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ON THE LOWER BOUND OF THE K-ENERGY AND F-FUNCTIONAL

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Abstract

Using Perelman's results on Kähler-Ricci flow, we prove that the K-energy is bounded from below if and only if the F-functional is bounded from below in the canonical Kähler class.

1. Introduction

One of the central problems in Kähler geometry is to study the existence of Kähler-Einstein metrics, which is closely related to the behavior of several energy functionals. During the last few decades, these energy functionals have been intensely studied and there are many interesting results. The K-energy, which was introduced by Mabuchi in [10], plays an important role in Kähler geometry.

Let (M, ω) be an n-dimensional compact Kähler manifold with $c_1(M) > 0$. We define the space of Kähler potentials by

$$\mathcal{P}(M, \omega) = \{\varphi \in C^\infty(M, \mathbb{R}) \mid \omega + \sqrt{-1}\,\partial\bar{\partial}\varphi > 0\},$$

where $\omega \in 2\pi c_1(M)$. For any $\varphi \in \mathcal{P}(M, \omega)$, we define the K-energy by

$$\nu_\omega(\varphi) = -\frac{1}{V}\int_0^1\int_M \frac{\partial\varphi_t}{\partial t}(R_{\varphi_t} - \underline{R})\omega_{\varphi_t}^n \wedge dt \tag{1.1}$$

where φ_t ($t \in [0, 1]$) is a path in $\mathcal{P}(M, \omega)$ with $\varphi_0 = 0$ and $\varphi_1 = \varphi$, $\underline{R}$ is the average of scalar curvature, and $V = [\omega]^n$ is the volume. Bando-Mabuchi [1] showed that if M admits a Kähler-Einstein metric, then the K-energy is bouned from below. Later, Tian [16] [17] proved that the existence of Kähler-Einstein metrics is equivalent to the properness of the K-energy in the canonical Kähler class. In fact, Tian proved that the existence of Kähler-Einstein metrics is equivalent to the properness of the F-functional,

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After we posted our first version of this paper on the arXiv, we got some feedbacks that our result can be proved by the continuity method. However, the idea of the proof, which comes from our joint paper [4], is still interesting, and may have some other applications.

which was introduced by Ding-Tian [7] as follows

$$F_\omega(\varphi) = \frac{1}{V}\sum_{i=0}^{n-1}\frac{i+1}{n+1}\int_M \sqrt{-1}\,\partial\varphi\wedge\bar{\partial}\varphi\wedge\omega^i\wedge\omega_\varphi^{n-1-i} - \frac{1}{V}\int_M \varphi\omega^n - \log\left(\frac{1}{V}\int_M e^{h_\omega-\varphi}\omega^n\right). \tag{1.2}$$

To prove the convergence of Kähler-Ricci flow, Chen-Tian [5] [6] introduced a series of energy functionals $E_k(k = 0, 1, \ldots, n)$ defined by

$$\begin{aligned} E_{k,\omega}(\varphi) = {} & \frac{1}{V}\int_M\left(\log\frac{\omega_\varphi^n}{\omega^n} - h_\omega\right)\left(\sum_{i=0}^{k}\mathrm{Ric}_\varphi^i\wedge\omega^{k-i}\right)\wedge\omega_\varphi^{n-k} \\ & + \frac{1}{V}\int_M h_\omega\left(\sum_{i=0}^{k}\mathrm{Ric}_\omega^i\wedge\omega^{k-i}\right)\wedge\omega^{n-k} \\ & + \frac{n-k}{V}\int_0^1\int_M\frac{\partial\varphi_t}{\partial t}(\omega_{\varphi_t}^{k+1}-\omega^{k+1})\wedge\omega_{\varphi_t}^{n-k-1}\wedge dt, \end{aligned}$$

where h_ω is the Ricci potential defined by

$$\mathrm{Ric}(\omega) - \omega = \sqrt{-1}\,\partial\bar{\partial}h_\omega, \quad \text{and} \quad \int_M (e^{h_\omega}-1)\omega^n = 0, \tag{1.3}$$

and φ_t $(t \in [0, 1])$ is a path from 0 to φ in $\mathcal{P}(M, \omega)$. The first energy E_0 of these series is exactly the K-energy, and the second E_1 is the Liouville energy on Riemann surfaces.

There are many relations between these energy functionals. Pali [12] prove that E_1 is bounded from below if the K-energy is bounded from below. Recently, Chen-Li-Wang [4] proved the converse is also true. There are also some results on the lower bound of E_k. Following a question proposed by X.X. Chen [3], Song-Weinkove [15] showed that the existence of Kähler-Einstein metrics is equivalent to the properness of E_1 in the canonical class, and they also showed that E_k are bounded from below under some additional curvature conditions. Recently, following suggestion of X.X. Chen, the author [9] found new relations between all these functionals and generalized Pali-Song-Weinkove's results.

In summary, the relations between the existence of Kähler-Einstein metrics and these energy functionals can be roughly written as follows: M admits Kähler-Einstein metrics $\Longleftrightarrow$ the F-functional is proper $\Longleftrightarrow$ the K-energy is proper $\Longleftrightarrow$ E_1 is proper. A natural question is what will happen if these energy functionals are just bounded from below instead of proper. In this paper, we prove

Theorem 1.1. *The K-energy is bounded from below if and only if F is bounded from below on $\mathcal{P}(M, \omega)$. Moreover, we have*

$$\inf_{\omega' \in [\omega]} F_\omega(\omega') = \inf_{\omega' \in [\omega]} \nu_\omega(\omega') - \frac{1}{V} \int_M h_\omega \omega^n,$$

where h_ω is the Ricci potential with respect to the metric ω.

Combining this with the results in [4], we actually prove that F is bounded from below $\Longleftrightarrow$ the K-energy is bounded from below $\Longleftrightarrow$ E_1 is bounded from below. We expect that the lower boundedness of all energy functionals E_k is equivalent, and perhaps the lower boundedness implies the existence of singular Kähler-Einstein metrics and certain stabilities.

The idea of the proof of Theorem 1.1 is essentially due to our joint paper [4]. The key point is to estimate the difference of F and ν_ω along the Kähler-Ricci flow, and we show that the difference of these two functionals at infinity is a uniform constant independent of the initial metric of the flow. However, the proof needs Perelman's deep estimates on the Kähler-Ricci flow, while in [4] the equivalence of the K-energy and E_1 doesn't. This is because we can compare the derivatives of these energy functionals along the Kähler-Ricci flow in [4], but we don't have similar estimates in this paper. The readers are referred to [4] for details. We expect that this flow method can be used to prove the equivalence of all E_k functionals in the future.

2. Kähler-Ricci flow and the K-energy

Let (M, ω) be an n-dimensional compact Kähler manifold with $\omega \in 2\pi c_1(M) > 0$. The Kähler-Ricci flow with the initial metric $\omega_0 = \omega + \sqrt{-1}\,\partial\bar{\partial}\varphi_0$ is of the form

$$\frac{\partial \omega_\varphi}{\partial t} = \omega_\varphi - \mathrm{Ric}_\varphi\,, \quad \varphi(0) = \varphi_0. \tag{2.1}$$

It follows that on the level of Kähler potentials, the Kähler-Ricci flow becomes

$$\frac{\partial \varphi}{\partial t} = \log \frac{\omega_\varphi^n}{\omega^n} + \varphi - h_\omega, \tag{2.2}$$

where h_ω is defined by (1.3). Notice that for any solution $\varphi(t)$ of (2.2), the function $\tilde{\varphi}(t) = \varphi(t) + Ce^t$ is also a solution for any constant C. Since

$$\frac{\partial \tilde{\varphi}}{\partial t}(0) = \frac{\partial \varphi}{\partial t}(0) + C,$$

we have

$$\frac{1}{V} \int_M \frac{\partial \tilde{\varphi}}{\partial t} \omega_{\tilde{\varphi}}^n \Big|_{t=0} = \frac{1}{V} \int_M \frac{\partial \varphi}{\partial t} \omega_\varphi^n \Big|_{t=0} + C.$$

Thus we can normalize the solution $\varphi(t)$ such that the average of $(\partial\varphi/\partial t)(0)$ is any given constant.

Next we recall some basic facts on energy functionals. The K-energy, which is defined by (1.1), can be explicitly expressed as (cf. [2] [17])

$$\begin{aligned}\nu_\omega(\varphi) = {} & \frac{1}{V}\int_M \log\frac{\omega_\varphi^n}{\omega^n}\omega_\varphi^n + \frac{1}{V}\int_M h_\omega(\omega^n - \omega_\varphi^n) \\ & - \frac{1}{V}\sum_{i=0}^{n-1}\frac{n-i}{n+1}\int_M \sqrt{-1}\,\partial\varphi\wedge\bar\partial\varphi\wedge\omega^i\wedge\omega_\varphi^{n-1-i}.\end{aligned} \tag{2.3}$$

By direct calculation, the K-energy is decreasing along the Kähler-Ricci flow. In fact, for the solution $\varphi(t)$ of (2.2) we have[1]

$$\frac{d}{dt}\nu_\omega(\varphi(t)) = -\frac{1}{V}\int_M \left|\nabla\frac{\partial\varphi}{\partial t}\right|^2\omega_\varphi^n \le 0. \tag{2.4}$$

The following lemma tells us that if the K-energy is bounded from below, we can normalize the solution such that the average of $\partial\varphi/\partial t$ can be controlled. Since the normalization is crucial in Section 3, we include a proof here.

Lemma 2.1 (cf. [5]). *Suppose that the K-energy is bounded from below along the Kähler-Ricci flow. Then we can normalize the solution $\varphi(t)$ so that*

$$c(0) = \frac{1}{V}\int_0^\infty e^{-t}\int_M \left|\nabla\frac{\partial\varphi}{\partial t}\right|^2\omega_\varphi^n\wedge dt < \infty,$$

where $c(t) = (1/V)\int_M(\partial\varphi/\partial t)\omega_\varphi^n$. Then for all time $t > 0$, we have

$$c(t) > 0, \qquad \int_0^\infty c(t)\,dt < \nu_\omega(0) - \nu_\omega(\infty),$$

where $\nu_\omega(\infty) = \lim_{t\to\infty}\nu_\omega(t)$.

Proof. A simple calculation yields

$$c'(t) = c(t) - \frac{1}{V}\int_M |\nabla\dot\varphi|^2\omega_\varphi^n.$$

Define

$$\epsilon(t) = \frac{1}{V}\int_M |\nabla\dot\varphi|^2\omega_\varphi^n.$$

[1]Throughout this paper, the expressions such as $|\nabla f|$ and Δf are with respect to the metric $\omega_{\varphi(t)}$.

Since the K energy has a lower bound along the Kähler Ricci flow, we have

$$\int_0^\infty \epsilon(t)\,dt = \frac{1}{V}\int_0^\infty\int_M |\nabla\dot{\varphi}|^2\omega_\varphi^n \wedge dt = \nu_\omega(0) - \nu_\omega(\infty).$$

Now we normalize our initial value of $c(t)$ as

$$\begin{aligned} c(0) &= \int_0^\infty \epsilon(t)e^{-t}\,dt \\ &= \frac{1}{V}\int_0^\infty e^{-t}\int_M |\nabla\dot{\varphi}|^2\omega_\varphi^n \wedge dt \\ &\le \frac{1}{V}\int_0^\infty\int_M |\nabla\dot{\varphi}|^2\omega_\varphi^n \wedge dt \\ &= \nu_\omega(0) - \nu_\omega(\infty). \end{aligned}$$

From the equation for $c(t)$, we have

$$(e^{-t}c(t))' = -\epsilon(t)e^{-t}.$$

Thus, we have

$$0 < c(t) = \int_t^\infty \epsilon(\tau)e^{-(\tau-t)}\,d\tau \le \nu_\omega(0) - \nu_\omega(\infty)$$

and

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty}\int_t^\infty \epsilon(\tau)e^{-(\tau-t)}\,d\tau = 0.$$

Since the K energy is bounded from below, we have

$$\int_0^\infty c(t)\,dt = \frac{1}{V}\int_0^\infty\int_M |\nabla\dot{\varphi}|^2\omega_\varphi^n \wedge dt - c(0) \le \nu_\omega(0) - \nu_\omega(\infty). \qquad \square$$

Now we recall the following result, which was proved by Perelman using the $\mathcal{W}$-functional and the gradient estimates for $\partial\varphi/\partial t$.

Lemma 2.2 (cf. [13] [11]). *For the solution $\varphi(t)$ of* (2.2), *we choose a_t by the condition $h_t = -(\partial\varphi/\partial t) + a_t$ such that*

$$\int_M e^{h_t}\omega_\varphi^n = V. \tag{2.5}$$

Then there is a uniform constant A independent of t such that

$$|h_t| \le A, \quad |\nabla h_t|^2(t) \le A, \quad \text{and} \quad |\Delta h_t| \le A. \tag{2.6}$$

Finally, we state the following Poincaré inequality, which is well-known in literature (cf. [8], [18]).

Lemma 2.3. *For any Kähler metric ω_g and any function $\phi \in C^\infty(M, \mathbb{C})$, we have*

$$\int_M |\nabla \phi|^2 e^h \omega_g^n \geq \int_M |\phi - \underline{\phi}|^2 e^h \omega_g^n,$$

where h is the Ricci potential function with respect to ω_g and

$$\underline{\phi} = \frac{1}{V} \int_M \phi e^h \omega_g^n.$$

3. Proof of Theorem 1.1

In this section, we prove the main theorem. First, by the expression (2.3) and (1.2), we can show the following lemma, which directly implies the K-energy is bounded from below if F is bounded from below.

Lemma 3.1.

$$\nu_\omega(\varphi) \geq F_\omega(\varphi) + \frac{1}{V} \int_M h_\omega \omega^n. \tag{3.1}$$

Proof. By the expression (2.3), the K-energy can be written as

$$\begin{aligned} \nu_\omega(\varphi) = {} & \frac{1}{V} \int_M u \omega_\varphi^n - \frac{1}{V} \int_M \varphi \omega_\varphi^n + \frac{1}{V} \int_M h_\omega \omega^n \\ & - \frac{1}{V} \sum_{i=0}^{n-1} \frac{n-i}{n+1} \int_M \sqrt{-1}\, \partial\varphi \wedge \bar{\partial}\varphi \wedge \omega^i \wedge \omega_\varphi^{n-1-i}, \end{aligned} \tag{3.2}$$

where

$$u = \log \frac{\omega_\varphi^n}{\omega^n} + \varphi - h_\omega.$$

By direct calculation, we have

$$\begin{aligned} \nu_\omega(\varphi) - F_\omega(\varphi) &= \frac{1}{V} \int_M u \omega_\varphi^n + \frac{1}{V} \int_M h_\omega \omega^n + \log\left(\frac{1}{V} \int_M e^{h_\omega - \varphi} \omega^n \right) \\ &= \frac{1}{V} \int_M u \omega_\varphi^n + \frac{1}{V} \int_M h_\omega \omega^n + \log\left(\frac{1}{V} \int_M e^{-u} \omega_\varphi^n \right). \end{aligned} \tag{3.3}$$

Using Jensen's inequality, we have

$$\log\left(\frac{1}{V}\int_M e^{-u}\omega_\varphi^n\right) \geq -\frac{1}{V}\int_M u\omega_\varphi^n.$$

Thus, we have

$$\nu_\omega(\varphi) \geq F_\omega(\varphi) + \frac{1}{V}\int_M h_\omega\omega^n. \qquad \square$$

Now we assume that the K-energy is bounded from below. For any metric $\omega' = \omega + \sqrt{-1}\,\partial\bar{\partial}\varphi_0$, we consider the solution $\varphi(t)$ of Kähler-Ricci flow with the initial metric ω':

$$\frac{\partial\varphi}{\partial t} = u = \log\frac{\omega_\varphi^n}{\omega^n} + \varphi - h_\omega, \quad \varphi(0) = \varphi_0.$$

Since $F(t) = F_\omega(\varphi(t))$ is decreasing along the Kähler-Ricci flow (cf. [5]), we will prove that $\nu_\omega(t) - F(t)$ has a uniform bound as $t \to \infty$, and the bound is independent of the initial metric ω'. Thus, F is also bounded from below.

Since $F(t)$ is decreasing along the Kähler-Ricci flow, for any $s < t$ by the equality (3.3) we have

$$\begin{aligned} F_\omega(\omega') = F(0) &\geq F(t) - \nu_\omega(t) + \nu_\omega(t) \\ &= F(t) - \nu_\omega(t) + \nu_\omega(s) - \frac{1}{V}\int_s^t\int_M |\nabla u|^2\omega_\varphi^n \\ &= -f(t) + \nu_\omega(s) - \frac{1}{V}\int_s^t\int_M |\nabla u|^2\omega_\varphi^n - \frac{1}{V}\int_M h_\omega\omega^n. \end{aligned} \tag{3.4}$$

where

$$f(t) = \frac{1}{V}\int_M u\omega_\varphi^n + \log\left(\frac{1}{V}\int_M e^{-u}\omega_\varphi^n\right). \tag{3.5}$$

If we can find a sequence of times $t_m \to \infty$ such that

$$\lim_{m\to\infty} f(t_m) = 0, \tag{3.6}$$

then we can take $t = t_m$ in (3.4), and let $m \to \infty$,

$$F_\omega(\omega') \geq \nu_\omega(s) - \frac{1}{V}\int_s^\infty\int_M |\nabla u|^2\omega_\varphi^n - \frac{1}{V}\int_M h_\omega\omega^n.$$

Since the K-energy is decreasing along Kähler-Ricci flow, taking $s \to \infty$ in the above inequality we have

$$F_\omega(\omega') \geq \inf v_\omega - \frac{1}{V}\int_M h_\omega \omega^n. \tag{3.7}$$

Then F is bounded from below. Thus, it suffices to show that (3.6) holds.

Now we are ready to prove (3.6). Since the K-energy is bounded from below, by Lemma 2.1 we can normalize the solution $\varphi(t)$ such that $c(t) > 0$ for all t, and

$$\lim_{t\to\infty} c(t) = \lim_{t\to\infty} \frac{1}{V}\int_M u\omega_\varphi^n = 0. \tag{3.8}$$

By Lemma 2.2, we prove

Lemma 3.2. *There exists a constant B independent of t such that $|u| \leq B$.*

Proof. We use the notations in Lemma 2.2. By the equality (2.5), we have

$$\int_M e^{-u+a_t}\omega_\varphi^n = V.$$

It follows that

$$a_t = -\log\left(\frac{1}{V}\int_M e^{-u}\omega_\varphi^n\right).$$

Then Lemma 2.2 implies

$$-A \leq u + \log\left(\frac{1}{V}\int_M e^{-u}\omega_\varphi^n\right) \leq A. \tag{3.9}$$

Since the K-energy is bounded from below, by Lemma 2.1 the integral $\int_M u\omega_\varphi^n$ is uniformly bounded from above and below. Thus, integrating (3.9) we have

$$\left|\log\left(\frac{1}{V}\int_M e^{-u}\omega_\varphi^n\right)\right| \leq C, \tag{3.10}$$

for some constant C. Combining (3.9) with (3.10), the lemma is proved. □

Next, we prove the following lemma

Lemma 3.3. *For time $t \to \infty$, we have*

$$\underline{u}(t) \to 0,$$

where $\underline{u}(t) = (1/V)\int_M ue^{h_t}\omega_\varphi^n$. Here we choose h_t as in Lemma 2.2.

Proof. Observe that

$$\left(\frac{1}{V}\int_M u e^{h_t}\omega_\varphi^n\right)^2 \le \frac{1}{V}\int_M u^2 e^{h_t}\omega_\varphi^n \le \frac{e^A}{V}\int_M u^2\omega_\varphi^n. \tag{3.11}$$

Let

$$b(t) = \int_M u^2\omega_\varphi^n.$$

Then

$$\begin{aligned}\frac{d}{dt}b(t) &= \int_M (2u(\Delta u + u) + u^2\Delta u)\omega_\varphi^n \\ &= \int_M (-2|\nabla u|^2 + 2u^2 - 2u|\nabla u|^2)\omega_\varphi^n \\ &\ge \int_M (-2|\nabla u|^2 + 2u^2 - u^2 - |\nabla u|^4)\omega_\varphi^n \\ &\ge b(t) - (2 + A)\int_M |\nabla u|^2\omega_\varphi^n\end{aligned}$$

where we use $|\nabla u|^2 \le A$ in the last inequality. Thus, integrating the above inequality from 0 to ∞ we have

$$\int_0^\infty b(t)\,dt \le \limsup_{t\to\infty} b(t) - b(0) + (2 + A)\int_0^\infty\int_M |\nabla u|^2\omega_\varphi^n < \infty.$$

Here the last inequality comes from Lemma 3.2 and the fact that the K-energy is bounded from below. By Lemma 2.2, we have $|(d/dt)b(t)| \le C$. Hence, we have $b(t) \to 0$ as $t \to \infty$. Therefore, by the inequality (3.11) we have $\underline{u}(t) \to 0$. □

Now we can prove

Lemma 3.4. *There is a sequence of times $t_m \to \infty$ such that*

$$\lim_{m\to\infty} f(t_m) \to 0,$$

where f is defined by (3.5).

Proof. By the equalities (3.5) and (3.8), it suffices to find a sequence of times t_m such that

$$\lim_{m\to\infty} \log\left(\frac{1}{V}\int_M e^{-u}\omega_\varphi^n\right)\Big|_{t=t_m} = 0. \tag{3.12}$$

Since u and $\underline{u}$ are bounded by Lemma 3.2 and Lemma 3.3, we have the Taylor expansion

$$e^{-(u-\underline{u})} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}(u-\underline{u})^k. \tag{3.13}$$

Therefore,

$$\log\left(\frac{1}{V}\int_M e^{-u}\omega_\varphi^n\right) = -\underline{u} + \log\left(1 + \frac{1}{V}\int_M \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}(u-\underline{u})^k\omega_\varphi^n\right). \tag{3.14}$$

Now by Lemma 2.2, we have

$$\begin{aligned} \left|\int_M \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}(u-\underline{u})^k\omega_\varphi^n\right| &\le \int_M \sum_{k=1}^{\infty} \frac{1}{k!}|u-\underline{u}|^k\omega_\varphi^n \\ &\le \sum_{k=1}^{\infty} \frac{e^A}{k!}\int_M |u-\underline{u}|^k e^{h_t}\omega_\varphi^n. \end{aligned} \tag{3.15}$$

Then by the Poincaré inequality in Lemma 2.3, we know

$$\begin{aligned} &\sum_{k=1}^{\infty} \frac{e^A}{k!}\int_M |u-\underline{u}|^k e^{h_t}\omega_\varphi^n \\ &\le e^A\int_M |u-\underline{u}|e^{h_t}\omega_\varphi^n + \sum_{k=2}^{\infty} \frac{e^A(2B)^{k-2}}{k!}\int_M |u-\underline{u}|^2 e^{h_t}\omega_\varphi^n \\ &\le e^A\sqrt{V}\sqrt{\int_M |u-\underline{u}|^2 e^{h_t}\omega_\varphi^n} + \frac{e^{A+2B}}{(2B)^2}\int_M |u-\underline{u}|^2 e^{h_t}\omega_\varphi^n \\ &\le e^A\sqrt{V}\sqrt{\int_M |\nabla u|^2 e^{h_t}\omega_\varphi^n} + \frac{e^{A+2B}}{(2B)^2}\int_M |\nabla u|^2 e^{h_t}\omega_\varphi^n \\ &\le e^{3/2A}\sqrt{V}\sqrt{\int_M |\nabla u|^2\omega_\varphi^n} + \frac{e^{2A+2B}}{(2B)^2}\int_M |\nabla u|^2\omega_\varphi^n. \end{aligned} \tag{3.16}$$

Since the K-energy is bounded from below, by (2.4) we can find a sequence of times $t_m \to \infty$ such that

$$\int_M |\nabla u|^2\omega_\varphi^n\Big|_{t=t_m} \to 0.$$

Combining this with (3.14)–(3.16), we know (3.12) holds. The lemma is proved. □

By Lemma 3.4, the equality (3.6) holds. This implies F_ω is bounded from below and the inequality (3.7) holds. Combining this with Lemma 3.1, the main theorem is proved.

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