Harris, M.E.
Osaka J. Math.
46 (2009), 557–562

THE GREEN CORRESPONDENCE AND ORDINARY INDUCTION OF BLOCKS IN FINITE GROUP MODULAR REPRESENTATION THEORY

MORTON E. HARRIS

(Received November 16, 2007, revised February 25, 2008)

Abstract

The first step in the fundamental Clifford theoretic approach to general block theory of finite groups reduces to: H is a subgroup of the finite group G and e is a central idempotent of H such that $e({}^{g}e) = 0$ for all $g \in G - H$. Then $Tr_H^G(e)$ is a central idempotent of G and induction from H to G, Ind_H^G, is part of a Morita equivalence between the categories of e-modules and of $Tr_H^G(e)$-modules. Let W be an indecomposable e-module, so that $V = Ind_H^G(W)$ is an indecomposable $Tr_H^G(e)$-module. We present results that relate the Green correspondents of W and V via induction and restriction.

1. Introduction and results

Our notation and terminology are standard and tend to follow [1] and [5]. All rings have identities and are Noetherian and all modules over a ring are unitary and finitely generated left modules.

Let R be a ring. Then R-mod will denote the abelian category of left R-modules. Let U and V be left R-modules. Then $U|V$ in R-mod signifies that U is isomorphic to a direct summand of V in R-mod. Also if R has the unique decomposition property (cf. [1, p. 37]), then U is a component of V if U is indecomposable in R-mod and $U|V$.

In this paper, G denotes a finite group, p is a prime integer and let $(\mathcal{O}, K, k = \mathcal{O}/J(\mathcal{O}))$ be a p-modular system that is "large enough" for all subgroups of G (i.e., $\mathcal{O}$ is a complete discrete valuation ring, $k = \mathcal{O}/J(\mathcal{O})$ is an algebraically closed field of characteristic p and the field of fractions K of $\mathcal{O}$ is of characteristic zero and is a splitting field for all subgroups of G).

Let A be a finitely generated $\mathcal{O}$-algebra. Then A has the unique decomposition property by the Krull-Schmidt theorem ([1, I, Theorem 11.4] or [5, Theorem 4.4]). Also the natural ring epimorphism $\overline{}\colon \mathcal{O} \to \mathcal{O}/J(\mathcal{O}) = k$ induces an $\mathcal{O}$-algebra epimorphism $\overline{}\colon A \to A/(J(\mathcal{O})A) = \overline{A}$.

2000 Mathematics Subject Classification. 20C20.

The author is grateful for the comments of the referee, especially his suggestions for Proposition 5 and Question 6.

Let $H < G$ and let e be an idempotent of $Z(\mathcal{O}H)$. We shall need an extension of [2, Remark 1.3]:

Lemma 1. *Let $g \in G$. The following six conditions are equivalent*:
(i) $\overline{e}(k(HgH))\overline{e} = (0)$;
(ii) $\overline{e}({}^g\overline{e}) = (0)$;
(iii) $\overline{e}(k(HgH)) \bigotimes_{kH} \overline{V} = (0)$ *for all modules* $\overline{V}$ *of* $(kH)\overline{e}$*-mod*;
(iv) $e(\mathcal{O}(HgH))e = (0)$;
(v) $e({}^g e) = 0$; *and*
(vi) $e\big(\mathcal{O}(HgH) \bigotimes_{\mathcal{O}H} V\big) = (0)$; *for all modules V of $(\mathcal{O}H)e$-mod.*

Proof. From [2, Remark 1.3] we conclude that (iv), (v) and (vi) are equivalent and (i), (ii) and (iii) are equivalent. Clearly (vi) implies (i). Assume (i) and note that $e(\mathcal{O}(HgH)e)$ is $\mathcal{O}$-free and $\overline{e(\mathcal{O}(HgH))e} = (0)$. Thus (iv) holds and we are done. □

Let V be an indecomposable $\mathcal{O}G$-module with vertex P and $\mathcal{O}P$-source X. Let K be a subgroup of G such that $N_G(P) \le K$. Thus the Green correspondent $\mathcal{G}r_K^G(V)$ of V in $\mathcal{O}K$-mod also has vertex P and $\mathcal{O}P$-source X. Let L be a subgroup of K such that $P \le L$.

Lemma 2. *Let U be an indecomposable direct summand of $Res_K^G(V)$ in $\mathcal{O}K$-mod such that $Res_L^K(U)$ has a component W in $\mathcal{O}L$-mod with vertex P. Then $U \cong \mathcal{G}r_K^G(V)$ in $\mathcal{O}K$-mod.*

Proof. Assume that U is not isomorphic to $\mathcal{G}r_K^G(V)$ in $\mathcal{O}K$-mod. Then, as in [1, III, Lemma 5.3], there is an $x \in G - K$ and a subgroup $A \le K \cap (P^x)$ such that A is a vertex of U. Since $W|Res_L^K(U)$ in $\mathcal{O}L$-mod, [1, III, Lemma 4.1] implies that there is a $y \in K$ such that W is $L \cap (A^y)$-projective. But then there is a $z \in L$ such that $P^z \le L \cap (A^y)$. Here $A^y \le K \cap (P^{(xy)})$, so that $P^z = L \cap (A^y) = P^{(xy)}$. Thus $xyz^{-1} \in N_G(P) \le K$ and $x \in K$. This contradiction establishes the lemma. □

The following two propositions are the main results of this paper. For the remainder of this paper, we assume that $e({}^g e) = 0$ for all $g \in G - H$. Hence $E = Tr_H^G(e)$ is an idempotent in $Z(\mathcal{O}G)$ and the functors Ind_H^G: $(\mathcal{O}H)e$-mod $\to (\mathcal{O}G)E$-mod and $e\,Res_H^G$: $(\mathcal{O}G)E$-mod $\to$ $(\mathcal{O}H)e$-mod demonstrate a Morita equivalence between $(\mathcal{O}H)e$-mod and $(\mathcal{O}G)E$-mod as is well-known (cf. [4, Case 1], [5, Theorem 9.9] or [2, Proposition 1.2]).

Let W be an indecomposable $(\mathcal{O}H)e$-module with vertex P and $\mathcal{O}P$-source X. Then $V = Ind_H^G(W)$ is an indecomposable $(\mathcal{O}G)E$-module and P is a vertex of V and X is an $\mathcal{O}P$-source of V (cf. [1, III,Corollary 4.7]). Here $P \le N_H(P) \le N_G(P)$.

Let $b \in Bl((\mathcal{O}H)e)$ be such that $bW = W$. Then $Tr_H^G(b) = B \in Bl((\mathcal{O}G)E)$ and $BV = V$. Also $b\,Res_H^G(V) \cong W$ in $\mathcal{O}H$-mod and $b({}^g b) = 0$ for all $g \in G - H$.

Under these conditions, we have the Green correspondents $\mathcal{G}r_{N_G(P)}^G(V)$ and $\mathcal{G}r_{N_H(P)}^H(W)$ of V in $\mathcal{O}N_G(P)$-mod and of W in $\mathcal{O}N_H(P)$-mod, resp., where both indecomposable modules $\mathcal{G}r_{N_G(P)}^G(V)$ and $\mathcal{G}r_{N_H(P)}^H(W)$ have P as a vertex and $\mathcal{O}P$-source X.

Proposition 3. *Let e_P be the unique block of $\mathcal{O}N_H(P)$ such that $e_P\,\mathcal{G}r_{N_H(P)}^H(W) = \mathcal{G}r_{N_H(P)}^H(W)$. Then*
(a) *$e_P({}^x e_P) = 0$ for all $x \in N_G(P) - N_H(P)$, $E_P = Tr_{N_H(P)}^{N_G(P)}(e_P)$ is a block of $\mathcal{O}N_G(P)$ and the conclusions of* [6, Theorem 1] *and* [2, Theorem 1.6] *hold.*
(b) *$E_P\,\mathcal{G}r_{N_G(P)}^G(V) = \mathcal{G}r_{N_G(P)}^G(V)$, $Ind_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_H(P)}^H(W)\big) \cong \mathcal{G}r_{N_G(P)}^G(V)$ in $\mathcal{O}N_G(P)$-mod and $e_P\,Res_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_G(P)}^G(V)\big) \cong \mathcal{G}r_{N_H(P)}^H(W)$ in $\mathcal{O}N_H(P)$-mod; and*
(c) *exactly one component of $Res_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_G(P)}^G(V)\big)$ in $\mathcal{O}N_H(P)$-mod is isomorphic to $\mathcal{G}r_{N_H(P)}^H(W)$.*

Proof. From [1, III, Theorem 7.8], we conclude that $Br_P(e)\bar{e}_P = \bar{e}_P$. Let $x \in N_G(P) - N_H(P)$. Then $\bar{e}_P({}^x\bar{e}_P) = \bar{e}_P Br_P(e)({}^x Br_P(e))({}^x\bar{e}_P) = \bar{e}_P Br_P(e)({}^x Br_P(e))({}^x\bar{e}_P) = \bar{e}_P Br_P(e({}^x e))^x\bar{e}_P = 0$. We conclude from Lemma 1 that $e_P({}^x e_P) = 0$ for all $x \in N_G(P) - N_H(P)$. Then (a) follows from [2, Proposition 1.2]. Here $W | Ind_{N_H(P)}^H\big(\mathcal{G}r_{N_H(P)}^H(W)\big)$ in $\mathcal{O}H$-mod. Thus

$$V \cong Ind_H^G(W) | Ind_{N_H(P)}^G\big(\mathcal{G}r_{N_H(P)}^H(W)\big)$$

in $\mathcal{O}G$-mod. Since $Ind_{N_H(P)}^G\big(\mathcal{G}r_{N_H(P)}^H(W)\big) \cong Ind_{N_G(P)}^G\big(Ind_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_H(P)}^H(W)\big)\big)$ in $\mathcal{O}G$-mod and $Ind_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_H(P)}^H(W)\big)$ is indecomposable in $\mathcal{O}N_G(P)$-mod with vertex P and $\mathcal{O}P$-source X by [2, Theorem 1.6 (c)], we conclude from [1, III, Theorem 5.6 (iii)] that $\mathcal{G}r_{N_G(P)}^G(V) \cong Ind_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_H(P)}^H(W)\big)$ in $\mathcal{O}N_G(P)$-mod. But then [2, Proposition 1.2] completes our proof of (b).

Clearly

$$Res_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_G(P)}^G(V)\big) = e_P\,Res_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_G(P)}^G(V)\big) \oplus (1 - e_P)\,Res_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_G(P)}^G(V)\big)$$

in $\mathcal{O}N_H(P)$-mod. Let $\mathcal{U}$ be a component of $(1-e_P)Res_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_G(P)}^G(V)\big)$ in $\mathcal{O}N_H(P)$-mod such that $\mathcal{U} \cong \mathcal{G}r_{N_H(P)}^H(W)$ in $\mathcal{O}N_H(P)$-mod. Let e_P^* be the unique block of $Z(\mathcal{O}N_H(P))$ such that $e_P^*\mathcal{U} = \mathcal{U}$. Since $e_P^*(1 - e_P) = e_P^*$, we have $e_P e_P^* = 0$. This contradiction completes our proof of Proposition 3. $\square$

For our next result, we shall investigate a more general situation than in Proposition 3. Consequently we assume that K is a subgroup of G such that $N_G(P) \le K$. Then $N_H(P) \le K \cap H \le H$, $\mathcal{G}r_K^G(V)$ is an indecomposable $\mathcal{O}K$-module with vertex P

and $\mathcal{O}P$-source X and $\mathcal{G}r^H_{K\cap H}(W)$ is an indecomposable $\mathcal{O}(K\cap H)$-module with vertex P and $\mathcal{O}P$-source X.

Proposition 4. (a) *Let U be a component of $Res^H_{K\cap H}(W)$ such that $Ind^K_{K\cap H}(U)$ has a component with vertex P. Then $U \cong \mathcal{G}r^H_{K\cap H}(W)$ in $\mathcal{O}(K\cap H)$-mod*;
(b) *in an indecomposable decomposition of $Ind^K_{K\cap H}\big(\mathcal{G}r^H_{K\cap H}(W)\big)$ in $\mathcal{O}K$-mod, exactly one component has P as a vertex and it is isomorphic to $\mathcal{G}r^G_K(V)$ in $\mathcal{O}K$-mod and all of the remaining components have a proper subgroup of P as a vertex*;
(c) *let Y be a component of $Res^G_K(V)$ such that $Res^K_{K\cap H}(Y)$ has a component with vertex P. Then $Y \cong \mathcal{G}r^G_K(V)$ in $\mathcal{O}K$-mod*; *and*
(d) *in an indecomposable decomposition of $Res^K_{K\cap H}\big(\mathcal{G}r^G_K(V)\big)$ in $\mathcal{O}(K\cap H)$-mod, exactly one component is isomorphic to $\mathcal{G}r^H_{K\cap H}(W)$.*

Proof. For (a), assume that $U \ncong \mathcal{G}r^H_{K\cap H}(W)$ in $\mathcal{O}(K\cap H)$-mod. Then [1, III, Lemma 5.3] implies that there is an $x \in H-(K\cap H)$ and a vertex A of U such that $A \le (K\cap H)\cap(P^x)$. Let Y be a component of $Ind^K_{K\cap H}(U)$ with P as a vertex. Then, as $Ind^K_{K\cap H}(U)$ is A-projective, there is a $k \in K$ such that $P^k \le A$. But then $P^k = A = P^x$ and so $xk^{-1} \in N_G(P) \le K$. This contradiction establishes (a).

For (b), [1, III, Lemma 5.4] yields:

$$Ind^H_{K\cap H}\big(\mathcal{G}r^H_{K\cap H}(W)\big) \cong W \oplus \left(\bigoplus_{i\in I}\mathcal{U}_i\right) \quad \text{in} \quad \mathcal{O}H\text{-mod} \tag{1.1}$$

where I is a finite set and for each $i \in I$, $\mathcal{U}_i$ is an indecomposable $\mathcal{O}H$-module having a proper subgroup of P as a vertex.

Thus:

$$Ind^G_{K\cap H}\big(\mathcal{G}r^H_{K\cap H}(W)\big) \cong V \oplus \left(\bigoplus_{i\in I} Ind^G_H(\mathcal{U}_i)\right) \quad \text{in} \quad \mathcal{O}G\text{-mod}. \tag{1.2}$$

Clearly $Ind^G_{K\cap H}\big(\mathcal{G}r^H_{K\cap H}(W)\big) \cong Ind^G_K\big(Ind^K_{K\cap H}\big(\mathcal{G}r^H_{K\cap H}(W)\big)\big)$ in $\mathcal{O}G$-mod and all components of $Ind^K_{K\cap H}\big(\mathcal{G}r^H_{K\cap H}(W)\big)$ are P-projective. Let T be a component of $Ind^K_{K\cap H}\big(\mathcal{G}r^H_{K\cap H}(W)\big)$ in $\mathcal{O}K$-mod such that $V|Ind^G_K(T)$ in $\mathcal{O}K$-mod. Then P must be a vertex of T and $T \cong \mathcal{G}r^G_K(V)$ in $\mathcal{O}K$-mod. Let T_1 be a component of $Ind^K_{K\cap H}\big(\mathcal{G}r^H_{K\cap H}(W)\big)$ with P as a vertex and such that $(T\oplus T_1)|Ind^K_{K\cap H}\big(\mathcal{G}r^H_{K\cap H}(W)\big)$ in $(\mathcal{O}K)$-mod. Then $Ind^G_K(T_1)$ has a component with P as a vertex by [1, III, Corollary 4.7]. Thus (1.1) and (1.2) imply that $V|Ind^G_K(T_1)$ and (1.1) and (1.2) yield a contradiction. Thus (b) is proved.

Clearly (c) follows from Lemma 2.

For (d), note that

$$\mathcal{G}r^H_{K\cap H}(W)|Res^H_{K\cap H}(W)|Res^H_{K\cap H}\big(Res^G_H(V)\big) = Res^K_{K\cap H}\big(Res^G_K(V)\big)$$

in $\mathcal{O}(K \cap H)$-mod. Thus $Res_K^G(V)$ has a component T in $\mathcal{O}K$-mod such that $\mathcal{G}r_{K\cap H}^H(W)|Res_{K\cap H}^K(T)$ in $\mathcal{O}(K \cap H)$-mod. Now (c) implies that $T \cong \mathcal{G}r_K^G(V)$ in $\mathcal{O}K$-mod and so $\mathcal{G}r_{K\cap H}^H(W)|Res_{K\cap H}^K\big(\mathcal{G}r_K^G(V)\big)$ in $\mathcal{O}(K \cap H)$-mod.

Let r be the number of components in an indecomposable decomposition of $Res_{K\cap H}^K(\mathcal{G}r_K^G(V))$ in $\mathcal{O}(K \cap H)$-mod that are isomorphic to $\mathcal{G}r_{K\cap H}^H(W)$. Thus there are at least r components in an indecomposable decomposition of $Res_{N_H(P)}^K(\mathcal{G}r_K^G(V))$ that are isomorphic to $\mathcal{G}r_{N_H(P)}^{K\cap H}(\mathcal{G}r_{K\cap H}^H(W)) \cong \mathcal{G}r_{N_H(P)}^H(W)$ in $\mathcal{O}N_H(P)$-mod. But

$$Res_{N_H(P)}^K\big(\mathcal{G}r_K^G(V)\big) = Res_{N_H(P)}^{N_G(P)}\Big(Res_{N_G(P)}^K\big(\mathcal{G}r_K^G(V)\big)\Big)$$

in $\mathcal{O}N_H(P)$-mod and

$$Res_{N_G(P)}^K\big(\mathcal{G}r_K^G(V)\big) \cong \mathcal{G}r_{N_G(P)}^G(V) \oplus \left(\bigoplus_{i\in I} \mathcal{U}_i\right)$$

in $\mathcal{O}N_G(P)$-mod where I is a finite set and if $i \in I$, then $\mathcal{U}_i$ is an indecomposable $\mathcal{O}N_G(P)$-module with a vertex $A_i \le N_G(P) \cap P^{x_i}$ for some $x_i \in K - N_G(P)$. Let $i \in I$ be such that $Res_{N_H(P)}^{N_G(P)}(\mathcal{U}_i)$ has a component isomorphic to $\mathcal{G}r_{N_H(P)}^H(W)$ in $\mathcal{O}N_H(P)$-mod. Then there is a $y \in N_G(P)$ such that $P^y \le A_i$ by [1, III, Lemma 4.6]. Thus $P^y = P \le A_i = P^{x_i}$ and so $x_i \in N_G(P)$. This contradiction implies that $Res_{N_H(P)}^{N_G(P)}\big(\bigoplus_{i\in I} \mathcal{U}_i\big)$ does not have a component isomorphic to $\mathcal{G}r_{N_H(P)}^H(W)$ in $\mathcal{O}N_H(P)$-mod. Since, Proposition 3 (c) asserts that exactly one component of $Res_{N_H(P)}^{N_G(P)}\big(\mathcal{G}r_{N_G(P)}^G(V)\big)$ is isomorphic to $\mathcal{G}r_{N_H(P)}^H(W)$ in $\mathcal{O}N_H(P)$-mod, $r = 1$ and our proof of Proposition 4 is complete. □

We conclude with a discussion of the Brauer block induction (cf. [3, Chapter 5, Section 3]) in the context of Proposition 4 as suggested by the referee. So we assume the context of Proposition 4. Thus $b \in Bl((\mathcal{O}H)e)$, $b({}^g b) = 0$ for all $g \in G - H$, $bW = W$, $B = Tr_H{}^G(b) \in Bl((\mathcal{O}G)E)$, $V = Ind_H{}^G(W)$ and $BW = W$. Here b^G is defined and $b^G = B$ by [3, Chapter 5, Theorem 3.1 (ii)] and [2, Proposition 1.7].

Let B_K be the block idempotent of $\mathcal{O}K$ such that $B_K\, \mathcal{G}r_K^G(V) = \mathcal{G}r_K^G(V)$, let B_P be the block idempotent of $\mathcal{O}N_G(P)$ such that $B_P\, \mathcal{G}r_{N_G(P)}^G(V) = \mathcal{G}r_{N_G(P)}^G(V)$, let $b_{K\cap H}$ be the block idempotent of $\mathcal{O}(K \cap H)$ such that $b_{(K\cap H)}\, \mathcal{G}r_{(K\cap H)}^H(W) = \mathcal{G}r_{(K\cap H)}^H(W)$ and let b_P be the block idempotent of $\mathcal{O}N_H(P)$ such that $b_P\, \mathcal{G}r_{N_H(P)}^H(W) = \mathcal{G}r_{N_H(P)}^H(W)$.

Clearly

$$N_K(P) = N_G(P), \quad N_H(P) = N_{(K\cap H)}(P), \quad \mathcal{G}r_{N_K(P)}^K(\mathcal{G}r_K^G(V)) \cong \mathcal{G}r_{N_G(P)}^G(V)$$

in $\mathcal{O}N_G(P)$-mod and $\mathcal{G}r_{N_H(P)}^{K\cap H}\big(\mathcal{G}r_{K\cap H}^H(W)\big) \cong \mathcal{G}r_{N_H(P)}^H(W)$ in $\mathcal{O}N_H(P)$-mod. From [3, Chapter 5, Theorem 3.12], we conclude that $(b_P)^{K\cap H}$ is defined and $(b_P)^{(K\cap H)} = b_{(K\cap H)}$ and that $(B_P)^K$ is defined and $(B_P)^K = B_K$. Also from [3, Chapter 5, Theorem 3.1 (ii)], [2, Proposition 1.7] and Proposition 3 (a), we deduce that $(b_P)^{N_G(P)}$ is defined and $(b_P)^{N_G(P)} = B_P$.

Here $(B_P)^K = B_K = ((b_P)^{N_G(P)})^K$ and so [3, Chapter 5, Lemma 3.4] implies that $(b_P)^K = B_K$. Since $(b_P)^{(K\cap H)}$ is defined and $(b_P)^{(K\cap H)} = b_{(K\cap H)}$, the same lemma forces $((b_P)^{(K\cap H)})^K = B_K = (b_{(K\cap H)})^K$. This is the proof given by the referee of:

Proposition 5. *As in* Proposition 4 *and with the notation above,* $(b_{K\cap H})^K$ *is defined and* $(b_{K\cap H})^K = B_K$.

Finally a question:

QUESTION 6. In the situation of Proposition 5, is $b_{(K\cap H)}({}^x(b_{(K\cap H)})) = 0$ for all $x \in K - (K \cap H)$?

References

[1] W. Feit: The Representation Theory of Finite Groups, North-Holland, Amsterdam, 1982.

[2] M.E. Harris: *Ordinary induction from a subgroup and finite group block theory*, Osaka J. Math. **44** (2007), 147–158.

[3] H. Nagao and Y. Tsushima: Representations of Finite Groups, Academic Press, Boston, MA, 1989.

[4] L. Puig: *Local block theory in p-solvable groups*; in The Santa Cruz Conference on Finite Groups (Univ. California, Santa Cruz, Calif., 1979), Proc. Sympos. Pure Math. **37**, Amer. Math. Soc., Providence, R.I, 1980, 385–388.

[5] J. Thévenaz: G-Algebras and Modular Representation Theory, Oxford University Press, New York, 1995.

[6] A. Watanabe: *On generalized decomposition numbers and Fong's reductions*, Osaka J. Math. **22** (1985), 393–400

Department of Mathematics
Statistics and Computer Science (M/C 249)
University of Illinois at Chicago
851 South Morgan Street
Chicago, IL 60607–7045
U.S.A.
e-mail: harris@math.uic.edu