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CORRECTED ENERGY OF THE REEB DISTRIBUTION OF A 3-SASAKIAN MANIFOLD

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Abstract

In this paper we show that the Reeb distribution on a spherical space form which admits a 3-Sasakian structure minimizes the corrected energy. Analogously for the characteristic distribution of the normal complex contact structure on the complex projective space $\mathbb{C}P^{2m+1}$ induced via the Hopf fibration $S^1 \hookrightarrow S^{4m+3} \to \mathbb{C}P^{2m+1}$. This last result is a consequence of a more general result on the corrected energy of the characteristic distribution of a compact twistor space over a quaternionic-Kähler manifold with positive scalar curvature (equipped with a normal complex contact metric structure).

1. Introduction

Let (M, g) be a compact Riemannian manifold. The question of to measure how far from being parallel a unit vector field, has been studied by several authors and in many different contexts. In [4] Chacon, Naveira and Weston, extending this question, defined the energy $\mathcal{E}(\mathcal{V})$ of a k-dimensional distribution $\mathcal{V}$ on M and studied the first and the second variation of the energy. Gil-Medrano, Gonzalez-Davila and Vanhecke [8] studied k-dimensional distributions as harmonic maps between the Riemannian manifold (M, g) and the Grassmann bundle $(G(k, M), g_s)$, where g_s is the induced Sasaki metric. The (quaternionic) Hopf distribution $S^3 \hookrightarrow S^{4m+3} \to \mathbb{H}P^m$, that is, the Reeb distribution of the natural 3-Sasakian structure on S^{4m+3}, is an instable critical point [4]. Then, Chacon and Naveira [5] defined a corrected energy $\mathcal{D}(\mathcal{V})$ of a k-dimensional distribution and proved, by using a result of [6], that the Hopf distribution is a minimum of $\mathcal{D}(\mathcal{V})$ in the set of all integrable 3-dimensional distributions on S^{4m+3}. In [8] the authors proved that the Reeb distribution of a 3-Sasakian manifold (M, η_i, ξ_i, g) defines a harmonic map between the Riemannian manifold (M, g) and the Grassmann bundle $(G(3, M), g_s)$.

Since the result of minimality of the corrected energy for the Hopf distribution was a single application of the corrected energy, Blair and Turgut Vanli [2] considered the question of extending this result for the Reeb distribution of an arbitrary compact 3-Sasakian manifold and for the characteristic distribution of a compact normal

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complex contact manifold. Unfortunately, their demonstrations don't prove the results enunciated in Theorems 1, 2 of [2], more precisely they prove only that for the Reeb distribution of a compact 3-Sasakian manifold holds the equality in Theorem 1 of [5] (Theorem A in our Section 2), similarly for a compact normal complex contact metric manifold. So, the result related to the Hopf distribution is the only result which gives a minimum for the corrected energy.

In this paper, as a consequence of a more general result (Theorem 3.1), we show, by using a direct method, that the Reeb distribution on a spherical space form which admits a 3-Sasakian structure minimizes the corrected energy in the set of all integrable 3-dimensional distributions. In particular, we get that for the natural 3-Sasakian structures on the sphere S^{4m+3}, on the real projective space $\mathbb{R}P^{4m+3}$ and on the lens spaces L^{4m+3}, the Reeb distribution is a minimum of $\mathcal{D}(\mathcal{V})$. Moreover, as a consequence of Theorem 4.1, we show that the characteristic distribution of a compact twistor space over a quaternionic-Kähler manifold with positive scalar curvature (equipped with a IK normal complex contact metric structure) is a minimum for the corrected energy in the set of all integrable 2-dimensional distributions $\mathcal{V}$ with curvature $K(\mathcal{V}) \leq 4$. In particular, the characteristic distribution of the natural complex contact metric structure on the complex projective space $\mathbb{C}P^{2m+1}$ induced via the Hopf fibration $S^1 \hookrightarrow S^{4m+3} \to \mathbb{C}P^{2m+1}$, is a minimum for the corrected energy in the set of all integrable 2-dimensional distributions.

2. Energy of Distributions

Let (M, g) be a compact oriented Riemannian manifold of dimension n with a k-dimensional distribution $\mathcal{V}$ and let $\mathcal{H}$ be the orthogonal complementary distribution of dimension $n-k$. Let $\{E_1, \dots, E_n\}$ be a positive orthonormal local frame such that $\{E_1, \dots, E_k\}$ span $\mathcal{V}$ and $\{E_{k+1}, \dots, E_n\}$ span $\mathcal{H}$. We assume the following index convention: $a, b = 1, \dots, n$; $i, j = 1, \dots, k$; $\alpha, \beta = k+1, \dots, n$. The second fundamental form of the distribution $\mathcal{V}$ in the direction of E_α and the second fundamental form of the distribution $\mathcal{H}$ in the direction of E_i are defined, respectively, by the coefficients

$$h^\alpha_{ij} = g(\nabla_{E_i} E_j, E_\alpha) \quad \text{and} \quad h^i_{\alpha\beta} = g(\nabla_{E_\alpha} E_\beta, E_i).$$

The mean curvature vectors $\vec{H}_\mathcal{V}$ and $\vec{H}_\mathcal{H}$ are defined by

$$\vec{H}_\mathcal{V} = \frac{1}{k} \sum_\alpha \left(\sum_i h^\alpha_{ii} \right) E_\alpha, \quad \vec{H}_\mathcal{H} = \frac{1}{n-k} \sum_i \left(\sum_\alpha h^i_{\alpha\alpha} \right) E_i.$$

The vector fields E_i $(i = 1, 2, \dots, k)$ are called *$\mathcal{H}$-conformal* if they are conformal vector fields for horizontal ones, that is,

$$(\mathcal{L}_{E_i} g)(X, Y) = f_i g(X, Y), \quad \forall X, Y \in \mathcal{H},$$

where $\mathcal{L}_Z$ denotes the Lie derivative and f_i is a function on M. Killing vector fields are $\mathcal{H}$-conformal with $f = 0$. If $G(k, M)$ denotes the Grassmann bundle of oriented k-planes in the tangent spaces of M, then the distribution $\mathcal{V}$ gives a section $\sigma : M \to G(k, M)$ of the Grassmann bundle and may be considered as a global smooth section of the tensor bundle $\bigwedge^k(M)$, also denoted by σ. It can be expressed locally as $\sigma = E_1 \wedge \cdots \wedge E_k$. The energy of the distribution $\mathcal{V}$ is then defined as the energy of the corresponding unit section σ, where $G(k, M)$ is considered with the induced Sasaki metric from $\bigwedge^k(M)$ (see [8], [4], [14]):

$$\mathcal{E}(\mathcal{V}) = \frac{n}{2}\operatorname{vol}(M) + \frac{1}{2}\int_M \|\nabla\sigma\|^2 v_g,$$

where the norm of the covariant derivative of the unit section σ is given by:

$$\|\nabla\sigma\|^2 = \sum_{\alpha}\|\nabla_{E_\alpha}\sigma\|^2 = \sum_{i,j,\alpha}(h_{ij}^{\alpha})^2 + \sum_{i,\alpha,\beta}(h_{\alpha\beta}^{i})^2.$$

We note that $\|\nabla\sigma\| = \|\nabla\sigma^{\perp}\|$ and hence $\mathcal{E}(\mathcal{V}) = \mathcal{E}(\mathcal{H})$. If $\mathcal{V}$ is defined by a unit vector field, then the energy of $\mathcal{V}$ is the energy studied by Wood [20]. Wiegmink [19] defined the *total bending* of a unit vector field U as

$$\mathcal{B}(U) = \frac{1}{(n-1)\operatorname{vol}(S^n)}\int_M \|\nabla U\|^2 v_g.$$

So, to study the possible minima of the total bending $\mathcal{B}(U)$ is the same as to study the possible minima of the energy. Chacon and Naveira [5] introduced the corrected energy of a distribution $\mathcal{V}$ as

$$\mathcal{D}(\mathcal{V}) = \int_M (\|\sigma\|^2 + (n-k)(n-k-2)\|\vec{H}_{\mathcal{H}}\|^2 + k^2\|\vec{H}_{\mathcal{V}}\|^2)v_g.$$

This corrected energy is not an extension of the corrected total bending defined in [3]. The main results of Chacon and Naveira [5] are the following theorems.

Theorem A. *If $\mathcal{V}$ is integrable, then*

$$\mathcal{D}(\mathcal{V}) \geq \int_M \left(\sum_{i,\alpha} K(E_i, E_\alpha)\right) v_g, \tag{2.1}$$

where $K(E_i, E_\alpha)$ is the sectional curvature of the plane spanned by $E_i \in \mathcal{V}$ and $E_\alpha \in \mathcal{H}$.

Moreover (see [5], p.103), the equality in (2.1) holds if and only if $\mathcal{V}$ is totally geodesic and $E_1, \ldots, E_k$ are $\mathcal{H}$-conformal.

Theorem B. *Among the integrable distributions of dimension* 3 *of* S^{4m+3}, *the* (*quaternionic*) *Hopf distribution* $S^3 \hookrightarrow S^{4m+3} \to \mathbb{H}P^m$ *minimizes the corrected energy.*

3. Corrected energy and 3-Sasakian manifolds

We start recalling some basic definitions and properties about contact metric manifolds and 3-Sasakian manifolds (for further details and informations, we refer to [1]). A *contact manifold* is a $(2n+1)$-dimensional manifold M equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. Given a contact form η, there exists a unique vector field ξ, called the *characteristic vector field* or the *Reeb vector field*, satisfying $\eta(\xi) = 1$ and $d\eta(\xi, \cdot) = 0$. A Riemannian metric g is said to be an associated metric if there exists a tensor field ϕ of type (1, 1) such that

$$\eta = g(\xi, \cdot), \quad d\eta = g(\cdot, \phi\,\cdot), \quad \phi^2 = -I + \eta \otimes \xi.$$

In this case (η, g), or (η, g, ξ, ϕ), is called a *contact metric structure* and M a *contact metric manifold*. If the almost complex structure J on $M \times \mathbb{R}$ defined by

$$J\left(X,\ f\frac{d}{dt}\right) = \left(\phi X - f\xi,\ \eta(X)\frac{d}{dt}\right)$$

is integrable, M is said to be *Sasakian*. If ξ is a Killing vector field, or equivalently if the tensor $\mathcal{L}_\xi\phi$ vanishes, M is said to be K-contact. A Sasakian manifold is K-contact, moreover we have

$$\nabla\xi = -\phi \quad \text{and} \quad K(\xi, E) = 1, \tag{3.1}$$

where $E \in \ker\eta$ is a unit vector field and $K(\xi, E)$ denotes the sectional curvature along the plane section containing E and ξ. An *almost contact metric structure* is defined by a tensor field ϕ of type (1, 1), a vector field ξ, a 1-form η and a metric g satisfying

$$\eta(\xi) = 1, \quad \phi^2 = -I + \eta \otimes \xi, \quad g(\phi\,\cdot, \phi\,\cdot) = g - \eta \otimes \eta.$$

Note that these conditions imply $\phi(\xi) = 0$, $\eta \circ \phi = 0$ and $\eta = g(\cdot, \xi)$. Of course, a contact metric structure is an almost contact metric structure.

An *almost contact metric* 3-*structure* is defined as three almost contact metric structures $(g, \eta_i, \xi_i, \phi_i)$, $i = 1, 2, 3$, such that

$$\phi_i\phi_j - \xi_i \otimes \eta_j = \phi_k = -\phi_j\phi_i + \xi_j \otimes \eta_i, \quad \phi_i\xi_j = \xi_k, \quad \eta_i\phi_j = \eta_k, \tag{3.2}$$

for cyclic permutation (i, j, k) of (1, 2, 3). In this case M has to be of dimension $4m + 3$ for a non-negative integer m. A *contact metric* 3-*structure* is defined as three contact metric structures $(g, \eta_i, \xi_i, \phi_i)$, satisfying (3.2). In such case the 3-dimensional distribution ξ determined by the tri-vector $\xi = \xi_1 \wedge \xi_2 \wedge \xi_3$ is called the *Reeb distribution* or the *characteristic distribution*. If each contact metric structure $(g, \eta_i, \xi_i, \phi_i)$ is Sasakian, then the contact metric 3-structure is called a 3-*Sasakian structure* and the manifold is called a 3-*Sasakian manifold*.

Now, we suppose that M is a compact 3-Sasakian manifold of dimension $4m + 3$. Using $(3.1)_1$ and (3.2), we get

$$\nabla_{\xi_j} \xi_i = -\phi_i \xi_j = -\xi_k \quad \text{and} \quad [\xi_i, \xi_j] = 2\xi_k.$$

Thus, the Reeb distribution ξ is integrable and totally geodesic (i.e., $h^{\alpha}_{ij} = 0$). Moreover, the Reeb vector fields ξ_i $(i = 1, 2, 3)$ are Killing, and using $(3.1)_2$, we obtain readily $\sum_{i,\alpha} K(\xi_i, E_\alpha) = 12m$. On the other hand (see [5], p.103) the equality in (2.1) holds if and only if the distribution $\mathcal{V}$ is totally geodesic and the vector fields E_i are $\mathcal{H}$-conformal. Consequently, in our case, we get (see also the proof of Theorem 1 in [2]):

$$\mathcal{D}(\xi) = \int_M \sum_{i,\alpha} K(\xi_i, E_\alpha) v_g = 12m \operatorname{vol}(M). \tag{3.3}$$

Let $\mathcal{V}$ be an arbitrary integrable 3-dimensional distribution on M. Suppose that $\mathcal{V}$ is expressed locally by the tri-vector $V = E_1 \wedge E_2 \wedge E_3$, where $\{E_1, E_2, E_3, E_4, \dots, E_n\}$, $n = 4m + 3$, is a positive orthonormal local frame. We show that the scalar

$$K(\mathcal{V}) := K(E_1, E_2) + K(E_1, E_3) + K(E_2, E_3),$$

that we call *the curvature of the distribution* $\mathcal{V}$, depends only on the distribution, that is, is invariant under adapted orthonormal frame changes. In dimension 3, $2K(\mathcal{V})$ is exactly the scalar curvature of the Riemannian manifold. We consider in general $m \geq 0$, and denote the dual basis of $\{E_i\}$ and the curvature 2-forms respectively by

$$\{\vartheta^1, \dots, \vartheta^n\} \quad \text{and} \quad \Omega_{ab}(X, Y) = g(R(X, Y)E_a, E_b),$$

where R is the curvature tensor with the convention sign $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$. Consider the following n-form

$$\Omega := \sum_{\sigma \in \mathfrak{C}_3, \tau \in \mathfrak{C}_3^-} \epsilon(\sigma)\epsilon(\tau)\Omega_{\sigma(1)\tau(4)} \vartheta^{\sigma(2)} \wedge \vartheta^{\sigma(3)} \wedge \vartheta^{\tau(5)} \wedge \cdots \wedge \vartheta^{\tau(n)},$$

where $\mathfrak{C}_3$ denotes the group of permutations of $\{1, 2, 3\}$, $\mathfrak{C}_3^-$ denotes the group of permutations of $\{4, 5, \dots, n\}$, and $\epsilon(\sigma)$ denotes the signature of the permutation σ. Such

n-form is invariant under adapted orthonormal frame changes and satisfies ([5], p.102, formula (13))

$$\Omega(E_1, \ldots, E_n) = -2(4m-1)! \sum_{i,\alpha} K(E_i, E_\alpha). \tag{3.4}$$

Moreover,

$$\begin{aligned}\sum_{i,\alpha} K(E_i, E_\alpha) &= \sum_{i,\alpha} R(E_i, E_\alpha, E_i, E_\alpha)\\ &= \sum_{i,a} R(E_i, E_a, E_i, E_a) - \sum_{i,j} R(E_i, E_j, E_i, E_j)\\ &= \sum_i \mathrm{Ric}(E_i, E_i) - 2\sum_{i<j} K(E_i, E_j)\\ &= \sum_i \mathrm{Ric}(E_i, E_i) - 2K(\mathcal{V}).\end{aligned}$$

Since, any 3-Sasakian manifold is Einstein (Kashiwada [11]) with scalar curvature $r = (4m+2)(4m+3)$, the above formula gives

$$\sum_{i,\alpha} K(E_i, E_\alpha) = 3(4m+2) - 2K(\mathcal{V}). \tag{3.5}$$

From (3.4) and (3.5), we deduce that $K(\mathcal{V})$ is invariant under adapted orthonormal frame changes. If we assume $K(\mathcal{V}) \leq 3$, from (3.3), (3.5) and Theorem A we obtain

$$\mathcal{D}(\mathcal{V}) \geq \int_M \left(\sum_{i,\alpha} K(E_i, E_\alpha)\right) v_g \geq 12m \operatorname{vol}(M) = \mathcal{D}(\xi), \tag{3.6}$$

where the equality holds if and only if $K(\mathcal{V}) = 3$, $\mathcal{V}$ is totally geodesic and E_1, E_2, E_3 are $\mathcal{H}$-conformal. Besides, we recall that Kashiwada [12] proved the remarkable result that every contact metric 3-structure is 3-Sasakian (see also [17] for a direct proof of such result in dimension three). Thus, we get the following

Theorem 3.1. *Let M be a compact 3-contact metric manifold. Then, among the integrable 3-dimensional distributions $\mathcal{V}$ of M with curvature $K(\mathcal{V}) \leq 3$, the Reeb distribution ξ minimizes the corrected energy $\mathcal{D}(\mathcal{V})$. Moreover, $\mathcal{D}(\mathcal{V}) = \mathcal{D}(\xi)$ if and only if $\mathcal{V}$ is totally geodesic, E_1, E_2, E_3 are $\mathcal{H}$-conformal and $K(\mathcal{V}) = 3$.*

Now, we give an interesting application of Theorem 3.1. Each compact Riemannian manifold of constant sectional curvature $+1$, $\dim M = 4m+3$, is a spherical space form $(S^{4m+3}/\Gamma, g)$, where Γ is a finite group of $O(4m+4)$ in which only the identity has $+1$ as an eigenvalue, and g is the metric on the quotient space induced by the canonical metric g_0 on S^{4m+3}. If $(\eta_i, \phi_i, \xi_i, g_0)$ is the standard 3-Sasakian structure on S^{4m+3}, the spherical space forms S^{4m+3}/Γ which admit a 3-Sasakian structure are defined by the groups Γ that leave invariant each of the three Sasakian structures (η_i, ϕ_i, ξ_i). Of course, on such spaces $K(\mathcal{V}) = 3$ for any 3-dimensional distribution. Then, Theorem 3.1 implies the following

Theorem 3.2. *Let M be a spherical space form which admits a 3-Sasakian structure. Then, among the integrable 3-dimensional distributions of M, the Reeb distribution ξ minimizes the corrected energy.*

Since a 3-Sasakian manifold is Einstein, then a 3-Sasakian manifold of dimension three is of constant sectional curvature +1. Therefore, we get that: *for a compact 3-contact metric manifold of dimension three, the Reeb distribution minimizes the corrected energy*.

EXAMPLES. For spherical space forms of dimension 3, Sasaki [18] classified completely the groups Γ that leave invariant each of three Sasakian structures. More precisely, the groups Γ are all finite subgroups of Clifford translations on S^3 and are equivalent to either one of

(a) $\Gamma = \{I\}$;

(b) $\Gamma = \{\pm I\}$;

(c) Γ is the cyclic group of order $q > 2$ generated by

$$\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad \text{where} \quad A = \begin{pmatrix} \cos\dfrac{2\pi}{q} & -\sin\dfrac{2\pi}{q} \\ \sin\dfrac{2\pi}{q} & \cos\dfrac{2\pi}{q} \end{pmatrix};$$

(d) Γ is a group of Clifford translations corresponding to a binary dihedral group or the binary polyhedral groups of the regular tetrahedron T^*, octahedron O^* or icosahedron I^*.

In dimension $4m + 3 > 3$, examples of spherical space forms which admit a 3-Sasakian structure are given by $M = S^{4m+3}/\Gamma_r$,

$$\Gamma_r = \Gamma \times \cdots \times \Gamma \quad (r = m + 1 \text{ factors}),$$

where Γ is any one of the groups classified in (a)–(d). In particular, the sphere S^{4m+3}, the real projective space $\mathbb{R}P^{4m+3}$ and the lens spaces $L^{4m+3} = S^{4m+3}/\Gamma_r$, where Γ is of type (c), admit a 3-Sasakian structure. Therefore, in all these cases the Reeb distribution minimizes the corrected energy and so, Theorem 3.2 extends Theorem B.

REMARK 3.1. Let M be a 3-Sasakian manifold. If M has constant ϕ_i-holomorphic sectional curvature c (for a fixed $i = 1, 2, 3$), then $c = +1$ and hence M has constant sectional curvature +1. In fact a Sasakian manifold of constant ϕ-holomorphic sectional curvature c is η-Einstein ([1], p.113) and it is Einstein if and only if $c = +1$.

4. The case of complex contact metric manifolds

We now consider the case of the characteristic distribution of a complex contact metric manifold. We start recalling some basic definitions and properties about complex contact metric manifolds and refer to [1] and [15] for further details and information on such spaces. A *complex contact manifold* is a complex manifold M of complex dimension $2m + 1$ together with an open covering $\{\mathcal{U}\}$ by coordinate neighborhoods such that

a) on each $\mathcal{U}$, there is a holomorphic 1-form θ with $\theta \wedge (d\theta)^m \neq 0$ everywhere;

b) if $\mathcal{U} \cap \mathcal{U}' \neq \emptyset$, there is a non-vanishing holomorphic function f such that $\theta' = f\theta$.

The complex contact form determines a non-integrable (horizontal) distribution $\mathcal{H}_0$ by the equation $\theta = 0$. A complex contact structure, that we denote by $\{\theta\}$, is given by a global 1-form if and only if the first Chern class vanishes. Let $(M, \{\theta\})$ be a complex contact manifold. The local contact form θ is $u - iv$ to within a non-vanishing complex-valued function multiple. Thus $v = u \circ J$ since θ is holomorphic, where J is complex structure on M. Locally we can define a vector field U satisfying the conditions: $(du)(U, X) = 0$ for all $X \in \mathcal{H}_0$, $u(U) = 1$ and $v(U) = 0$. Then, we have a global distribution $\mathcal{V}_0$ locally defined by the bi-vector $U \wedge V$, where $V = -JU$, with $TM = \mathcal{V}_0 \oplus \mathcal{H}_0$. $\mathcal{V}_0$ is called the *characteristic distribution* or the *vertical distribution*. The characteristic distribution is usually assumed integrable because for all known examples this condition is satisfied.

Let $(M, J, \{\theta\})$ be a complex contact manifold. A Hermitian metric g is called an *associated metric* if:

1) on each $\mathcal{U}$, there exist tensor fields G and $H = GJ$ of type (1, 1) such that

$$H^2 = G^2 = -I + u \otimes U + v \otimes V, \quad GJ = -JG, \quad GU = 0, \quad g(GX, Y) = -g(X, GY);$$

$u(X) = g(U, X)$, $(du)(X, Y) = g(X, GY) + (\sigma \wedge v)(X, Y)$, $(dv)(X, Y) = g(X, HY) - (\sigma \wedge u)(X, Y)$, where $\sigma(X) = g(\nabla_X U, V)$;

2) on $\mathcal{U} \cap \mathcal{U}' \neq$, we have

$$u' = au - bv, \quad v' = bu + av, \quad G' = aG - bH, \quad H' = bG + aH,$$

where a, b are functions on $\mathcal{U} \cap \mathcal{U}'$ with $a^2 + b^2 = 1$.

In such case $(J, \{\theta\}, g)$, or (u, v, U, V, G, H, g), is called complex contact metric structure and M a *complex contact metric manifold*. Foreman in his thesis (cf. [1], p.192) proved that each complex contact manifold has a complex contact metric structure. If $X \in \mathcal{H}_{0p}$ is a unit vector field, the plane in T_pM spanned by X and $X' = aGX + bHX$,

$a, b \in \mathbb{R}$, $a^2 + b^2 = 1$, is called GH-plane and its sectional curvature the *GH-sectional curvature*. A complex contact metric manifold M is said to be *normal* if ([15], [1]):

$$S(X, Y) = T(X, Y) = 0, \quad \forall X, Y \in \mathcal{H}_0 \quad \text{and} \quad S(U, X) = T(V, X) = 0, \quad \forall X, \tag{4.1}$$

where S, T are $(1, 2)$ tensors defined by

$$\begin{aligned} S(X, Y) &= [G, G](X, Y) + 2g(X, GY)U - 2g(X, HY)V + 2v(Y)HX - 2v(X)HY \\ &\quad + \sigma(GY)HX - \sigma(GX)HY + \sigma(X)GHY - \sigma(Y)GHX, \\ T(X, Y) &= [H, H](X, Y) + 2g(X, GY)U + 2g(X, HY)V + 2u(Y)GX - 2u(X)GY \\ &\quad + \sigma(HX)GY - \sigma(HY)GX + \sigma(X)GHY - \sigma(Y)GHX. \end{aligned}$$

One consequence of normality is that all the sectional curvatures of plane sections spanned by a vector in $\mathcal{V}_0$ and a vector in $\mathcal{H}_0$ are equal to $+1$. Thus, if $\{E_\alpha\}$ is an orthonormal basis of the horizontal distribution $\mathcal{H}_0$, we have

$$\sum_{\alpha=1}^{4m} (K(U, E_\alpha) + K(V, E_\alpha)) = 8m. \tag{4.2}$$

Consequences of normality are also

$$\nabla_X U = -GX + \sigma(X)V, \quad \nabla_X V = -HX - \sigma(X)U.$$

Thus,

$$(\mathcal{L}_U g)(X, Y) = g(-GX + \sigma(X)V, Y) + g(X, -GY + \sigma(Y)V) = 0, \quad \forall X, Y \in \mathcal{H}_0,$$

similarly for V, that is, U, V are $\mathcal{H}_0$-Killing vector fields. Moreover,

$$g(\nabla_U U, X) = g(\nabla_V V, X) = g(\nabla_U V, X) = g(\nabla_V U, X) = 0, \quad \forall X \in \mathcal{H}_0,$$

that is, $\mathcal{V}_0$ is totally geodesic. Consequently, as in the 3-Sasakian case, using (4.1) we get

$$\mathcal{D}(\mathcal{V}_0) = \int_M \sum_\alpha (K(U, E_\alpha) + K(V, E_\alpha)) v_g = 8m \operatorname{vol}(M). \tag{4.3}$$

This result was also obtained in [2], more precisely (4.3) is the corrected statement of Theorem 2 of [2]. Of course, (4.3) does not imply, in general, that $\mathcal{V}_0$ minimizes the corrected energy. However, in special cases this property is true. We note that the Ricci curvatures $\operatorname{Ric}(U, U)$ and $\operatorname{Ric}(V, V)$, are given by

$$\operatorname{Ric}(U, U) = \operatorname{Ric}(V, V) = \sum_{\alpha=1}^{4m} K(V, E_\alpha) + K(U, V) = 4m + K(\mathcal{V}_0), \tag{4.4}$$

where $\{E_\alpha\}$ is an orthonormal basis of the horizontal distribution $\mathcal{H}_0$. Now, let $\mathcal{V}$ be a 2-dimensional integrable distribution on M and let $\mathcal{H}$ be the orthogonal complementary distribution of dimension $4m$. Let $\{V_1, V_2, W_1, \ldots, W_{4m}\}$ be a positive orthonormal local frame such that $\{V_1, V_2\}$ span $\mathcal{V}$ and $\{W_1, \ldots, W_{4m}\}$ span $\mathcal{H}$. Using Theorem A, we get

$$\mathcal{D}(\mathcal{V}) \geq \int_M \sum_{\alpha=1}^{4m} (K(V_1, W_\alpha) + K(V_2, W_\alpha)) v_g. \tag{4.5}$$

Suppose that the complex contact metric manifold M is Einstein, then (4.4) gives that the Ricci tensor is given by

$$\mathrm{Ric} = (4m + K(\mathcal{V}_0))g,$$

and $K(\mathcal{V}_0)$ is a constant. Consequently

$$\begin{aligned} \sum_{\alpha=1}^{4m} (K(V_1, W_\alpha) + K(V_2, W_\alpha)) &= \sum_{\alpha=1}^{4m} (R(V_1, W_\alpha, V_1, W_\alpha) + R(V_2, W_\alpha, V_2, W_\alpha)) \\ &= \mathrm{Ric}(V_1, V_1) + \mathrm{Ric}(V_2, V_2) - 2K(V_1, V_2) \\ &= 8m + 2(K(\mathcal{V}_0) - K(\mathcal{V})). \end{aligned} \tag{4.6}$$

Moreover (see [5], p.103), the equality in (4.5) holds if and only if $\mathcal{V}$ is totally geodesic and V_1, V_2 are $\mathcal{H}$-conformal. Therefore, using (4.3), (4.5) and (4.6), we obtain the following

Theorem 4.1. *Let M be a compact Einstein normal complex contact metric manifold. Then, among the integrable 2-dimensional distributions $\mathcal{V}$ of M with curvature $K(\mathcal{V}) \leq K(\mathcal{V}_0)$, the characteristic distribution $\mathcal{V}_0$ minimizes the corrected energy $\mathcal{D}(\mathcal{V})$. Moreover, $\mathcal{D}(\mathcal{V}) = \mathcal{D}(\mathcal{V}_0)$ if and only if $\mathcal{V}$ is totally geodesic, V_1, V_2 are $\mathcal{H}$-conformal and $K(\mathcal{V}) = K(\mathcal{V}_0)$.*

There exist interesting examples of Einstein normal complex contact metric manifolds. We recall that a complex contact metric manifold M is said to be IK-normal, that is, in the sense of Ishihara-Konishi [10], if the tensors S and T vanish. Of course an IK-normal complex contact metric structure is a normal complex contact metric structure in the sense of (4.1). Ishihara and Konishi in the same paper proved that a such manifold is Kähler-Einstein with first Chern class $c_1(M) > 0$. Then, Foreman ([7], Theorem 6.1 and Proposition 6.3) proved that M is isometric to a twistor space of a quaternionic-Kähler manifold (of positive scalar curvature), moreover the curvature tensor of M satisfies

$$R(X,Y)U = -u(Y)X + u(X)Y - v(Y)JX + v(X)JY - 2g(JX,Y)V. \tag{4.7}$$

Then,

$$K(\mathcal{V}_0) = g(R(U,V)U,V) = 4 \quad \text{and} \quad \mathrm{Ric} = (4m+1)g. \tag{4.8}$$

Conversely, Foreman [7], using a result of Ishihara-Konishi [9], proved that every twistor space $\mathcal{Z}$ of a quaternionic-Kähler manifold with positive scalar curvature has an IK-normal complex contact metric structure satisfying (4.7). Now, let $\mathcal{Z}$ be a compact complex $(2m+1)$-dimensional manifold with a complex contact structure. LeBrun [16] proved that if $\mathcal{Z}$ admits a Kähler-Einstein metric of positive scalar curvature, then it is the twistor space of a quaternionic-Kähler manifold with positive scalar curvature. Consequently, using the above results, we get that: *a compact Kähler-Einstein manifold $\mathcal{Z}$ of positive scalar curvature*, $\dim_{\mathbb{C}} \mathcal{Z} = 2m+1$, *with a complex contact structure, admits an Einstein normal complex contact metric structure with scalar curvature $r =$* $2(2m+1)(4m+1)$. Another way to build twistor spaces that admit an Einstein normal complex contact metric structure is the following. If $\bar{M}$ is a 3-Sasakian manifold and one of the Reeb vector fields ξ_i, say ξ_1, is regular, then the orbit space $M = \bar{M}/\xi_1$ admits an IK-normal complex contact metric structure which is Kähler-Einstein of positive scalar curvature (see [9]). Thus, M is isometric to a twistor space of a quaternionic-Kähler manifold with positive scalar curvature. So, the class of twistor spaces of a quaternionic-Kähler manifold with positive scalar curvature is a class of Einstein normal complex contact metric manifolds satisfying (4.8). Then, from Theorem 4.1 we get

Theorem 4.2. *Let $\mathcal{Z}$ be a compact twistor space of a quaternionic-Kähler manifold with positive scalar curvature* (*equipped with an IK-normal complex contact metric structure*). *Then, among the* 2*-dimensional integrable distributions $\mathcal{V}$ on $\mathcal{Z}$ with curvature $K(\mathcal{V}) \le 4$, the characteristic distribution $\mathcal{V}_0$ minimizes the corrected energy.*

A particular case of the previous examples is the odd-dimensional complex projective space $\mathbb{C}P^{2m+1}$ equipped with the standard Fubini-Study metric g of constant holomorphic sectional curvature +4. In fact, $\mathbb{C}P^{2m+1}$ is the twistor space of the quaternionic-Kähler manifold $\mathbb{Q}P^{2m+1}$. Ishihara and Konishi [9] proved that $\mathbb{C}P^{2m+1}$ admits a normal complex contact metric structure $(J, \{\theta\}, g)$ closely related to the standard Sasakian 3-structure on the sphere S^{4m+3}. More precisely this structure is induced via the Hopf fibration $S^1 \hookrightarrow S^{4m+3} \to \mathbb{C}P^{2m+1}$. Let $\mathcal{V}$ be a 2-dimensional integrable distribution on $\mathbb{C}P^{2m+1}$, locally defined by the bi-vector $V_1 \wedge V_2$. In such case, since $\mathbb{C}P^{2m+1}$ has constant holomorphic sectional curvature $c = +4$, the sectional curvature

$K(\mathcal{V})$ satisfies (see, for example, [13] p.167)

$$K(\mathcal{V}) = 1 + 3\cos\zeta(\mathcal{V}) \leq 4,$$

where $\cos\zeta(\mathcal{V}) = |g(V_1, JV_2)|$, and $K(\mathcal{V}) = 4$ if and only if $V_2 = \pm JV_1$. Then, from Theorem 4.2 we get

Corollary 4.1. *Among the 2-dimensional integrable distributions $\mathcal{V}$ on $\mathbb{C}P^{2m+1}$, the characteristic distribution $\mathcal{V}_0$ of the normal complex contact metric structure induced via the Hopf fibration $S^1 \hookrightarrow S^{4m+3} \to \mathbb{C}P^{2m+1}$ minimizes the corrected energy. Moreover, if $\mathcal{V}$ is locally defined by the bi-vector $V_1 \wedge V_2$, then $\mathcal{D}(\mathcal{V}) = \mathcal{D}(\mathcal{V}_0)$ iff $\mathcal{V}$ is totally geodesic, $V_2 = \pm JV_1$ and V_1, V_2 are $\mathcal{H}$-conformal.*

REMARK 4.1. We note that if M is a compact normal complex contact metric manifold satisfying one of the following conditions:
a) M has constant holomorphic sectional curvature c,
b) M has constant GH-sectional curvature +1 and $K(\mathcal{V}_0) = 4$,
then it is holomorphically isometric to the complex projective space $\mathbb{C}P^{2m+1}$ with the Fubini-Study metric of constant holomorphic sectional curvature +4. In fact, if we assume a), Proposition 5.7 of [15] gives that the manifold is Kähler and $c = +4$. Moreover, if we assume b), Theorem 5.8 (first part) of [15] gives that M is a Kähler manifold of constant holomorphic sectional curvature +4. On the other hand, a compact Kähler manifold of positive holomorphic sectional curvature is necessarily simply connected (see, for example [13] p.171). Therefore, in both cases, we get that M is holomorphically isometric to $\mathbb{C}P^{2m+1}$ with the Fubini-Study metric of constant holomorphic sectional curvature +4.

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