed a foliation on a manifold and deformed the transverse calibrations on it. One of the advantage of our approach was that we could use the Hodge theory on a foliated manifold [6]. As a result, we obtained a generalization of Moser's theorem and a smooth deformation space of transverse calibrations. El Kacimi-Alaoui, Guasp and Nicolau give a deformation theory of transversely homogeneous foliations defined by systems of 1-forms, which are not transverse calibrations [7].

In this paper, we discuss the moduli space of transverse calibrations and provide a criterion for the moduli space to be a Hausdorff and smooth manifold of finite dimension. If the foliation is taut, then we can construct a Riemannian metric on the set of transverse Riemannian structures. This result is a generalization of Ebin's results in Riemannian geometry to effect that there exists a Riemannian metric on the set of Riemannian structures on a closed manifold [5]. The metric on the set of transverse Riemannian structures induces a distance on the moduli space of transverse calibrations. As a result, the moduli space becomes Hausdorff.

Let M be a closed manifold of dimension $(p+q)$ and $\mathcal{F}$ a foliation on M of codimension q. The foliation $\mathcal{F}$ is defined by data $\{U_i,\ f_i,\ T,\ \gamma_{ij}\}$ consisting of an open covering $\{U_i\}_i$ of M, a q-dimensional transverse manifold T, submersions $f_i\colon U_i \to T$ and diffeomorphisms $\gamma_{ij}\colon\ f_i(U_i \cap U_j) \to\ f_j(U_i \cap U_j)$ for $U_i \cap U_j \neq \emptyset$ satisfying $f_j = \gamma_{ij} \circ f_i$. A *transverse structure* on $(M, \mathcal{F})$ is a geometric structure on T which is invariant by γ_{ij}. For example, a transverse Kähler structure is defined by a Kähler structure on T preserved by γ_{ij}. A foliation $\mathcal{F}$ is called transverse Kähler if there exists a transverse Kähler structure on $(M, \mathcal{F})$. On a closed manifold M with a transverse Kähler foliation $\mathcal{F}$, if the basic canonical line bundle is trivial, then there exists a transverse Calabi–Yau structure on $(M, \mathcal{F})$ by applying the basic version of Yau's theorem [6]. Remark that we can give alternative definitions for such transverse structures in terms of basic sections of basic bundles over $(M, \mathcal{F})$ (see Section 2). In particular, any transverse Calabi–Yau structure is characterized by a pair of two closed basic forms (see Definition 6.3).

We apply Goto's method to transverse structures on a foliated manifold $(M,\mathcal{F})$. Our idea is to consider basic differential forms on $(M, \mathcal{F})$ instead of differential forms on M. Let W be a q-dimensional vector space and $\bigwedge^p W^*$ the space of skew-symmetric tensor of the dual space W^*. Then the group $G = \mathrm{GL}(W)$ acts on diagonally the direct sum $\bigoplus_{i=1}^{l} \bigwedge^{p_i} W^*$. Let $\Phi_W = (\phi_1, \dots, \phi_l)$ be an element of $\bigoplus_{i=1}^{l} \bigwedge^{p_i} W^*$ and $\mathcal{O}$ $(= \mathcal{A}_{\mathcal{O}}(W))$ the G-orbit through Φ_W with an isotropy group H, so $\mathcal{O}$ is the homogeneous space G/H. On the foliated manifold $(M, \mathcal{F})$, we have a completely integrable distribution

F of dimension p and the quotient bundle $Q = TM/F$ over M. Let $\mathcal{A}_{\mathcal{O}}(M, \mathcal{F})$ be a fiber bundle $\bigcup_{x\in M} \mathcal{A}_{\mathcal{O}}(Q_x)$ and $\mathcal{E}_{\mathcal{O}}$ the set $\Gamma(M, \mathcal{A}_{\mathcal{O}}(M, \mathcal{F})) \cap \bigoplus_{i=1}^{l} \bigwedge_B^{p_i}$ of sections of $\mathcal{A}_{\mathcal{O}}(M, \mathcal{F})$ which are basic forms, where $\bigwedge_B^{p_i}$ denotes the space of basic p_i-forms on M.

DEFINITION 1.1. A system Φ of differential forms on $(M, \mathcal{F})$ is called a *transverse calibration associated with the orbit* $\mathcal{O}$ if Φ is an element of $\mathcal{E}_{\mathcal{O}}$ whose components are closed as differential forms.

Let $\tilde{\mathfrak{M}}_{\mathcal{O}}(M, \mathcal{F})$ be the set of transverse calibrations associated with $\mathcal{O}$. We denote by $\mathrm{Diff}(M, \mathcal{F})$ the group of diffeomorphisms preserving the foliation $\mathcal{F}$. We define $\mathfrak{M}_{\mathcal{O}}(M, \mathcal{F})$ to be the quotient of $\tilde{\mathfrak{M}}_{\mathcal{O}}(M, \mathcal{F})$ divided by the action of $\mathrm{Diff}_0(M, \mathcal{F})$:

$$\mathfrak{M}_{\mathcal{O}}(M, \mathcal{F}) = \tilde{\mathfrak{M}}_{\mathcal{O}}(M, \mathcal{F})/\mathrm{Diff}_0(M, \mathcal{F})$$

where $\mathrm{Diff}_0(M, \mathcal{F})$ denotes the identity component of $\mathrm{Diff}(M, \mathcal{F})$. The set $\mathfrak{M}_{\mathcal{O}}(M, \mathcal{F})$ is the moduli space of transverse calibrations associated with $\mathcal{O}$. We also give definitions of an orbit $\mathcal{O}$ being elliptic (Definition 3.1), metrical (Definition 3.4) and topological (Definition 3.6). We can consider the map $\tilde{P}\colon \tilde{\mathfrak{M}}_{\mathcal{O}}(M, \mathcal{F}) \to \bigoplus_i H_B^{p_i}(M)$ which is defined by corresponding Φ to the basic de Rham cohomology class $[\Phi]$. This map $\tilde{P}$ induces a map

$$P : \mathfrak{M}_{\mathcal{O}}(M, \mathcal{F}) \to \bigoplus_i H_B^{p_i}(M)$$

since $\mathrm{Diff}_0(M, \mathcal{F})$ acts trivially on the basic de Rham cohomology groups. The map P is called a *period map*. We assume that M is a closed oriented manifold and $\mathcal{F}$ is a Riemannian foliation. Then we can show the local Torelli type theorem:

Theorem 1.2. *If $\mathcal{O}$ is elliptic and topological, then the period map P is locally injective.*

We can also prove

Theorem 1.3. *We suppose that $\mathcal{F}$ is taut. If an orbit $\mathcal{O}$ is elliptic, metrical and topological, then the moduli space $\mathfrak{M}_{\mathcal{O}}(M, \mathcal{F})$ is a Hausdorff and smooth manifold of finite dimension.*

We can regard a transverse Calabi–Yau structure on $(M, \mathcal{F})$ as a transverse calibration associated with the orbit $\mathcal{O}_{\mathrm{CY}}$ of Calabi–Yau structures. Then we obtain

Theorem 1.4. *The moduli space of transverse Calabi–Yau structures on $(M, \mathcal{F})$ is a Hausdorff and smooth manifold of finite dimension if $\mathcal{F}$ is taut.*

This paper is organized as follows. In Section 2, we prepare some definitions and results in foliated geometry. In Section 3, we introduce transverse calibrations on $(M, \mathcal{F})$. Each transverse calibration induces a deformation complex. Then we see that the deformation complex is a subcomplex of the basic de Rham complex. In Section 4, we construct a Riemannian metric on the set of transverse Riemannian structures on $(M, \mathcal{F})$. In Section 5, we provide a sufficient condition for the moduli space to be a Hausdorff and smooth manifold (Theorem 5.6) and also show the local Torelli type theorem (Theorem 5.5). In the last section, as an application of Theorem 5.6, we prove that the moduli space of transverse Calabi–Yau structures on $(M, \mathcal{F})$ is a Hausdorff and smooth manifold (Theorem 6.5). We study some examples of transverse Calabi–Yau structures and compute the dimension of their moduli spaces.

2. Preparations on foliated geometry

In this section, we will give a brief review of some elementary results in foliated geometry. For much of this material, we refer to [6], [15] and [17]. We assume that M is a closed manifold of dimension $(p+q)$ and $\mathcal{F}$ is a foliation on M of codimension q. We denote by F a completely integrable distribution of dimension p associated to the foliation $\mathcal{F}$.

2.1. Basic vector fields and basic forms. A vector field $u \in \Gamma(TM)$ is *foliated* if $[u, v] \in \Gamma(F)$ for any $v \in \Gamma(F)$. We denote by $\Gamma(M, \mathcal{F})$ the set of foliated vector fields on $(M, \mathcal{F})$. Let $\mathfrak{X}(M, \mathcal{F})$ be the quotient space of $\Gamma(M, \mathcal{F})$ by $\Gamma(\mathcal{F})$:

$$\mathfrak{X}(M, \mathcal{F}) = \Gamma(M, \mathcal{F})/\Gamma(F).$$

We call an element u of $\mathfrak{X}(M, \mathcal{F})$ a *basic vector field* on $(M, \mathcal{F})$.

A differential k-form $\phi \in \bigwedge^k$ on M is a *basic form* on $(M, \mathcal{F})$ if the interior product $i_v\phi$ and the Lie derivative $L_v\phi$ vanish for any $v \in \Gamma(F)$. Let $\bigwedge_B^k$ be the set of basic k-forms on $(M, \mathcal{F})$:

$$\bigwedge\nolimits_B^k = \left\{\phi \in \bigwedge\nolimits^k \;\middle|\; i_v\phi = L_v\phi = 0,\ \forall v \in \Gamma(F)\right\}.$$

For a section $u \in \Gamma(TM/F)$ and a basic k-form $\phi \in \bigwedge_B^k$, the interior product $i_u\phi$ and the Lie derivative $L_u\phi$ are defined by the $(k-1)$-form $i_{\tilde{u}}\phi$ and the k-form $L_{\tilde{u}}\phi$ for a lift $\tilde{u} \in \Gamma(TM)$ of u, respectively. If u is a basic vector field, then $i_u\phi$ and $L_u\phi$ are basic forms.

We define a *foliated diffeomorphism* f as a diffeomorphism f of M preserving the foliation $\mathcal{F}$, i.e., $f_*(F) = F$. We denote by $\mathrm{Diff}(M, \mathcal{F})$ the group of foliated diffeomorphisms:

$$\mathrm{Diff}(M, \mathcal{F}) = \{ f \in \mathrm{Diff}(M) \mid f_*(F) = F\}.$$

We can define an action of Diff$(M, \mathcal{F})$ on the space of basic forms $\bigwedge_B^*$ by pull-back. For $u \in \mathfrak{X}(M, \mathcal{F})$, any lift $\tilde{u} \in \Gamma(TM)$ of u induces a one parameter family of foliated diffeomorphisms f_t. Then the Lie derivative $L_u\phi$ for $\phi \in \bigwedge_B^k$ may be regarded as the limit $(d/dt) f_t^*\phi|_{t=0}$ by the one-parameter family f_t.

2.2. Basic bundles and basic sections. Let $\iota\colon P \to M$ be a principal fiber bundle and ω a connection form on P. The horizontal subbundle H is defined by the subbundle $\operatorname{Ker}\omega$ of the tangent bundle TP. Then the derivative ι_* restricted to H is the isomorphism from H to TM. Hence we have the subbundle $\tilde{F} = \iota_*^{-1}(F)$ of H over P. If $\tilde{F}$ is integrable, then $\tilde{F}$ induces the foliation $\tilde{\mathcal{F}}$ on P.

DEFINITION 2.1. A principal fiber bundle P is *foliated* if there exists a connection form ω on P such that the bundle $\tilde{F}$ is integrable. Moreover, if the form ω is basic with respect to the induced foliation $\tilde{\mathcal{F}}$, then the bundle P is called *basic*.

A vector bundle $\pi : E \to M$ is called *foliated* (resp. *basic*) if the associated principal bundle P_E is a foliated (resp. basic) bundle. In the case $\pi : E \to M$ is a foliated vector bundle, the bundle $P_E \to M$ admits a foliation $\tilde{\mathcal{F}}$ on the total space P_E by the definition. This foliation $\tilde{\mathcal{F}}$ induces a foliation $\tilde{\mathcal{F}}_E$ on E. In addition, if E is basic then there exists a connection ∇ of E whose connection form is basic. Such a connection ∇ is called a *basic connection* on E.

DEFINITION 2.2. Let E be a basic vector bundle with a basic connection ∇. A section $s \in \Gamma(E)$ is called *basic* if $\nabla_v s = 0$ for any $v \in \Gamma(F)$.

We denote by $\Gamma_B(E)$ the set of basic sections of E. Remark that for a basic bundle E, the dual bundle E^*, exterior powers $\bigwedge^k E^*$ and symmetric covariant tensors $S^k E^*$ are also basic bundles, where k is non-negative integer. We consider a hermitian metric h on E as the section of a basic bundle. Then we call E a *basic Hermitian bundle* if the hermitian metric h is basic.

2.3. Riemannian foliations. Let Q be the normal bundle TM/F and $\pi : TM \to Q$ the natural projection. We define an action of $\Gamma(F)$ on any section $u \in \Gamma(Q)$ as follows:

$$L_v u = \pi[\tilde{u}, v] \in \Gamma(Q)$$

for any vector field $v \in \Gamma(F)$ where $\tilde{u} \in \Gamma(TM)$ is a lift of u, i.e., a vector field $u \in \Gamma(TM)$ with $\pi(\tilde{u}) = u$. This action is independent of the choice of lifts $\tilde{u} \in \Gamma(TM)$ of u. Let g be a Riemannian metric on M. Then we have an orthogonal decomposition $TM = F^\perp \oplus_g F$ and the isomorphism $\sigma\colon Q \to F^\perp$. Set a metric $g_Q = \sigma^* g_{F^\perp}$ for the induced metric $g_{F^\perp}$ on $F^\perp$. Then the map $\sigma\colon (Q, g_Q) \to (F^\perp, g_{F^\perp})$ is an isometry. Let

∇^M be the Levi-Civita connection with respect to g. Then we introduce a connection ∇ on Q as follows:

$$\nabla_v u = \begin{cases} L_v u, & v \in \Gamma(F), \\ \pi(\nabla^M_v \tilde{u}), & v \in \Gamma(F^\perp) \end{cases} \tag{1}$$

for $u \in \Gamma(Q)$, where $\tilde{u} \in \Gamma(TM)$ is a lift of u. In general, the connection (1) is not necessary basic.

A foliation $\mathcal{F}$ is Riemannian if the date $\{U_i, f_i, T, \gamma_{i,j}\}$ satisfies that T is a Riemannian manifold and each $\gamma_{i,j}$ is an isometry. A Riemannian metric g is called *bundle like* if $L_v g_Q = 0$ for any $v \in \Gamma(F)$ where the tensor $L_v g_Q \in \Gamma(S^2 Q^*)$ is defined by

$$(L_v g_Q)(u, w) = v(g_Q(u, w)) - g_Q(L_v u, w) - g_Q(u, L_v w) \tag{2}$$

for $u, w \in \Gamma(Q)$. It turns out that $\mathcal{F}$ is a Riemannian foliation if and only if there exists a bundle like Riemannian metric g on M. For a bundle like metric g, the connection ∇ in (1) is basic. Hence Q is a basic vector bundle for a Riemannian foliation $\mathcal{F}$. It is easy to see that any basic section of $\bigwedge^k Q^*$ is a basic k-form on M:

$$\bigwedge\nolimits^k_B = \Gamma_B\Big(\bigwedge\nolimits^k Q^*\Big).$$

The space $\Gamma_B(Q)$ is nothing but $\mathfrak{X}(M, \mathcal{F})$:

$$\mathfrak{X}(M, \mathcal{F}) = \Gamma_B(Q).$$

So we also call an element s of $\Gamma_B(Q)$ a basic vector field. Moreover, a basic vector field $s \in \Gamma_B(Q)$ is identified with a foliated vector field $u_s = \sigma(s) \in \Gamma(F^\perp)$ by the isomorphism σ. Therefore we have the following identification:

$$\Gamma_B(Q) \simeq \{u \in \Gamma(F^\perp) \mid [u, v] \in \Gamma(F), \forall v \in \Gamma(F)\}. \tag{3}$$

From now, we consider any basic section of Q as a vector field on M under the identification of (3). Then a basic vector field $u \in \Gamma_B(Q)$ induces a one parameter family of foliated diffeomorphisms f_t since a vector field $u \in \Gamma(F^\perp)$ associates a one parameter family of diffeomorphisms.

2.4. Transversely elliptic operators. Let E be a basic bundle of rank N. A linear map $D\colon \Gamma_B(E) \to \Gamma_B(E)$ is called a *basic differential operator of order l* if, in local coordinates $(x_1, \ldots, x_p, y_1, \ldots, y_q)$ for which $\mathcal{F}$ is given by the equations $dy_1 = \cdots = dy_q = 0$, D has the following expression:

$$D = \sum_{|s| \le l} a_s(y) \frac{\partial^{|s|}}{\partial^{s_1} y_1 \cdots \partial^{s_q} y_q}$$

where $s = (s_1, \dots, s_q) \in \mathbb{N}^q$ and each a_s is an $N \times N$-matrix valued basic function. We define the principal symbol of D at $z = (x, y)$ and the basic covector $\xi \in Q_z^*$ as the linear map $\sigma(D)(z, \xi)\colon E_z \to E_z$ given by

$$\sigma(D)(z, \xi)(\eta) = \sum_{|s|=l} \xi_1^{s_1} \cdots \xi_q^{s_q} a_s(y)(\eta)$$

for any $\eta \in E_z$.

DEFINITION 2.3. A basic differential operator D is *transversely elliptic* if $\sigma(D)(z, \xi)$ is an isomorphism for every $z \in M$ and $\xi\ (\neq 0) \in Q_z^*$.

We suppose that E is a Hermitian basic bundle with a hermitian metric h and $l = 2l'$. Then a quadratic form $A(D)(z, \xi)\colon E_z \to \mathbb{C}$ is given by

$$A(D)(z, \xi)(\eta) = (-1)^{l'} \langle \sigma(D)(z, \xi)(\eta), \eta \rangle.$$

DEFINITION 2.4. A basic differential operator D is *strongly transversely elliptic* if $A(D)(z, \xi)$ is positive definite for every $z \in M$ and every non-zero $\xi \in Q_z^*$.

Let $\{(E^k, D_k)\}_{k=0,1,\dots,q}$ be a family of Hermitian basic bundles and basic differential operators of order 1 with the differential complex

$$\cdots \xrightarrow{D_{k-1}} \Gamma_B(E^k) \xrightarrow{D_k} \Gamma_B(E^{k+1}) \xrightarrow{D_{k+1}} \cdots \tag{4}$$

where $D_k\colon \Gamma_B(E^k) \to \Gamma_B(E^k)$ for $k = 0, 1, \dots, q$. We denote by σ_k the principal symbol $\sigma(D_k)(z, \xi)$ of D_k. Then the complex (4) is *transversely elliptic* if the symbol sequence

$$\cdots \xrightarrow{\sigma_{k-1}} E_z^k \xrightarrow{\sigma_k} E_z^{k+1} \xrightarrow{\sigma_{k+1}} \cdots$$

is exact for any z and any non-zero ξ. Remark that the complex (4) is transversely elliptic if and only if the basic operator $L_k = D_k^* D_k + D_{k-1} D_{k-1}^*$ is strongly transversely elliptic, where D_k^* is the formal adjoint operator. We have the Hodge theory for the transversely elliptic complex (4) with the cohomology $H_B^k(E^*)$:

Proposition 2.5 ([6, Theorem 2.8.7]). (i) *The kernel $\mathcal{H}_B^k$ of L_k is finite dimensional and we have an orthogonal decomposition*

$$\Gamma_B(E^k) = \mathcal{H}_B^k \oplus \mathrm{Im}(D_{k-1}) \oplus \mathrm{Im}(D_k^*).$$

(ii) *The orthogonal projection $\Gamma_B(E^k) \to \mathcal{H}_B^k$ induces an isomorphism from $H_B^k(E^*)$ to $\mathcal{H}_B^k$.*

2.5. Transverse Riemannian structures. A Riemannian foliation is characterized by the following structure:

DEFINITION 2.6. A symmetric 2-tensor $\tilde{g} \in \Gamma(S^2 Q^*)$ is a *transverse Riemannian structure on* $(M, \mathcal{F})$ if $\tilde{g}$ is positive definite on Q and $L_v \tilde{g} = 0$ for any $v \in \Gamma(F)$ where $L_v \tilde{g}$ is defined by (2).

A bundle like metric g induces a transverse Riemannian structure g_Q on $(M, \mathcal{F})$. Conversely, for a transverse Riemannian structure $\tilde{g}$, we can take a bundle like metric g such that $g_Q = \tilde{g}$. Given a transverse Riemannian structure g_Q on $(M, \mathcal{F})$, then the complexification $Q \otimes \mathbb{C}$ is a basic hermitian bundle, and so $\bigwedge^k Q^* \otimes \mathbb{C}$ is. Hence from Proposition 2.5 we have

Proposition 2.7 ([6, Theorem 3.2.5]). (i) *The kernel $\mathcal{H}_B^k$ of the basic Laplacian $dd^* + d^*d$ on $\bigwedge_B^k$ is finite dimensional and we have an orthogonal decomposition*

$$\bigwedge\nolimits_B^k = \mathcal{H}_B^k \oplus \mathrm{Im}(d) \oplus \mathrm{Im}(d^*).$$

(ii) *The orthogonal projection $\bigwedge_B^k \to \mathcal{H}_B^k$ induces an isomorphism from the basic de Rham cohomology $H_B^k(M)$ to $\mathcal{H}_B^k$.*

2.6. Transverse Kähler structures. We can associate an action of $\Gamma(F)$ to any section $J \in \Gamma(\mathrm{End}(Q))$ as follows:

$$(L_v J)(u) = L_v(J(u)) - J(L_v u)$$

for $v \in \Gamma(F)$ and $u \in \Gamma(Q)$. If $J \in \Gamma(\mathrm{End}(Q))$ is a complex structure of Q, i.e. $J^2 = -\mathrm{id}_Q$, and satisfies that $L_v J = 0$ for any $v \in \Gamma(F)$, then a tensor $N_J \in \Gamma\big(\bigotimes^2 Q^* \otimes Q\big)$ can be defined by

$$N_J(u, w) = [Ju, Jw]_Q - [u, w]_Q - J[u, Jw]_Q - J[Ju, w]_Q$$

for $u, w \in \Gamma(Q)$, where $[u, w]_Q$ denotes the bracket $\pi[\tilde{u}, \tilde{w}]$ for each lift $\tilde{u}$ and $\tilde{w}$.

DEFINITION 2.8. A section $J \in \Gamma(\mathrm{End}(Q))$ is a *transverse complex structure on* $(M, \mathcal{F})$ if J is a complex structure of Q, i.e. $J^2 = -\mathrm{id}_Q$, such that $L_v J = 0$ for any $v \in \Gamma(F)$ and $N_J = 0$.

A foliation $\mathcal{F}$ is transversely holomorphic if and only if there exists a transverse complex structure on $(M, \mathcal{F})$. Thus we may regard a transverse complex structure as a generalization of complex structures on complex manifolds. A transverse complex structure J on $(M, \mathcal{F})$ give rises to the decomposition $\bigwedge_B^k \otimes \mathbb{C} = \bigoplus_{r+s=k} \bigwedge_B^{r,s}$ in the same manner as complex geometry. We denote by $H_B^{r,s}(M)$ the (r, s)-basic Dolbeault cohomology group. We provide the following remark about the integrability condition of transverse complex structure.

REMARK 2.9. Let J be a complex structure of Q such that $L_v J = 0$ for any $v \in \Gamma(F)$. Then J is a transverse complex structure, i.e. $N_J = 0$, if and only if $d\big(\bigwedge_B^{1,0}\big) \subset \bigwedge_B^{2,0} \oplus \bigwedge_B^{1,1}$, which is equivalent to $d\big(\bigwedge_B^{0,1}\big) \subset \bigwedge_B^{1,1} \oplus \bigwedge_B^{0,2}$, where d denotes the exterior derivative.

DEFINITION 2.10. A pair of sections $(\tilde{g}, J) \in \Gamma(S^2 Q^*) \times \Gamma(\mathrm{End}(Q))$ is a *transverse Kähler structure on* $(M, \mathcal{F})$ if $\tilde{g}$ is a transverse Riemannian structure and J is a transverse complex structure on $(M, \mathcal{F})$ satisfying

$$\begin{aligned}&\tilde{g}(\,\cdot\,, J\,\cdot\,) \quad \text{is a } d\text{-closed form}\\ &\tilde{g}(Ju, Jw) = \tilde{g}(u, w)\end{aligned}$$

for $u, w \in \Gamma(Q)$.

A transversely Kähler foliation $\mathcal{F}$ is defined by date $\{U_i, f_i, T, \gamma_{i,j}\}$ with a Kähler manifold T and local diffeomorphisms $\gamma_{i,j}$ preserving the Kähler structure. We remark that there exists a transverse Kähler structure on $(M,\mathcal{F})$ if and only if $\mathcal{F}$ is a transverse Kähler foliation. Given a transverse Kähler structure $(\tilde{g}, J)$, then $\bigwedge_B^k \otimes \mathbb{C}$ and $\bigwedge_B^{r,s}$ are all basic hermitian bundles. Then Proposition 2.5 implies that each basic Dolbeault cohomology group $H_B^{r,s}(M)$ is finite dimensional. Moreover, the basic de Rham–Hodge decomposition holds:

Proposition 2.11 ([6, Theorem 3.4.6]). *Let $\mathcal{F}$ be a transverse Kähler foliation on M. Then there exists an isomorphism*

$$H_B^k(M, \mathbb{C}) = \bigoplus_{r+s=k} H_B^{r,s}(M).$$

3. Transverse calibrations

3.1. Orbits in vector spaces. Let W be a vector space of dimension q. We denote by ρ the representation of $G = \mathrm{GL}(W)$ on the space $\bigoplus_{i=1}^{l} \bigwedge^{p_i} W^*$ where each $\bigwedge^{p_i} W^*$ is the space of skew-symmetric tensor of degree p_i of the dual space W^*. We fix an element $\Phi_W = (\phi_1, \ldots, \phi_l) \in \bigoplus_{i=1}^{l} \bigwedge^{p_i} W^*$ and denote by H the isotropy group of Φ_W:

$$H = \{g \in G \mid \rho_g \Phi_W = \Phi_W\}.$$

The G-orbit space $\mathcal{O} = \left\{\rho_g \Phi_W \in \bigoplus_{i=1}^{l} \bigwedge^{p_i} W^* \;\middle|\; g \in G\right\}$ through Φ_W is regarded as the homogeneous space G/H. We denote by $\mathcal{A}_{\mathcal{O}}(W)$ the G-orbit space $\mathcal{O}$:

$$\mathcal{A}_{\mathcal{O}}(W) = \left\{\rho_g \Phi_W \in \bigoplus_{i=1}^{l} \bigwedge^{p_i} W^* \;\middle|\; g \in G\right\}.$$

For an element $\Phi_0 \in \mathcal{A}_{\mathcal{O}}(W)$, the tangent space $T_{\Phi_0}\mathcal{A}_{\mathcal{O}}(W)$ is given by

$$E^1_{\Phi_0}(W) = \left\{\hat{\rho}_\xi \Phi_0 \in \bigoplus_{i=1}^{l} \bigwedge^{p_i} W^* \;\middle|\; \xi \in \mathfrak{g}\right\}$$

where $\mathfrak{g}$ is the Lie algebra of G and $\hat{\rho}$ is the differential representation of $\mathfrak{g}$. We also define vector spaces $E^0_{\Phi_0}(W)$ and $E^k_{\Phi_0}(W)$ by

$$E^0_{\Phi_0}(W) = \left\{i_v \Phi_0 = (i_v\phi_1, \dots, i_v\phi_l) \in \bigoplus_{i=1}^{l} \bigwedge^{p_i-1} W^* \;\middle|\; v \in W\right\},$$
$$E^k_{\Phi_0}(W) = \left\{\alpha \wedge i_v \Phi_0 \in \bigoplus_{i=1}^{l} \bigwedge^{p_i+k-1} W^* \;\middle|\; \alpha \in \bigwedge^k W^*,\, v \in W\right\}$$

for integers $k \geq 2$, respectively. Then we have a complex

$$(\sharp_{\Phi_0}) \qquad 0 \to E^0_{\Phi_0}(W) \xrightarrow{\wedge u} E^1_{\Phi_0}(W) \xrightarrow{\wedge u} E^2_{\Phi_0}(W) \xrightarrow{\wedge u} \cdots$$

for a form $u \in W^*$.

DEFINITION 3.1. An orbit $\mathcal{O}$ is *elliptic* if the complex $(\sharp_{\Phi_0})$ is exact for any nonzero element $u \in W^*$ at $E^1_{\Phi_0}(W)$ and $E^2_{\Phi_0}(W)$.

We give some examples of elliptic orbits. Now we assume that W is even dimensional, that is $q = 2n$.

EXAMPLE 3.2. The set of all symplectic forms on W is an orbit space $\mathcal{O}_{\mathrm{symp}}$, which is isomorphic to the quotient space $\mathrm{GL}(2n, \mathbb{R})/\mathrm{Sp}(2n, \mathbb{R})$. For any $\Phi_0 \in \mathcal{O}_{\mathrm{symp}}$, the complex $(\sharp_{\Phi_0})$ is

$$0 \to \bigwedge^1 W^* \xrightarrow{\wedge u} \bigwedge^2 W^* \xrightarrow{\wedge u} \bigwedge^3 W^* \xrightarrow{\wedge u} \cdots$$

for any element $u \in W^*$. Thus the orbit $\mathcal{O}_{\mathrm{symp}}$ is elliptic.

EXAMPLE 3.3. A non-zero complex n-form $\Omega \in \bigwedge^n \otimes \mathbb{C}$ is called an $\mathrm{SL}_n(\mathbb{C})$ *structure on* W if the form Ω satisfies that

$$W \otimes \mathbb{C} = \mathrm{Ker}_{\mathbb{C}}\, \Omega \oplus \overline{\mathrm{Ker}_{\mathbb{C}}\, \Omega}.$$

where $\mathrm{Ker}_{\mathbb{C}}\, \Omega$ denotes the space $\{v \in W \otimes \mathbb{C} \mid i_v \Omega = 0\}$. We remark that an $\mathrm{SL}_n(\mathbb{C})$ structure Ω induces a complex structure J_Ω on W defined by

$$J_\Omega(v) = \begin{cases} -\sqrt{-1}v & \text{for} \quad v \in \mathrm{Ker}_{\mathbb{C}}\, \Omega, \\ \sqrt{-1}v & \text{for} \quad v \in \overline{\mathrm{Ker}_{\mathbb{C}}\, \Omega}. \end{cases} \tag{5}$$

Then Ω is an $(n, 0)$-form with respect to the complex structure J_Ω. Let $\mathcal{O}_{\mathrm{SL}}$ be the set of $\mathrm{SL}_n(\mathbb{C})$ structures on W. Then it turns out that $\mathcal{O}_{\mathrm{SL}}$ is an orbit space such that

$$\mathcal{O}_{\mathrm{SL}} = \mathrm{GL}(2n, \mathbb{R})/\mathrm{SL}(n, \mathbb{C}).$$

For any $\Phi_0 \in \mathcal{O}_{\mathrm{SL}}$, the complex $(\sharp_{\Phi_0})$ is

$$0 \to \bigwedge^{n-1,0} W^* \xrightarrow{\wedge u} \bigwedge^{n,0} W^* \oplus \bigwedge^{n-1,1} W^* \xrightarrow{\wedge u} \bigwedge^{n,1} W^* \oplus \bigwedge^{n-1,2} W^* \xrightarrow{\wedge u} \cdots$$

for any $u \in W^*$. Here we regard the element u as an element of $\bigwedge^{1,0} W^* \oplus \bigwedge^{0,1} W^*$ such that $\bar{u} = u$. So this orbit $\mathcal{O}_{\mathrm{SL}}$ is elliptic.

DEFINITION 3.4. An orbit $\mathcal{O}$ is *metrical* if the isotropy group H is a subgroup of the orthonormal group $O(W)$ with respect to a metric g_W on W.

The above two examples $\mathcal{O}_{\mathrm{symp}}$ and $\mathcal{O}_{\mathrm{SL}}$ are not metrical. However, we may have an example of an elliptic and metrical orbit:

EXAMPLE 3.5. A pair $(\Omega, \omega) \in \bigwedge_B^n \otimes \mathbb{C} \oplus \bigwedge_B^2$ is called a *Calabi–Yau structure on* W if Ω is an $\mathrm{SL}_n(\mathbb{C})$ structure and ω is a symplectic structure on W such that

$$\begin{aligned} &\Omega \wedge \omega = \bar{\Omega} \wedge \omega = 0, \\ &\Omega \wedge \bar{\Omega} = c_n \omega^n, \\ &\omega(\,\cdot\,, J_\Omega\,\cdot\,) \quad \text{is positive definite} \end{aligned}$$

where $c_n = (1/n!)(-1)^{n(n-1)/2}(2/\sqrt{-1})^n$. Let $\mathcal{O}_{\mathrm{CY}}$ be the set of Calabi–Yau structures on W. Then $\mathcal{O}_{\mathrm{CY}}$ is an elliptic orbit such that

$$\mathcal{O}_{\mathrm{CY}} = \mathrm{GL}(2n, \mathbb{R})/\mathrm{SU}(n)$$

([10, Proposition 4.9]). Thus the orbit $\mathcal{O}_{\mathrm{CY}}$ is metrical.

3.2. Transverse calibrations in foliated manifolds. Let M be a closed manifold of dimension $p+q$ and $\mathcal{F}$ a Riemannian foliation on M of codimension q. We consider the completely integrable distribution F associated to $\mathcal{F}$ and the quotient bundle $Q = TM/F$ over M. For each $x \in M$, we identify Q_x with $W = \mathbb{R}^q$. Then, as in Section 3.1, we have an orbit $\mathcal{A}_{\mathcal{O}}(Q_x) = \mathcal{A}_{\mathcal{O}}(W)$ at $x \in M$ for an orbit $\mathcal{O}$. Note that the orbit $\mathcal{A}_{\mathcal{O}}(Q_x) \subset \bigoplus_i \bigwedge^{p_i} Q_x^*$ does not depend on the choice of the identification $h\colon Q_x \simeq W$. Then we can define G/H-bundle $\mathcal{A}_{\mathcal{O}}(M, \mathcal{F})$ by

$$\mathcal{A}_{\mathcal{O}}(M, \mathcal{F}) = \bigcup_{x \in M} \mathcal{A}_{\mathcal{O}}(Q_x) \to M.$$

Since $\mathcal{A}_{\mathcal{O}}(M, \mathcal{F}) \subset \bigoplus_i \bigwedge^{p_i} Q^*$, we can consider the Lie derivative and the exterior derivative for any section of $\mathcal{A}_{\mathcal{O}}(M,\mathcal{F})$ as a differential form. We denote by $\mathcal{E}_{\mathcal{O}}(M,\mathcal{F})$ the space of sections of $\mathcal{A}_{\mathcal{O}}(M, \mathcal{F})$ which are basic forms:

$$\mathcal{E}_{\mathcal{O}}(M, \mathcal{F}) = \Gamma(\mathcal{A}_{\mathcal{O}}(M, \mathcal{F})) \cap \bigoplus_i \bigwedge^{p_i} Q_x^*.$$

Let $\operatorname{Ker}\Phi$ be a space $\{v \in TM \mid i_v\Phi = 0\}$ for $\Phi \in \mathcal{E}_{\mathcal{O}}(M, \mathcal{F})$.

DEFINITION 3.6. A section $\Phi \in \mathcal{E}_{\mathcal{O}}(M, \mathcal{F})$ is called *a transverse calibration associated with the orbit* $\mathcal{O}$ if Φ is a closed form such that $\operatorname{Ker}\Phi = F$.

We denote by $\tilde{\mathfrak{M}}_{\mathcal{O}}(M, \mathcal{F})$ the space of transverse calibrations associated with $\mathcal{O}$. The group $\operatorname{Diff}(M, \mathcal{F})$ acts on $\tilde{\mathfrak{M}}_{\mathcal{O}}(M, \mathcal{F})$ by pull-back. Given a transverse calibration $\Phi \in \tilde{\mathfrak{M}}_{\mathcal{O}}(M, \mathcal{F})$, we can consider the vector spaces $E^k_{\Phi_x}(Q_x)$ at each point $x \in M$, and define vector bundles

$$E^k_{\Phi}(M, \mathcal{F}) = \bigcup_{x \in M} E^k_{\Phi_x}(Q_x) \to M$$

for integers $k \geq 0$. Each bundle $E^k_\Phi(M,\mathcal{F})$ is a basic bundle since its associated principal bundle is that of the normal bundle Q^*. It follows that a section $i_v\Phi \in \Gamma(E^0_\Phi(M, \mathcal{F}))$ is basic if and only if $v \in \Gamma(Q)$ is a basic section since $\operatorname{Ker}\Phi = F$ and $L_w(i_v\Phi) = i_{L_w v}\Phi$ for any $w \in \Gamma(F)$. Hence we have

$$\Gamma_B(E^0_\Phi(M, \mathcal{F})) = \left\{ i_v\Phi \in \bigoplus_{i=1}^{l} \bigwedge^{p_i - 1} Q^* \;\middle|\; v \in \Gamma_B(Q) \right\},$$

$$\Gamma_B(E^1_\Phi(M, \mathcal{F})) = \left\{ \hat{\rho}_\xi\Phi \in \bigoplus_{i=1}^{l} \bigwedge^{p_i} Q^* \;\middle|\; \xi \in \Gamma_B(\operatorname{End}(Q)) \right\}.$$

We introduce the graded vector spaces $E_\Phi(M, \mathcal{F}) = \bigoplus_k E^k_\Phi(M, \mathcal{F})$. For simplicity, we shall denote by E^k and E the spaces $E^k_\Phi(M, \mathcal{F})$ and $E_\Phi(M, \mathcal{F})$, respectively.

Proposition 3.7. *The module* $\Gamma_B(E)$ *is a differential graded module in* $\bigoplus_k\big(\bigoplus_i \bigwedge_B^{p_i+k-1}\big)$ *with respect to the derivative* d_B, *where* d_B *is the exterior derivative* d *restricted to the space of the basic forms.*

Proof. We prove that $d_B a \in \Gamma_B(E^k)$ for all $a \in \Gamma_B(E^{k-1})$. To show this, it is sufficient to prove that $d_B i_v \Phi \in \Gamma_B(E^1)$ for any element $i_v\Phi \in \Gamma_B(E^0)$, since $\Gamma_B(E)$ is generated by $\Gamma_B(E^0)$. The basic vector field v induces a one-parameter transformation $\{f_t\}$ such that each f_t is an element of $\mathrm{Diff}(M, \mathcal{F})$. Then it follows from $d\Phi = 0$ that

$$d i_v \Phi = L_v \Phi = \frac{d}{dt} f_t^* \Phi\Big|_{t=0}.$$

The right hand side $(d/dt) f_t^*\Phi|_{t=0}$ is contained in the tangent space of $\mathcal{E}_{\mathcal{O}}(M, \mathcal{F})$ at Φ since $f_t^*\Phi$ is in $\mathcal{E}_{\mathcal{O}}(M, \mathcal{F})$. Recall that the tangent space of $\mathcal{E}_{\mathcal{O}}(M, \mathcal{F})$ at Φ is the space $\Gamma_B(E^1)$. This implies that $d i_v\Phi = (d/dt) f_t^*\Phi|_{t=0} \in \Gamma_B(E^1)$. $\square$

Thus we obtain a complex

$$(\sharp_\Phi) \qquad 0 \to \Gamma_B(E^0) \xrightarrow{d_0} \Gamma_B(E^1) \xrightarrow{d_1} \Gamma_B(E^2) \xrightarrow{d_2} \cdots$$

where $d_i = d_B|_{E^i}$ for each i. The complex $(\sharp_\Phi)$ is a subcomplex of the basic de Rham complex:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_B(E^0) \arrow[r, "d_0"] \arrow[d] & \Gamma_B(E^1) \arrow[r, "d_1"] \arrow[d] & \Gamma_B(E^2) \arrow[r, "d_2"] \arrow[d] & \cdots \\
\cdots \arrow[r] & \bigoplus_i \bigwedge_B^{p_i-1} \arrow[r, "d_B"] & \bigoplus_i \bigwedge_B^{p_i} \arrow[r, "d_B"] & \bigoplus_i \bigwedge_B^{p_i+1} \arrow[r, "d_B"] & \cdots .
\end{tikzcd}$$

We denote by $H^k(\sharp_\Phi)$ the cohomology groups of the complex $(\sharp_\Phi)$:

$$H^k(\sharp_\Phi) = \{\alpha \in \Gamma_B(E^k) \mid d_k\alpha = 0\}/\{d_{k-1}\beta \in \Gamma_B(E^k) \mid \beta \in \Gamma_B(E^{k-1})\}.$$

Then we can obtain a map

$$p_\Phi^k \colon H^k(\sharp_\Phi) \to \bigoplus_i H_B^{p_i+k-1}(M)$$

for each $k \geq 0$.

DEFINITION 3.8. A section $\Phi \in \mathcal{E}_{\mathcal{O}}(M, \mathcal{F})$ is *topological* if p_Φ^1 and p_Φ^2 are injective. An orbit $\mathcal{O}$ is *topological* if any $\Phi \in \mathcal{E}_{\mathcal{O}}(M, \mathcal{F})$ is topological.

REMARK 3.9. If $\mathcal{O}$ is an elliptic orbit, then the complex $(\sharp_\Phi)$ is transverse elliptic at $\Gamma_B(E^1)$ and $\Gamma_B(E^2)$, and the operators $\Delta_\sharp^k = d_{k-1}d_{k-1}^* + d_k^* d_k$ are strongly transversely elliptic for $k = 1, 2$.

4. Riemannian metrics on the set of transverse Riemannian structures

We assume that M is a closed oriented manifold of dimension m $(= p + q)$ and $\mathcal{F}$ is a Riemannian foliation of codimension q. Let $\mathcal{M}(M, \mathcal{F})$ be the set of transverse Riemannian structures on $(M, \mathcal{F})$. We denote by $S^2 Q^*$ the bundle of symmetric covariant 2-tensors on Q.

4.1. Completions of $\mathcal{M}(M, \mathcal{F})$ and Diff$(M, \mathcal{F})$. At first, we may regard $\mathcal{M}(M, \mathcal{F})$ as a Fréchet manifold which is an open subset of the Fréchet space $\Gamma_B(S^2 Q^*)$. Now we consider the completion $\Gamma^s(S^2 Q^*)$ of $\Gamma(S^2 Q^*)$ with respect to the Sobolev norm $\| \ , \ \|_s$. This space $\Gamma^s(S^2 Q^*)$ is a Banach space (in fact, a Hilbert space), and $\Gamma^s(S^2 Q^*) \subset C^k \Gamma(S^2 Q^*)$ for $s > k + m/2$. From now, we assume that $s > 1 + m/2$. We define

$$\Gamma^s_B(S^2 Q^*) = \Gamma^s(S^2 Q^*) \cap C^1 \Gamma_B(S^2 Q^*)$$

where $C^1 \Gamma_B(S^2 Q^*)$ denotes the set $\{u \in C^1 \Gamma(S^2 Q^*) \mid L_v u = 0, \ \forall v \in \Gamma(F)\}$. Then the vector space $\Gamma^s_B(S^2 Q^*)$ is a closed subspace of $\Gamma^s(S^2 Q^*)$, so it is a Banach space. We introduce the set

$$\mathcal{M}^s_B(M, \mathcal{F}) = \{\tilde{g} \in \Gamma^s_B(S^2 Q^*) \mid \tilde{g}\colon \text{positive definite}\}.$$

Then the set $\mathcal{M}^s_B(M, \mathcal{F})$ is an open subset of the Banach space $\Gamma^s_B(S^2 Q^*)$, and so a Banach manifold.

Secondly, we study the properties of the set $\mathrm{Diff}(M, \mathcal{F})$. In [19], Omori shows that $\mathrm{Diff}(M, \mathcal{F})$ is an ILB-Lie group with the model $\{\Gamma(M, \mathcal{F}), \Gamma^s(M, \mathcal{F}),\ s \geq 1\}$. Then we may obtain a Banach manifold $\mathrm{Diff}^s(M, \mathcal{F})$ with the model $\Gamma^s(M, \mathcal{F})$ for each $s \geq 1$. The group $\mathrm{Diff}(M, \mathcal{F})$ acts on $\mathcal{M}(M, \mathcal{F})$ by pull-back. This action naturally extends that of $\mathrm{Diff}^{s+1}(M, \mathcal{F})$ on $\mathcal{M}^s(M, \mathcal{F})$. Then we prove that

Proposition 4.1. *The action of* $\mathrm{Diff}^{s+1}(M, \mathcal{F})$ *on* $\mathcal{M}^s(M, \mathcal{F})$ *is continuous.*

Proof. Let $\mathcal{M}(M)$ be the set of Riemannian metrics on M. The group $\mathrm{Diff}(M)$ acts on $\mathcal{M}(M)$ by pull-back. We use the fact that the action of $\mathrm{Diff}(M)$ on $\mathcal{M}(M)$ can be extended to continuous one of $\mathrm{Diff}^{s+1}(M)$ on $\Gamma^s(S^2 T^* M)$, which is proved by Ebin in [5]. Let $\tilde{A}$ denote the extended action, that is, the continuous map

$$\tilde{A}\colon \mathrm{Diff}^{s+1}(M) \times \Gamma^s(S^2 T^* M) \to \Gamma^s(S^2 T^* M). \tag{6}$$

Now the inclusions $\mathrm{Diff}^{s+1}(M, \mathcal{F}) \subset \mathrm{Diff}^{s+1}(M)$ and $\Gamma^s_B(S^2 Q^*) \subset \Gamma^s(S^2 T^* M)$ are continuous. Hence we have a continuous map

$$\mathrm{Diff}^{s+1}(M, \mathcal{F}) \times \Gamma^s_B(S^2 Q^*) \to \Gamma^s(S^2 T^* M)$$

by restricting the map $\tilde{A}$ to $\mathrm{Diff}^{s+1}(M, \mathcal{F})$ and $\Gamma^s_B(S^2Q^*)$. The image of this map is in $\Gamma^s_B(S^2Q^*)$ and the topology of $\Gamma^s_B(S^2Q^*)$ coincides with the relative topology as a subspace of $\Gamma^s(S^2T^*M)$. Hence we obtain a continuous map

$$A\colon \mathrm{Diff}^{s+1}(M, \mathcal{F}) \times \Gamma^s_B(S^2Q^*) \to \Gamma^s_B(S^2Q^*).$$

This map A induces the continuous action of $\mathrm{Diff}^{s+1}(M, \mathcal{F})$ on $\mathcal{M}^s(M, \mathcal{F})$. This finishes the proof. □

For any element $\Phi \in \Gamma^s_B(S^2Q^*)$, we define a map

$$A_\Phi\colon \mathrm{Diff}^{s+1}(M, \mathcal{F}) \to \Gamma^s_B(S^2Q^*)$$

by $A_\Phi(\,\cdot\,) = A(\,\cdot\,, \Phi)$. Then we have the

Proposition 4.2. *If Φ is a smooth element of $\Gamma_B(S^2Q^*)$, then $A_\Phi\colon \mathrm{Diff}^{s+1}(M,\mathcal{F}) \to \Gamma^s_B(S^2Q^*)$ is a smooth map.*

Proof. The map $\tilde{A}$ given by (6) induces the smooth map

$$\tilde{A}_\phi\colon \mathrm{Diff}^{s+1}(M) \to \Gamma^s(S^2T^*M)$$

for a smooth element $\Phi \in \Gamma(S^2T^*M)$ (cf. [5, p. 18]). Since any smooth element Φ of $\Gamma_B(S^2Q^*)$ can be regarded as smooth one of $\Gamma(S^2T^*M)$, the map $\tilde{A}_\phi$ is smooth for any element $\Phi \in \Gamma_B(S^2Q^*)$. By restricting $\tilde{A}_\phi$ to $\mathrm{Diff}^{s+1}(M, \mathcal{F})$, we consider the map

$$\tilde{A}_\phi|_{\mathrm{Diff}^{s+1}(M,\mathcal{F})}\colon \mathrm{Diff}^{s+1}(M, \mathcal{F}) \to \Gamma^s(S^2T^*M).$$

Then this map $\tilde{A}_\phi|_{\mathrm{Diff}^{s+1}(M,\mathcal{F})}$ is smooth since $\mathrm{Diff}^{s+1}(M, \mathcal{F})$ is a Banach submanifold of $\mathrm{Diff}^{s+1}(M)$. The image of $\tilde{A}_\phi|_{\mathrm{Diff}^{s+1}(M,\mathcal{F})}$ is in $\Gamma_B(S^2Q^*)$ which is a Banach subspace of $\Gamma^s(S^2T^*M)$. Thus we can get a smooth map

$$\tilde{A}_\phi|_{\mathrm{Diff}^{s+1}(M,\mathcal{F})}\colon \mathrm{Diff}^{s+1}(M, \mathcal{F}) \to \Gamma_B(S^2Q^*). \tag{7}$$

The smooth map (7) is nothing but the map A_Φ, which completes the proof. □

4.2. Riemannian structures on $\mathcal{M}^s_B(M, \mathcal{F})$. We assume the $s > 1 + m/2$. We recall that $\mathcal{M}^s_B(M, \mathcal{F})$ is a Banach manifold whose tangent space is identified with $\Gamma^s_B(S^2Q^*)$. Each element $\gamma \in \mathcal{M}^s_B(M,\mathcal{F})$ induces a metric $\langle\ ,\ \rangle_\gamma$ on S^2Q^* and a transverse volume form μ_γ on $(M, \mathcal{F})$. For elements α and β of $\Gamma^s_B(S^2Q^*)$ $(= T_\gamma M)$, we obtain the basic q-form $\langle \alpha, \beta \rangle_\gamma \mu_\gamma$ on $(M, \mathcal{F})$. To integrate this form, we need a volume form along the foliation. Fix a bundle like metric g on $(M, \mathcal{F})$. Then a *characteristic form* $\chi_{\mathcal{F}}$ is defined by

$$\chi_{\mathcal{F}}(X_1, \dots, X_p) = \det(g(X_i, e_j)_{i,j})$$

for $X_i \in \Gamma(TM)$, where $\{e_j\}_{j=1,\dots,p}$ is an orthonormal basis of F with respect to g.

Now we define the Riemannian structure $(\ ,\ )_\gamma$ on $\mathcal{M}^s_B(M, \mathcal{F})$ as follows.

$$(\alpha, \beta)_\gamma = \int_M \langle \alpha, \beta \rangle_\gamma \mu_\gamma \wedge \chi_{\mathcal{F}}$$

for any $\alpha, \beta \in \Gamma^s_B(S^2 Q^*)$. The bilinear form $(\ ,\ )_\gamma$ is positive definite and smooth for $\gamma \in \mathcal{M}^s_B(M, \mathcal{F})$. However, for any $s > 0$ the space $\Gamma^s_B(S^2 Q^*)$ is not complete under $(\ ,\ )_\gamma$. We denote the inner product $(\ ,\ )_\gamma$ by $(\ ,\ )^0_\gamma$. We can find a unique affine connection ∇ on $\mathcal{M}^s_B(M, \mathcal{F})$ by a similar argument in p. 19 of [5]. Then the connection ∇ associates an isomorphism

$$D^s \colon J^s_B(S^2 Q^*) \to \bigoplus_{i=0}^{s} S^i Q^* \otimes S^2 Q^*$$

where $J^s_B(S^2 Q^*)$ is a basic jet bundle ([6, Theorem 2.3.6]). For $\gamma \in \mathcal{M}^s_B(M, \mathcal{F})$, we have the positive definite bilinear form on $\bigoplus_{i=0}^s S^i Q^* \otimes S^2 Q^*$ induced by $(\ ,\ )^0_\gamma$. Hence, under the isomorphism D^s, we obtain a positive definite bilinear form $(\ ,\ )^s_\gamma$ on $\Gamma^s_B(S^2 Q^*)$ $(= T_\gamma \mathcal{M}^s_B(M, \mathcal{F}))$. Then the space $\Gamma^s_B(S^2 Q^*)$ is complete under $(\ ,\ )^s_\gamma$ for each $\gamma \in \mathcal{M}^s_B(M, \mathcal{F})$ (cf. [5, p. 21]). Thus we obtain the Riemannian metric $(\ ,\ )^s$ on $\mathcal{M}^s_B(M, \mathcal{F})$.

4.3. $\mathrm{Diff}_0^{s+1}(M, \mathcal{F})$-invariant Riemannian structures. Let $\mathrm{Diff}_0^{s+1}(M, \mathcal{F})$ be the identity component of $\mathrm{Diff}^{s+1}(M, \mathcal{F})$. In previous section, we construct a Riemannian structure $(\ ,\ )^s$ on $\mathcal{M}^s_B(M, \mathcal{F})$. In general, the structure $(\ ,\ )^s$ is not $\mathrm{Diff}_0^{s+1}(M, \mathcal{F})$-invariant. We will show that this structure $(\ ,\ )^s$ is $\mathrm{Diff}_0^{s+1}(M, \mathcal{F})$-invariant if $\mathcal{F}$ is a taut foliation.

DEFINITION 4.3. A foliation $\mathcal{F}$ is called *taut* if there exists a Riemannian metric g on M such that each leaf of $\mathcal{F}$ is a minimal submanifold of (M, g).

In this case, the Poincare duality holds on basic de Rham cohomology groups, i.e., if $\mathcal{F}$ is taut then there exists a non-degenerate pairing: $H^r_B(M) \otimes H^{q-r}_B(M) \to \mathbb{R}$ induced by the integral $\int_M \alpha \wedge \beta \wedge \chi_{\mathcal{F}}$ for $\alpha \in \bigwedge^r_B$ and $\beta \in \bigwedge^{q-r}_B$ ([15, Corollary 7.58]). It implies that

$$\int_M d\alpha \wedge \chi_{\mathcal{F}} = 0 \tag{8}$$

for any $\alpha \in \bigwedge_B^{q-1}$. We can prove

Proposition 4.4. *If $\mathcal{F}$ is taut, then the Riemannian structure* $(\ ,\ )^s$ *is* $\mathrm{Diff}_0^{s+1}(M, \mathcal{F})$-*invariant.*

Proof. At first, we check that $(\ ,\ )^0$ is $\mathrm{Diff}_0^{s+1}(M, \mathcal{F})$-invariant. For any $\phi \in \mathrm{Diff}_0^{s+1}(M, \mathcal{F})$ and $\eta \in \bigwedge_B^q$, we have

$$\phi^*\eta = \eta + K(d\eta) + dK(\eta) = \eta + dK(\eta) \tag{9}$$

where K is the homotopy operator associated to $\phi \in \mathrm{Diff}_0^{s+1}(M, \mathcal{F})$. Remark that the form $K(\eta)$ is basic for any basic form $\eta \in \bigwedge_B^q$. It follows from (8) and (9) that

$$\begin{aligned} \int_M (\phi^*\eta) \wedge \chi_{\mathcal{F}} &= \int_M (\eta + dK(\eta)) \wedge \chi_{\mathcal{F}} \\ &= \int_M \eta \wedge \chi_{\mathcal{F}}. \end{aligned}$$

It gives rise that

$$\begin{aligned} (\phi^*\alpha, \phi^*\beta)^0_{\phi^*\gamma} &= \int_M \langle \phi^*\alpha, \phi^*\beta\rangle_{\phi^*\gamma}\mu_{\phi^*\gamma} \wedge \chi_{\mathcal{F}} \\ &= \int_M \phi^*(\langle \alpha, \beta\rangle_\gamma \mu_\gamma) \wedge \chi_{\mathcal{F}} \\ &= \int_M \langle \alpha, \beta\rangle_\gamma \mu_\gamma \wedge \chi_{\mathcal{F}} \\ &= (\alpha, \beta)^0_\gamma. \end{aligned}$$

Hence ϕ preserves the structure $(\ ,\ )^0$ on $\mathcal{M}_B^s(M, \mathcal{F})$.

Secondly, we consider the metric $(\ ,\ )^s$ for $s > 0$. The connection ∇ in Subsection 4.2 satisfies

$$\phi^*(\nabla_X Y) = \nabla_{\phi^* X}\phi^* Y \tag{10}$$

for vector fields $X, Y \in T\mathcal{M}_B^s(M, \mathcal{F})$ since $(\ ,\ )^0$ is $\mathrm{Diff}_0^{s+1}(M, \mathcal{F})$-invariant. Equation (10) gives rise to

$$\phi^* \circ D^s = D^s \circ \phi^*.$$

Thus the action of ϕ commutes with the isomorphism D^s. By the definition of $(\ ,\ )^s$, the metric $(\ ,\ )^s$ is $\mathrm{Diff}_0^{s+1}(M, \mathcal{F})$-invariant since $(\ ,\ )^0$ is $\mathrm{Diff}_0^{s+1}(M, \mathcal{F})$-invariant. Hence we finish the proof. □

5. Moduli spaces of transverse calibrations

In this section, we provide a sufficiently condition for a moduli space $\mathfrak{M}_{\mathcal{O}}(M, \mathcal{F})$ of transverse calibrations to be a Hausdorff and smooth manifold. Moreover, we show a local Torelli type theorem for transverse calibrations. We assume that the manifold M is closed oriented and $\mathcal{F}$ is a Riemannian foliation.

5.1. Local coordinates of $\mathfrak{M}_{\mathcal{O}}(M,\mathcal{F})$. Let Φ be an element of $\mathfrak{M}_{\mathcal{O}}(M,\mathcal{F})$. We suppose that $\mathcal{O}$ is an elliptic orbit and p^2_Φ is injective. We consider a formal power series $a(t)$ in t:

$$a(t) = a_1 t + \frac{1}{2!}a_2 t^2 + \frac{1}{3!}a_3 t^3 + \cdots \in \Gamma_B(\mathrm{End}(Q))[[t]]$$

where each a_k is a basic section of $\mathrm{End}(Q)$. Then we obtain a formal power series

$$e^{a(t)} = \exp a(t) \in \Gamma_B(\mathrm{GL}(Q))[[t]].$$

Let $\mathbb{H}^1(\sharp_\Phi)$ be $\Delta^1_\sharp$-harmonic elements $\{\alpha \in \Gamma_B(E^1_\Phi) \mid \Delta^1_\sharp \alpha = 0\}$ where $\Delta^1_\sharp$ is the operator $d_0\, d_0^* + d_1^* d_1$. From Theorem 4.2 in [18], for an element $\hat{\rho}_{a_1}\Phi \in \mathbb{H}^1(\sharp_\Phi)$ there exists a smooth form $\rho_{e^{a(t)}}\Phi \in \tilde{\mathfrak{M}}_{\mathcal{O}}(M, \mathcal{F})$. Hence we have a map

$$\begin{aligned} \tilde{\kappa} : \mathbb{H}^1(\sharp_\Phi) &\to \tilde{\mathfrak{M}}_{\mathcal{O}}(M, \mathcal{F}) \\ \hat{\rho}_{a_1}\Phi &\mapsto \rho_{e^{a(1)}}\Phi \end{aligned}$$

where $e^{a(1)}$ is the value of the $e^{a(t)}$ at $t = 1$. We denote by π the projection:

$$\tilde{\mathfrak{M}}_{\mathcal{O}}(M, \mathcal{F}) \to \mathfrak{M}_{\mathcal{O}}(M, \mathcal{F}) = \tilde{\mathfrak{M}}_{\mathcal{O}}(M, \mathcal{F})/\mathrm{Diff}_0(M, \mathcal{F})$$

and consider the composition map

$$\kappa = \pi \circ \tilde{\kappa} : \mathbb{H}^1(\sharp_\Phi) \to \mathfrak{M}_{\mathcal{O}}(M, \mathcal{F}).$$

The map κ maps the origin of $\mathbb{H}^1(\sharp_\Phi)$ to the class of Φ in $\mathfrak{M}_{\mathcal{O}}(M, \mathcal{F})$.

Proposition 5.1. *If $p^1_\Phi \colon H^1(\sharp_\Phi) \to \bigoplus_i H^{p_i}_B(M)$ is injective, then there exists an open neighborhood S of the origin in $\mathbb{H}^1(\sharp_\Phi)$ such that $\kappa|_S \colon S \to \mathfrak{M}_{\mathcal{O}}(M,\mathcal{F})$ is injective.*

Proof. We define a map $P \colon \mathfrak{M}_{\mathcal{O}}(M,\mathcal{F}) \to \bigoplus_i H^{p_i}_B(M)$ by $P(\Phi) = [\Phi] \in \bigoplus_i H^{p_i}_B(M)$ for any $\Phi \in \mathfrak{M}_{\mathcal{O}}(M, \mathcal{F})$ and consider the composition

$$P \circ \kappa : \mathbb{H}^1(\sharp_\Phi) \to \bigoplus_i H^{p_i}_B(M).$$

Then the differential of $P \circ \kappa$ at the origin is given by the map p^1. Since p^1 is injective, there exists a small neighbourhood $S \subset \mathbb{H}^1(\sharp_\Phi)$ of the origin such that the restriction $P \circ \kappa|_S\colon S \to \bigoplus_i H_B^{p_i}(M)$ is injective. Hence $\kappa|_S\colon S \to \mathfrak{M}_{\mathcal{O}}(M, \mathcal{F})$ is also injective. $\square$

Let $\mathcal{E}^s$ be the set $C^1\Gamma(A_H(M, \mathcal{F})) \cap \Gamma^{s+1}\big(\bigoplus_i \bigwedge_B^{p_i}\big)$. Then $\mathrm{Diff}_0^{s+1}(M, \mathcal{F})$ acts on the set $\mathcal{E}^s$ by the pull-back. If we give a vector field $\xi \in \Gamma_B^{s+1}(Q)$ and the diffeomorphism $f_\xi \in \mathrm{Diff}_0^{s+1}(M, \mathcal{F})$ associated by ξ. Then, for $\rho_{e^a}\Phi \in \mathcal{E}^s$, there exists a section $b_\xi \in \Gamma_B(\mathrm{End}(Q))$ such that

$$f_\xi^* \rho_{e^a}\Phi = \rho_{e^{b_\xi}}\Phi \tag{11}$$

since the set $\mathcal{E}^s$ is $\mathrm{Diff}_0^{s+1}(M, \mathcal{F})$-invariant. For a d-closed element $\rho_{e^a}\Phi$, we can choose ξ such that $\hat{\rho}_{b_\xi}\Phi$ is in $\mathbb{H}^1(\sharp_\Phi)$:

Lemma 5.2. *If $\rho_{e^a}\Phi$ is an element of $\mathfrak{M}_{\mathcal{O}}(M, \mathcal{F})$ with $\|a\|_s < \epsilon$ for sufficiently small $\epsilon > 0$, then there exists a C^∞-vector field $\xi \in \Gamma_B(Q)$ satisfying $\hat{\rho}_{b_\xi}\Phi \in \mathbb{H}^1(\sharp_\Phi)$.*

Proof. We assume that a vector field $\xi \in \Gamma_B^{s+1}(Q)$ and the diffeomorphism $f_\xi \in \mathrm{Diff}_0^{s+1}(M, \mathcal{F})$ are given as in (11) for $\rho_{e^a}\Phi \in \mathfrak{M}_{\mathcal{O}}(M, \mathcal{F})$. It is sufficient to show that there exists a vector field ξ satisfying

$$d_0^* \hat{\rho}_{b_\xi}\Phi = 0. \tag{12}$$

Note that $\hat{\rho}_{b_\xi}\Phi = \rho_{e^{b_\xi}}\Phi - \Phi - \sum_{k\geq 2}(1/k!)\hat{\rho}_{b_\xi}^k\Phi$ and $f_\xi^*\rho_{e^a}\Phi = \rho_{e^a}\Phi + L_\xi\rho_{e^a}\Phi + \cdots = \Phi + \hat{\rho}_a\Phi + di_\xi\Phi + \tilde{W}(\xi, a)$ where $\tilde{W}(\xi, a)$ is the higher order term with respect to ξ and a. Therefore we obtain

$$\hat{\rho}_{b_\xi}\Phi = \hat{\rho}_a\Phi + di_\xi\Phi + W(\xi, a) \tag{13}$$

where $W(\xi, a)$ is defined by the higher term $\tilde{W}(\xi, a) - \sum_{k\geq 2}(1/k!)\hat{\rho}_{b_\xi}^k\Phi$. We remark that $W(\xi, a)$ is an element of E_Φ^1 and satisfies

$$\|W(\xi_2, a) - W(\xi_1, a)\|_s < \epsilon\|\xi_2 - \xi_1\|_{s+1} \tag{14}$$

for sufficiently small ξ_1, ξ_2, a and a positive constant $\epsilon < 1$ (see [10, Lemma 3.3]). We choose a vector field ξ such that the harmonic part of $i_\xi\Phi \in E_\Phi^0$ vanishes. Then it follows from equation (13) that $d_0^*\hat{\rho}_{b_\xi}\Phi = 0$ is equivalent to

$$i_\xi\Phi + G_\sharp\, d_0^*\hat{\rho}_a\Phi + G_\sharp\, d_0^* W(\xi, a) = 0 \tag{15}$$

where $G_\sharp$ is Green's operator of the complex $(\sharp_\Phi)$. Now we can take $\xi_1 \in \Gamma_B(Q)$ satisfying

$$i_{\xi_1}\Phi = -d_0^* G_\sharp \hat{\rho}_a\Phi.$$

Inductively, we define $\xi_k \in \Gamma_B(Q)$ for $k \geq 2$ as follows

$$i_{\xi_k}\Phi = -G_\sharp\, d_0^*\hat{\rho}_a\Phi - G_\sharp\, d_0^* W(\xi_{k-1}, a).$$

From the estimate $\|\xi_k\|_{s+1} = C\|i_{\xi_k}\Phi\|_{s+1}$ for a constant C, it follows that

$$\|\xi_{k+1} - \xi_k\|_{s+1} \leq C\|W(\xi_{k+1}, a) - W(\xi_k, a)\|_s.$$

By taking a sufficiently small $\epsilon < 1$ in (14), we have

$$\|\xi_{k+1} - \xi_k\|_{s+1} \leq \epsilon\|\xi_k - \xi_{k-1}\|_{s+1}.$$

Therefore the sequence $\{\xi_k\}_k$ converges uniformly to a vector field $\xi_\infty \in \Gamma_B^{s+1}(Q)$ with respect to the norm $\|\cdot\|_{s+1}$. It turns out that ξ_∞ satisfies equation (15). Hence ξ_∞ is in $\Gamma_B^{s+1}(Q)$ and satisfies (12). This completes the proof. □

From this Lemma 5.2, we immediately show the following proposition.

Proposition 5.3. *There exists an open neighborhood U_Φ of $\pi(\Phi)$ in $\mathfrak{M}_{\mathcal{O}}(M, \mathcal{F})$ such that $\kappa|_{S_\Phi}\colon S_\Phi \to U_\Phi$ is surjective for a small open neighbourhood $S_\Phi \subset S$ of the origin in $\mathbb{H}^1(\sharp_\Phi)$.*

Proof. If we define an open neighbourhood U_Φ of $\pi(\Phi)$ by

$$U_\Phi = \pi(\{\rho_{e^a}\Phi \in \tilde{\mathfrak{M}}_{\mathcal{O}}(M, \mathcal{F}) \mid \|a\|_s < \epsilon\})$$

for a small constant ϵ as in Lemma 5.2, then for any $\rho_{e^a}\Phi \in U_\Phi$, there exists an element $\hat{\rho}_{b_\xi}\Phi \in \mathbb{H}^1(\sharp_\Phi)$ such that $\tilde{\kappa}(\hat{\rho}_{b_\xi}\Phi) = \rho_{e^a}\Phi$. Hence κ is surjective. □

5.2. Distance on $\mathfrak{M}_{\mathcal{O}}(M, \mathcal{F})$. We assume that the orbit $\mathcal{O}$ is elliptic and topological. We construct a distance on the moduli space $\mathfrak{M}_{\mathcal{O}}(M, \mathcal{F})$.

Proposition 5.4. *We suppose that $\mathcal{F}$ is taut. If the orbit $\mathcal{O}$ is metrical, then there exists a distance on $\mathfrak{M}_{\mathcal{O}}(M, \mathcal{F})$.*

Proof. Since $\mathcal{O}$ is metrical, for each $\Phi \in \mathcal{E}^s$ there exists a metric g_Φ on Q. The metric g_Φ induces an L^2-metric on the tangent space $T_\Phi\mathcal{E}^s \subset \Gamma^s\big(\bigoplus_i \bigwedge_B^{p_i}\big)$. Hence we obtain a Riemannian metric $(\ ,\ )^s$ on $\mathcal{E}^s$ which is $\mathrm{Diff}_0^{s+1}(M, \mathcal{F})$-invariant by using the same argument in Proposition 4.4. Then $\tilde{\mathfrak{M}}^s_{\mathcal{O}}(M, \mathcal{F}) = \mathcal{H} \cap T_\Phi\mathcal{E}^s$ admits a distance $\tilde{d}$ given by the Riemannian structure of $\mathcal{E}^s$. Now we define a function $d\colon \mathfrak{M}_{\mathcal{O}}(M, \mathcal{F}) \times \mathfrak{M}_{\mathcal{O}}(M, \mathcal{F}) \to \mathbb{R}$ by

$$d(\pi(\Phi), \pi(\Phi')) = \inf_{f, g \in \mathrm{Diff}_0(M, \mathcal{F})} \tilde{d}(f^*\Phi, g^*\Phi')$$

for $\Phi, \Phi' \in \tilde{\mathfrak{M}}_{\mathcal{O}}(M, \mathcal{F})$, where π is the projection $\tilde{\mathfrak{M}}_{\mathcal{O}}(M, \mathcal{F}) \to \mathfrak{M}_{\mathcal{O}}(M, \mathcal{F})$. We remark that $d(\pi(\Phi), \pi(\Phi')) = \inf_{f \in \mathrm{Diff}_0(M,\mathcal{F})} \tilde{d}(\Phi, f^*\Phi')$.

We shall see that d is a distance on $\mathfrak{M}_{\mathcal{O}}(M, \mathcal{F})$. The triangle inequality holds:

$$\begin{aligned} d(\pi(\Phi), \pi(\Phi'')) &= \inf_{f,g} \tilde{d}(f^*\Phi, g^*\Phi'') \\ &\le \inf_{f,g} (\tilde{d}(f^*\Phi, \Phi') + \tilde{d}(\Phi', g^*\Phi'')) \\ &= \inf_{f} \tilde{d}(f^*\Phi, \Phi') + \inf_{g} \tilde{d}(\Phi', g^*\Phi'') \\ &= d(\pi(\Phi), \pi(\Phi')) + d(\pi(\Phi'), \pi(\Phi'')). \end{aligned}$$

To show the positivity of d, we suppose that $d(\pi(\Phi), \pi(\Phi')) = 0$ for $\Phi, \Phi' \in \mathfrak{M}_{\mathcal{O}}(M, \mathcal{F})$, that is, $\inf_{f \in \mathrm{Diff}_0(M,\mathcal{F})} \tilde{d}(\Phi, f^*\Phi') = 0$. Then there exists a sequence $\{f_j\}_{j \in \mathbb{N}}$ of $\mathrm{Diff}_0(M,\mathcal{F})$ such that

$$\tilde{d}(\Phi, f_j^*\Phi') \to 0, \quad (j \to \infty).$$

This implies that

$$\|\Phi - f_j^*\Phi'\|_{L^2} \to 0, \quad (j \to \infty) \tag{16}$$

since $\tilde{d}$ is locally equivalent to the L^2-metric induced by g_Φ. It follows from (16) that

$$[\Phi - f_j^*\Phi'] \to 0 \in \bigoplus_i H_B^{p_i}(M), \quad (j \to \infty) \tag{17}$$

where $[\Phi - f_j^*\Phi']$ is the basic cohomology class of $\Phi - f_j^*\Phi'$. Since $\mathrm{Diff}_0(M, \mathcal{F})$ acts trivially on the basic cohomology group $H_B^{p_i}(M)$, the cohomology class $[\Phi - f_j^*\Phi']$ is $[\Phi] - [\Phi']$, and independent of j. Hence, it follows from (17) that $[\Phi] - [\Phi']$ must be zero, so we obtain $[\Phi] = [\Phi'] \in \bigoplus_i H_B^{p_i}(M)$. We may assume that $\pi(\Phi')$ is included in an open set U_Φ given as in Proposition 5.3. Remark that the period map $P|_{U_\Phi}$ restricted to U_Φ is injective since $P \circ \kappa|_{S_\Phi} \colon S_\Phi \to \bigoplus_i H_B^{p_i}(M)$ is injective and $\kappa|_{S_\Phi} \colon S_\Phi \to U_\Phi$ is isomorphic (see Propositions 5.1 and 5.3). Now we have $P(\pi(\Phi)) = [\Phi] = [\Phi'] = P(\pi(\Phi'))$. Hence $\pi(\Phi) = \pi(\Phi') \in \mathfrak{M}_{\mathcal{O}}(M, \mathcal{F})$. Thus d is a distance on $\mathfrak{M}_{\mathcal{O}}(M, \mathcal{F})$. □

5.3. Main theorems. We recall that the period map

$$P : \mathfrak{M}_{\mathcal{O}}(M, \mathcal{F}) \to \bigoplus_{i=1}^{l} H_B^{p_i}(M)$$

is induced by taking the basic de Rham cohomology class $[\Phi]$. We can show the local Torelli type theorem:

Theorem 5.5. *If $\mathcal{O}$ is elliptic and topological, then the period map P is locally injective.*

Proof. It follows from Propositions 5.1 and 5.3 that a small open set of $\mathfrak{M}_{\mathcal{O}}(M,\mathcal{F})$ is isomorphic to an open neighbourhood S of origin in $\mathbb{H}^1(\sharp_\Phi)$ by the map $\kappa\colon \mathbb{H}^1(\sharp_\Phi) \to \mathfrak{M}_{\mathcal{O}}(M,\mathcal{F})$. Thus it is sufficient to show that the composition map $P \circ \kappa|_S$ is injective for a small open set S. However, as in the proof of Proposition 5.3, there exists a small open set S such that $P \circ \kappa|_S$ is injective, and the proof is finished. $\square$

We prove the main theorem:

Theorem 5.6. *We suppose that $\mathcal{F}$ is taut. If $\mathcal{O}$ is metrical, elliptic and topological, then the moduli space $\mathfrak{M}_{\mathcal{O}}(M,\mathcal{F})$ is a Hausdorff and smooth manifold.*

Proof. If $\mathcal{O}$ is elliptic and topological, Propositions 5.1 and 5.3 implies that the moduli space $\mathfrak{M}_{\mathcal{O}}(M,\mathcal{F})$ admits local coordinates given by the open neighbourhood of the origin in $H^1(\sharp_\Phi)$. The dimension $\dim H^1(\sharp_\Phi)$ is not independent of Φ in a connected component of $\mathfrak{M}_{\mathcal{O}}(M,\mathcal{F})$ by Proposition 5.1 in [18]. Thus $\mathfrak{M}_{\mathcal{O}}(M,\mathcal{F})$ is a manifold. In addition, if $\mathcal{O}$ is metrical then $\mathfrak{M}_{\mathcal{O}}(M,\mathcal{F})$ has a distance as in Proposition 5.4. Hence $\mathfrak{M}_{\mathcal{O}}(M,\mathcal{F})$ is Hausdorff. $\square$

6. The moduli space of transverse Calabi–Yau structures

In this section, we will show the moduli space of transverse Calabi–Yau structures is a Hausdorff and smooth manifold (Theorem 6.5). We assume that M is a closed oriented manifold and $\mathcal{F}$ is a Riemannian and taut foliation of codimension $2n$. Let F denote the integrable distribution induced by the foliation $\mathcal{F}$.

6.1. Transverse $\mathrm{SL}_n(\mathbb{C})$ structures.

DEFINITION 6.1. A nowhere vanishing complex n-form $\Omega \in \bigwedge^n \otimes \mathbb{C}$ is called a *transverse* $\mathrm{SL}_n(\mathbb{C})$ *structure on* $(M,\mathcal{F})$ if Ω is a basic form such that $d\Omega = 0$ and

$$Q \otimes \mathbb{C} = \mathrm{Ker}_{\mathbb{C}}\, \Omega/F \oplus \overline{\mathrm{Ker}_{\mathbb{C}}\, \Omega/F}$$

where $\mathrm{Ker}_{\mathbb{C}}\, \Omega/F$ is the space $\{v \in Q \otimes \mathbb{C} \mid i_v\Omega = 0\}$.

A transverse $\mathrm{SL}_n(\mathbb{C})$ structure Ω induces a complex structure J_Ω on Q such that Ω is an $(n,0)$-basic form on $(M,\mathcal{F})$ (see Example 3.3). Then we can check that $d\theta \in \bigwedge_B^{2,0} \oplus \bigwedge_B^{1,1}$ for any $\theta \in \bigwedge_B^{1,0}$ because of $(d\theta) \wedge \Omega = 0$. It follows from Remark 2.9 that J_Ω is a transverse complex structure on $(M,\mathcal{F})$. Hence $(\mathcal{F}, J_\Omega)$ is a transverse holomorphic foliation on M. Let $\tilde{\mathfrak{M}}_{\mathrm{SL}}(M,\mathcal{F})$ be the space of transverse $\mathrm{SL}_n(\mathbb{C})$

structures on $(M, \mathcal{F})$. Any element $\Omega \in \tilde{\mathfrak{M}}_{\mathrm{SL}}(M, \mathcal{F})$ induces a transverse calibration associated with the orbit $\mathcal{O}_{\mathrm{SL}}$, and converse is true. Thus, we can identify $\tilde{\mathfrak{M}}_{\mathrm{SL}}(M, \mathcal{F})$ with the set $\tilde{\mathfrak{M}}_{\mathcal{O}_{\mathrm{SL}}}(M, \mathcal{F})$ of transverse calibrations associated with the orbit $\mathcal{O}_{\mathrm{SL}}$. We recall that the orbit $\mathcal{O}_{\mathrm{SL}}$ is elliptic. For $\Omega \in \tilde{\mathfrak{M}}_{\mathrm{SL}}(M, \mathcal{F})$, the complex $(\sharp_\Omega)$ is

$$0 \to \bigwedge\nolimits_B^{n-1,0} \xrightarrow{d_0} \bigwedge\nolimits_B^{n,0} \oplus \bigwedge\nolimits_B^{n-1,1} \xrightarrow{d_1} \bigwedge\nolimits_B^{n,1} \oplus \bigwedge\nolimits_B^{n-1,2} \xrightarrow{d_2} \cdots .$$

Unfortunately, the maps

$$p_\Omega^1 \colon H^1(\sharp_\Omega) \to H_B^n(M, \mathbb{C}),$$
$$p_\Omega^2 \colon H^2(\sharp_\Omega) \to H_B^{n+1}(M, \mathbb{C})$$

are not always injective for $\Omega \in \tilde{\mathfrak{M}}_{\mathrm{SL}}(M, \mathcal{F})$. However, we obtain

Proposition 6.2. *If $(\mathcal{F}, J_\Omega)$ is a transverse Kähler foliation, then the element $\Omega \in \tilde{\mathfrak{M}}_{\mathrm{SL}}(M, \mathcal{F})$ is topological. Moreover, the period map P is injective on a neighbourhood of the equivalent class of Ω in $\mathfrak{M}_{\mathrm{SL}}(M, \mathcal{F})$.*

Proof. We suppose that $\Omega \in \tilde{\mathfrak{M}}_{\mathrm{SL}}(M, \mathcal{F})$ satisfies $(\mathcal{F}, J_\Omega)$ is a transverse Kähler foliation on M. By modifying the argument of Proposition 4.4 in [10], we obtain

$$H^1(\sharp_\Omega) = H_B^{n,0}(M) \oplus H_B^{n-1,1}(M),$$
$$H^2(\sharp_\Omega) = H_B^{n,1}(M) \oplus H_B^{n-1,2}(M).$$

The maps p_Ω^1 and p_Ω^2 are injective by Proposition 2.11, so Ω is topological. Moreover, it follows from Propositions 5.1 and 5.3 that there exists an open neighbourhood $U_\Omega \in \mathfrak{M}_{\mathrm{SL}}(M, \mathcal{F})$ of $\pi(\Omega)$ such that the period map $P|_{U_\Omega}$ restricted to U_Ω is injective. Hence we finish the proof. □

6.2. Transverse Calabi–Yau structures. We say that a real 2-form $\omega \in \bigwedge^2$ is a *transverse symplectic structure on* $(M, \mathcal{F})$ if ω is a basic form on $(M, \mathcal{F})$ such that $d\omega = 0$ and $\omega^n \neq 0$.

DEFINITION 6.3. A pair $(\Omega, \omega) \in \bigwedge_B^n \otimes \mathbb{C} \oplus \bigwedge_B^2$ is called a *transverse Calabi–Yau structure on* $(M, \mathcal{F})$ if Ω is a transverse $\mathrm{SL}_n(\mathbb{C})$ structure and ω is a transverse symplectic structure on $(M, \mathcal{F})$ such that

$$\Omega \wedge \omega = \bar{\Omega} \wedge \omega = 0,$$
$$\Omega \wedge \bar{\Omega} = c_n \omega^n,$$
$$\omega(\,\cdot\,, J_\Omega\,\cdot\,) \quad \text{is positive definite on} \quad Q$$

where $c_n = (1/n!)(-1)^{n(n-1)/2}(2/\sqrt{-1})^n$.

We denote by $\tilde{\mathfrak{M}}_{\mathrm{CY}}(M,\mathcal{F})$ the set of transverse Calabi–Yau structures on $(M,\mathcal{F})$. Any structure $(\Omega,\omega)\in\tilde{\mathfrak{M}}_{\mathrm{CY}}(M,\mathcal{F})$ is a transverse calibration associated with the orbit $\mathcal{O}_{\mathrm{CY}}$.

Proposition 6.4. *The orbit $\mathcal{O}_{\mathrm{CY}}$ is metrical, elliptic and topological.*

Proof. It suffices to show that $\mathcal{O}_{\mathrm{CY}}$ is topological. Given a structure $\Phi=(\Omega,\omega)\in\tilde{\mathfrak{M}}_{\mathrm{CY}}(M,\mathcal{F})$, then, by repeating a similar argument of the computation of cohomology groups ([10, Theorem 4.8]) to basic forms we obtain

$$\begin{aligned} H^1(\sharp_\Phi) &= H_B^{n,0}(M)\oplus H_B^{n-1,1}(M)\oplus \mathbb{P}_{B,\mathbb{R}}^{1,1},\\ H^2(\sharp_\Phi) &= H_B^{n,1}(M)\oplus H_B^{n-1,2}(M)\oplus (H_B^{2,1}(M)\oplus H_B^{1,2}(M))_{\mathbb{R}} \end{aligned}$$

where $(H_B^{2,1}(M)\oplus H_B^{1,2}(M))_{\mathbb{R}}$ and $\mathbb{P}_{B,\mathbb{R}}^{1,1}$ denote the real part of $H_B^{2,1}(M)\oplus H_B^{1,2}(M)$ and the space of real harmonic and primitive basic (1, 1)-forms, respectively. We refer to Section 3.4.7 in [6] for the Lefschetz decomposition theorem for a transverse Kähler foliation. Hence the maps

$$\begin{aligned} p_\Phi^1 &\colon H^1(\sharp_\Phi)\to H_B^n(M,\mathbb{C})\oplus H_B^2(M),\\ p_\Phi^2 &\colon H^2(\sharp_\Phi)\to H_B^{n+1}(M,\mathbb{C})\oplus H_B^3(M) \end{aligned}$$

are injective from Proposition 2.11 and the Lefschetz decomposition on basic differential forms. □

We obtain the following results:

Theorem 6.5. *The moduli space $\mathfrak{M}_{\mathrm{CY}}(M,\mathcal{F})$ is a Hausdorff and smooth manifold of dimension* $\dim_{\mathbb{R}}(H_B^{n,0}(M)\oplus H_B^{n-1,1}(M)\oplus\mathbb{P}_{B,\mathbb{R}}^{1,1})$.

Proof. It immediately follows from Theorem 5.6 and Proposition 6.4 that $\mathfrak{M}_{\mathrm{CY}}(M,\mathcal{F})$ is a Hausdorff and smooth manifold. The dimension of $\mathfrak{M}_{\mathrm{CY}}(M,\mathcal{F})$ is $\dim H^1(\sharp_\Phi)$ since $\mathfrak{M}_{\mathrm{CY}}(M,\mathcal{F})$ is locally diffeomorphic to an open subset of $H^1(\sharp_\Phi)$. In the proof of Proposition 6.4, we showed that $H^1(\sharp_\Phi)$ is equal to $H_B^{n,0}(M)\oplus H_B^{n-1,1}(M)\oplus\mathbb{P}_{B,\mathbb{R}}^{1,1}$. Hence this ends the proof. □

6.3. Examples.

6.3.1. Linear foliations on tori. Let T^{2n+1} be the real torus $\mathbb{R}^{2n+1}/\mathbb{Z}^{2n+1}$ of dimension $2n+1$. We take a local coordinate $(x_1,\ldots,x_n,y_1,\ldots,y_n,t)$ on T^{2n+1}, then a foliation $\mathcal{F}_{(\lambda,\mu)}$ is induced by the vector field

$$\xi=\sum_{i=1}^{n}\lambda_i\frac{\partial}{\partial x_i}+\mu_i\frac{\partial}{\partial y_i}-\frac{\partial}{\partial t}$$

for $(\lambda, \mu) = (\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_n) \in \mathbb{R}^{2n}$. The foliation $\mathcal{F}_{(\lambda,\mu)}$ is called a *linear foliation* on T^{2n+1}. Note that $\mathcal{F}_{(\lambda,\mu)}$ is a taut foliation with respect to the standard flat metric on T^{2n+1}. We define z_i by complex functions

$$z_i = x_i + \lambda_i t + \sqrt{-1}(y_i + \mu_i t)$$

for $i = 1, \dots, n$, then $(z_1, \dots, z_n)$ is a transverse coordinate on $(T^{2n+1}, \mathcal{F}_{(\lambda,\mu)})$. Now we define a pair (Ω, ω) of forms as

$$\Omega = dz_1 \wedge \cdots \wedge dz_n,$$
$$\omega = \frac{\sqrt{-1}}{2} \sum_{i=1}^{n} dz_i \wedge d\bar{z}_i,$$

then it is easy to see that (Ω, ω) is a transverse Calabi–Yau structure on $(T^{2n+1}, \mathcal{F}_{(\lambda,\mu)})$.

We start to compute the dimension of the moduli space $\mathfrak{M}_{\mathrm{CY}}(T^{2n+1}, \mathcal{F}_{(\lambda,\mu)})$. The vector space $\mathbb{H}_B^{p,q}(T^{2n+1})$ is generated by wedge products $dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge \bar{z}_{j_q}$, and thus we obtain

$$\dim_{\mathbb{C}} H_B^{p,q}(T^{2n+1}) = \binom{n}{p}\binom{n}{q}.$$

It follows that

$$\dim_{\mathbb{R}} \mathbb{P}_{B,\mathbb{R}}^{1,1} = \dim_{\mathbb{C}} \mathbb{P}_B^{1,1} = n^2 - 1$$

from $H_B^{1,1}(T^{2n+1}) = \mathbb{P}_B^{1,1} + \mathbb{C}\omega$. The moduli space $\mathfrak{M}_{\mathrm{CY}}(T^{2n+1}, \mathcal{F}_{(\lambda,\mu)})$ is a smooth manifold of dimension $\dim_{\mathbb{R}}(H_B^{n,0}(T^{2n+1}) \oplus H_B^{n-1,1}(T^{2n+1}) \oplus \mathbb{P}_{B,\mathbb{R}}^{1,1})$ by Theorem 6.5. Hence we can see that

$$\dim \mathfrak{M}_{\mathrm{CY}}(T^{2n+1}, \mathcal{F}_{(\lambda,\mu)}) = 2(1 + n^2) + n^2 - 1 = 3n^2 + 1.$$

We refer to [18] for deformations of transverse $\mathrm{SL}_n(\mathbb{C})$ and Calabi–Yau structures on $(T^{2n+1}, \mathcal{F}_{(\lambda,\mu)})$.

6.3.2. Null-Sasakian structures.

DEFINITION 6.6. A $(2n + 1)$-dimensional Riemannian manifold (M, g) is a *Sasakian manifold* if the metric cone $(C(M), \bar{g}) = (\mathbb{R}_{>0} \times M, dr^2 + r^2 g)$ is Kähler.

Under the identification M with $\{r = 1\} \subset C(M)$, we can obtain a vector field ξ and a 1-form η over M defined by

$$\xi = J\left(\frac{\partial}{\partial r}\right)\bigg|_M, \quad \eta(\,\cdot\,) = g(\xi,\,\cdot\,)$$

where J is the complex structure on $(C(M), \bar{g})$. Then the vector field ξ is a Killing vector field such that each integral curve is geodesic. Thus ξ induces a taut foliation $\mathcal{F}_\xi$ on M. The vector field ξ and the foliation $\mathcal{F}_\xi$ are called a *Reeb field* and a *Reeb foliation*, respectively. The 1-form η satisfies that $\eta(\xi) = 1$ and $d\eta \in \bigwedge_B^2$. We may consider the distribution D over M defined by $\operatorname{Ker}\eta$. Then D is the $2n$-dimensional distribution satisfying the orthogonal decomposition

$$TM = D \oplus F_\xi$$

where F_ξ is the trivial line bundle generated by ξ. We define a section Φ of $\operatorname{End}(TM)$ by setting $\Phi|_D = J|_D$ and $\Phi|_{F_\xi} = 0$. The data (ξ, η, Φ, g) is called a *Sasakian structure* on M. Under the identification D with the quotient bundle $Q = TM/F_\xi$, the basic form $d\eta$ and the section Φ induces a transverse Kähler structure on $(M, \mathcal{F}_\xi)$. We denote by Ric_Q the transverse Ricci tensor of the transverse Riemannian metric $g_Q = g|_D$. Then the transverse Ricci form ρ_Q is defined by

$$\rho_Q(\,\cdot\,,\ \cdot\,) = \mathrm{Ric}_Q(\,\cdot\,, \Phi\,\cdot\,).$$

The form ρ_Q is a basic closed $(1,1)$-form on $(M, \mathcal{F}_\xi)$, and defines the basic cohomology class $[\rho_Q] \in H_B^{1,1}(M)$. The basic class $[(1/2\pi)\rho_Q]$ is called the *basic first Chern class* and is denoted by $c_B^1(M)$.

DEFINITION 6.7. A Sasakian structure (ξ, η, Φ, g) is a *null-Sasakian structure* on M if $c_B^1(M) = 0$. We say that (M, ξ, η, Φ, g) is a *null-Sasakian manifold* if (ξ, η, Φ, g) is a null-Sasakian structure on M.

The class $c_B^1(M)$ is independent of the choice of a Sasakian structure whose Reeb foliation is $\mathcal{F}_\xi$. By the transverse version of Yau's theorem proved by El Kacimi-Alaoui [6], there exists a transverse Calabi–Yau structure (Ω, ω) on a null-Sasakian manifold (M, ξ, η, Φ, g).

Let us start to compute the dimension of the moduli space $\mathfrak{M}_{\mathrm{CY}}(M, \mathcal{F}_\xi)$ on a null-Sasakian manifold (M, ξ, η, Φ, g). We remark an important property of Sasakian structures that, on a compact Sasakian manifold M, a k-form is harmonic if and only if it is primitive and basic for $1 \leq k \leq n$ ([2, Proposition 7.4.13]):

$$H^k(M, \mathbb{C}) = \mathbb{P}_B^k \tag{18}$$

for $1 \leq k \leq n$.

On a null-Sasakian manifold M of 5-dimension (the case of $n = 2$), we will see that $\dim \mathfrak{M}_{\mathrm{CY}}(M, \mathcal{F}_\xi)$ is given by the Betti number of M. It is obvious that $\dim_{\mathbb{C}} H_B^{2,0} = \dim_{\mathbb{C}} H_B^{0,2} = 1$ since Ω is the basic holomorphic $(2,0)$-form. We have a decomposition

$$\mathbb{P}_B^2 = \mathbb{P}_B^{2,0} \oplus \mathbb{P}_B^{1,1} \oplus \mathbb{P}_B^{0,2} = H_B^{2,0} \oplus \mathbb{P}_B^{1,1} \oplus H_B^{0,2}. \tag{19}$$

Equations (18) and (19) give rise to

$$\dim_{\mathbb{C}} \mathbb{P}_B^{1,1} = \dim_{\mathbb{C}} \mathbb{P}_B^2 - 2\dim_{\mathbb{C}} H_B^{2,0} = b^2 - 2 \tag{20}$$

where b^2 is the second Betti number $\dim_{\mathbb{C}} H^2(M, \mathbb{C})$. It follows that

$$\dim_{\mathbb{C}} H_B^{1,1} = \dim_{\mathbb{C}} \mathbb{P}_B^{1,1} + 1 = b^2 - 1$$

from equation (20) and the basic Lefschetz decomposition $H_B^{1,1} = \mathbb{P}_B^{1,1} \oplus \mathbb{C}\omega$. Thus we obtain

$$\dim_{\mathbb{C}}(H_B^{2,0} \oplus H_B^{1,1}) = 1 + b^2 - 1 = b^2. \tag{21}$$

We remark that $H_B^{2,0} \oplus H_B^{1,1}$ can be regarded as the tangent space of deformations of transverse $\mathrm{SL}_n(\mathbb{C})$ structures on $(M, \mathcal{F}_\xi)$ (cf. [18]). The moduli space $\mathfrak{M}_{\mathrm{CY}}(M, \mathcal{F}_\xi)$ is a smooth manifold of dimension $\dim_{\mathbb{R}}(H_B^{2,0}(M) \oplus H_B^{1,1}(M) \oplus \mathbb{P}_{B,\mathbb{R}}^{1,1})$. Hence equations (20) and (21) yield

$$\dim \mathfrak{M}_{\mathrm{CY}}(M, \mathcal{F}_\xi) = 2b^2 + b^2 - 2 = 3b^2 - 2. \tag{22}$$

On a null-Sasakian manifold M of 7-dimension (the case of $n = 3$), we will find the dimension of the deformation space of transverse $\mathrm{SL}_n(\mathbb{C})$ structures is given by Betti numbers of M. Now we consider the basic Hodge decompositions

$$\begin{aligned} \mathbb{P}_B^3 &= \mathbb{P}_B^{3,0} \oplus \mathbb{P}_B^{2,1} \oplus \mathbb{P}_B^{1,2} \oplus \mathbb{P}_B^{0,3}, \\ \mathbb{P}_B^1 &= \mathbb{P}_B^{1,0} \oplus \mathbb{P}_B^{0,1}. \end{aligned}$$

Then it follows from (18) and $\dim_{\mathbb{C}} \mathbb{P}_B^{3,0} = \dim_{\mathbb{C}} H_B^{3,0} = 1$ that

$$2\dim_{\mathbb{C}} \mathbb{P}_B^{2,1} = b^3 - 2, \quad 2\dim_{\mathbb{C}} \mathbb{P}_B^{1,0} = b^1 \tag{23}$$

where b^1 and b^3 are Betti numbers $\dim_{\mathbb{C}} H^1(M, \mathbb{C})$ and $\dim_{\mathbb{C}} H^3(M, \mathbb{C})$, respectively. By equation (23) and the basic Lefschetz decomposition $H_B^{2,1} = \mathbb{P}_B^{2,1} \oplus \mathbb{P}_B^{1,0} \wedge \omega$, we have

$$\dim_{\mathbb{C}} H_B^{2,1} = \frac{1}{2}(b^3 - 2) + \frac{1}{2}b^1 = \frac{1}{2}(b^3 + b^1 - 2),$$

and

$$\dim_{\mathbb{C}}(H_B^{3,0} \oplus H_B^{2,1}) = 1 + \frac{1}{2}(b^3 + b^1 - 2) = \frac{1}{2}(b^3 + b^1). \tag{24}$$

The vector space $H_B^{3,0} \oplus H_B^{2,1}$ can be identified with the tangent space of deformations of transverse $\mathrm{SL}_n(\mathbb{C})$ structures on $(M, \mathcal{F}_\xi)$. We also have that

$$\dim_{\mathbb{C}} \mathbb{P}_B^{1,1} = \dim_{\mathbb{C}} H_B^{1,1} - 1$$

by the relation $H_B^{1,1} = \mathbb{P}_B^{1,1} \oplus \mathbb{C}\omega$. Hence the manifold $\mathfrak{M}_{\mathrm{CY}}(M, \mathcal{F}_\xi)$ has the dimension

$$\dim \mathfrak{M}_{\mathrm{CY}}(M, \mathcal{F}_\xi) = (b^3 + b^1) + \dim_{\mathbb{C}} \mathbb{P}_B^{1,1} = b^3 + b^1 + \dim_{\mathbb{C}} H_B^{1,1} - 1. \tag{25}$$

6.3.3. Calabi–Yau orbifolds. The geometry of Riemannian foliations is related to that of orbifolds, and so our method of transverse calibrations is useful for geometric structures on orbifolds. We will see this phenomenon on examples. We refer to [2] and [11] for some facts of Sasakian links and the notation of orbifolds, respectively.

DEFINITION 6.8. A singular real manifold X of dimension m is an *orbifold* if singularities are locally isomorphic to quotient singularities $\mathbb{R}^m/G$ for finite subgroups $G \subset \mathrm{GL}(m, \mathbb{R})$ such that each group G is small, that is, for any $\gamma \neq 1 \in G$ the subspace $V_\gamma \subset \mathbb{R}^m$ fixed by γ has codimension at least two.

We can also define a complex orbifold in a similar way. Any compact complex orbifold X is a leaf space of a Riemannian and transversely holomorphic foliation $\mathcal{F}$ on a smooth compact manifold $\tilde{X}$ (cf. [6, §4]). Therefore, we can regard geometric structures on an orbifold X as transverse geometric structures on a smooth foliated manifold $(\tilde{X}, \mathcal{F})$.

DEFINITION 6.9. Let X be an orbifold of dimension $2n$. A pair (Ω, ω) is a *Calabi–Yau structure* on X if (Ω, ω) is a Calabi–Yau structure on the non-singular set of X in the sense of Definition 6.3, and wherever X is locally isomorphic to $\mathbb{R}^{2n}/G$, (Ω, ω) is the quotient of a G-invariant Calabi–Yau structure defined near 0 in $\mathbb{R}^{2n}$. We say that (X, Ω, ω) is a *Calabi–Yau orbifold* if (Ω, ω) is a Calabi–Yau structure on X.

We denote by $\mathfrak{M}^{\mathrm{orb}}_{\mathrm{CY}}(X)$ the moduli space of Calabi–Yau structures on X, then $\mathfrak{M}^{\mathrm{orb}}_{\mathrm{CY}}(X)$ is a smooth manifold by Theorem 6.5. Any Calabi–Yau structure on X corresponds to a transverse Calabi–Yau structure on $(\tilde{X}, \mathcal{F})$. Thus the moduli space $\mathfrak{M}_{\mathrm{CY}}(\tilde{X}, \mathcal{F})$ can be identified with $\mathfrak{M}^{\mathrm{orb}}_{\mathrm{CY}}(X)$. We can easily compute the dimension of $\mathfrak{M}_{\mathrm{CY}}(\tilde{X}, \mathcal{F})$ in a special case.

We consider the $\mathbb{C}^*$-action on $\mathbb{C}^{n+1}$ defined by

$$(z_0, \dots, z_n) \mapsto (\lambda^{w_0} z_0, \dots, \lambda^{w_n} z_n)$$

where $\lambda \in \mathbb{C}^*$ and $\mathbf{w} = (w_0, \dots, w_n) \in (\mathbb{Z}_+)^{n+1}$. Let $\mathbb{C}^*(\mathbf{w})$ denote this $\mathbb{C}^*$-action.

DEFINITION 6.10. The *weighted projective space* $\mathbb{C}P(\mathbf{w})$ is defined as the quotient $(\mathbb{C}^{n+1} \setminus 0)/\mathbb{C}^*(\mathbf{w})$.

The weighted projective space $\mathbb{C}P(\mathbf{w})$ is a complex orbifold. However, it is not a Calabi–Yau orbifold. To obtain a Calabi–Yau orbifold, we consider hypersurfaces on

$\mathbb{C}P(\mathbf{w})$. A weighted homogeneous polynomial f of degree d and weight $\mathbf{w}$ is defined by a polynomial $f \in \mathbb{C}[z_0, \dots, z_n]$ satisfying

$$f(\lambda^{w_0} z_0, \dots, \lambda^{w_n} z_n) = \lambda^d f(z_0, \dots, z_n)$$

for any $\lambda \in \mathbb{C}^*$. Given a weighted homogeneous polynomial f, then we can define the subset X_f of $\mathbb{C}P(\mathbf{w})$ as the zero locus of f in $\mathbb{C}P(\mathbf{w})$. Such a variety X_f is called a *weighted hypersurface* of degree d in $\mathbb{C}P(\mathbf{w})$. Let $\pi : \mathbb{C}^{n+1} \setminus 0 \to \mathbb{C}P(\mathbf{w})$ be the natural projection. We denote by $C^*_{X_f}$ the punctured affine cone $\pi^{-1}(X_f)$, and define C_{X_f} as the completion of $C^*_{X_f}$ in $\mathbb{C}^{n+1}$. A weighted hypersurface X_f is called *quasi-smooth* if the cone C_{X_f} is smooth of dimension n outside the origin 0. A quasi-smooth weighted hypersurface X_f has a complex orbifold structure induced by that of $\mathbb{C}P(\mathbf{w})$. If $|\mathbf{w}| - d = 0$, then X_f becomes a Calabi–Yau orbifold. In [20], Reid provided a list of 95 K3 surfaces, i.e., Calabi–Yau orbifolds of complex dimension 2, given as weighted hypersurfaces in $\mathbb{C}P(w_0, w_1, w_2, w_3)$ (We refer to Appendix B in [2] for Reid's list). In the case of complex dimension 3, there exist more than 6000 examples of Calabi–Yau orbifolds in $\mathbb{C}P(w_0, w_1, w_2, w_3, w_4)$ (cf. [3]).

Let X_f be a quasi-smooth weighted hypersurface in $\mathbb{C}P(\mathbf{w})$ with $|\mathbf{w}| - d = 0$. Consider the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$, then the intersection $C_{X_f} \cap S^{2n+1}$ is a smooth manifold of dimension $2n - 1$. We denote $C_{X_f} \cap S^{2n+1}$ by L_f and call it a *link* of f. The link L_f has a null-Sasakian structure (ξ, η, Φ, g) such that X_f is the leaf space of the Reeb foliation $\mathcal{F}_\xi$. We shall compute the dimension of $\mathfrak{M}_{\mathrm{CY}}(L_f, \mathcal{F}_\xi)$, which coincides that of $\mathfrak{M}^{\mathrm{orb}}_{\mathrm{CY}}(X_f)$. Note that L_f is $(n-2)$-connected.

If X_f is a K3 surface, then the link L_f is a 5-dimensional null-Sasakian manifold with $b^2(L_f) = b^2(X_f) - 1$ (cf. [2, Section 10.3.2]). Applying equation (21) to the link L_f, then we obtain

$$\dim_{\mathbb{C}}(H^{2,0}_B \oplus H^{1,1}_B) = b^2(L_f) \quad (= b^2(X_f) - 1).$$

We remark the space $H^{2,0}_B(L_f) \oplus H^{1,1}_B(L_f)$ can be regarded as the tangent space of deformations of $\mathrm{SL}_n(\mathbb{C})$ structures on the orbifold X_f. Equation (22) implies that the moduli space $\mathfrak{M}_{\mathrm{CY}}(L_f, \mathcal{F}_\xi)$ has the dimension

$$\dim \mathfrak{M}_{\mathrm{CY}}(L_f, \mathcal{F}_\xi) = 3b^2(L_f) - 2 \quad (= 3b^2(X_f) - 5).$$

If X_f is a Calabi–Yau 3-fold, then the link L_f is a 7-dimensional null-Sasakian manifold. It follows from equation (24) that

$$\dim_{\mathbb{C}}(H^{3,0}_B(L_f) \oplus H^{2,1}_B(L_f)) = \frac{1}{2} b^3(L_f)$$

since L_f is 2-connected. Equation (25) implies that

$$\dim \mathfrak{M}_{\mathrm{CY}}(L_f, \mathcal{F}_\xi) = b^3(L_f) + \dim_{\mathbb{C}} H^{1,1}_B(L_f) - 1.$$

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