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ON REGULAR RINGS WHOSE CYCLIC FAITHFUL MODULES CONTAIN GENERATORS

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1. Introduction

The present paper may be considered as a continuation of [6], in which we studied nonsingular rings R satisfying the condition (C^*) that every cyclic faithful right R-module is a generator for the category Mod-R of all right R-modules. We proved in [6] that a (von Neumann) regular ring satisfies (C^*) if and only if it is isomorphic to a finite direct product of an abelian regular ring and full matrix rings over self-injective abelian regular rings (c.f. [4]). Concerning this, we note that there exists a regular ring R over which, although R fails to satisfy the condition (C^*), yet every cyclic faithful right R-module "*contains*" a submodule which is a generator for Mod-R. For instance, choose a division ring D_n containing a division subring E_n for $n = 1, 2, \dots$, let k (≥ 2) be an integer, and let S_n and T_n be the rings of all $k \times k$ matrices over D_n and E_n, respectively, for $n = 1, 2, \dots$. Now, consider the regular ring R which consists of all sequences $(x_n) \in \prod_{n=1}^{\infty} S_n$ such that $x_n \in T_n$ for all but finitely many n. Then, R satisfies (C^*) only when $D_n = E_n$ for all but finitely many n (see [4], or [6]), whence in case E_n is properly contained in D_n for infinitely many n, the ring R does not satisfy (C^*). However, it is shown that every cyclic (finitely generated) faithful right module over the ring R actually contains a generator. In fact, as will be noted in Example 3(2) of §3, the full matrix rings over any continuous abelian non-self-injective regular rings do not satisfy (C^*), but every cyclic (finitely generated) faithful module over the rings does contain a generator. This shows that the condition (C^*) is not equivalent to the one (C) that every cyclic faithful right module contains a submodule which is a generator.

In this paper, we shall consistently investigate regular rings satisfying the condition (C) above, and determine their structure. Section 2 is devoted to preliminary results on regular rings R satisfying (C), part of which would be derived from their more general property that every cyclic faithful right R-module is co-faithful. In Section 3 we shall present our main result (Theorem A) which asserts that the regular rings satisfying the condition (C) are precisely the finite direct products of abelian regular rings and full matrix rings over abelian regular rings S such that every finitely generated faithful right S-submodule of the maximal quotient ring

$Q(S)$ of S contains a unit in $Q(S)$. Furthermore, we show that the regular rings over which every finitely generated faithful right module contains a generator are just the finite direct products of full matrix rings over such abelian regular rings S (Theorem B). As corollaries, we obtain a structure theorem for regular rings satisfying the condition (C^*) as in [6] and the well-known structure theorem for FPF rings in [5](Corollary A, B). Also, we shall present some examples to illustrate these results.

NOTATION AND TERMINOLOGY. All rings considered in this paper are associative with identity and all modules are unitary.

Let R be a ring, M an R-module, and x and X an element and a subset of M. We denote by $Z(M)$ the (right) singular submodule of M, by $(X : x)$ the set $\{r \in R \mid xr \in X\}$, and by $r_R(X)$ (respectively, $l_R(X)$) the right (resp. left) annihilator of X in R. The notation $N \leq M$ (resp. $N \leq_e M$) means that N is a submodule (resp. an essential submodule) of M, while the notation $N \lesssim M$ means that N is isomorphic to a submodule of M. In particular, the notation $A \leq R_R$ signifies that A is a right ideal of R. Given a positive integer n, we denote by $M^{(n)}$ the direct sum of n copies of M, and by $M_n(R)$ the ring of all $n \times n$ matrices over R. The set of all central idempotents in R and the maximal (right) quotient ring of R are denoted by $B(R)$ and $Q(R)$, respectively. A *complement* for N in M is any submodule L of M which is maximal with respect to the property $N \cap L = 0$. A right R-module M is *co-faithful modulo its annihilator* if $R/r_R(M) \lesssim M^{(n)}$ for some integer n. In particular, if $r_R(M) = 0$, i.e., $R \lesssim M^{(n)}$, then we call M simply *co-faithful*.

In what follows we shall be concerned with rings R satisfying the condition:

(C) Every cyclic faithful right R-module contains a submodule which is a generator for Mod-R.

For brevity we referred to them as rings with (C).

2. Preliminaries

It is obvious that any ring R with (C) satisfies the condition:

(C_1) Every cyclic faithful right R-module is co-faithful.

Thus, for a while we shall examine the property of rings with (C_1).

The following remark is immediate.

REMARK 1. For any ring R, the following conditions are equivalent:

(a) R satisfies the condition (C_1);

(b) Every finitely generated faithful right R-module is co-faithful;

(c) For every finitely generated faithful right R-module M and for every finitely generated projective right R-module P, there exists a positive integer n such that $P \lesssim M^{(n)}$.

In particular, the property (C_1) of rings is Morita-invariant.

A ring R is *right (essentially) bounded* provided that every essential right ideal of R contains a two-sided ideal which is essential as a right ideal.

Lemma 1. *For a semiprime ring R, the following conditions are equivalent:*

(a) *R satisfies (C_1);*

(b) *R is right bounded, and for every two-sided ideal I such that $(R/I)_R$ is nonsingular, the ring R/I satisfies (C_1);*

(c) *R is right bounded, and every cyclic faithful nonsingular right R-module is co-faithful.*

Proof. (a) $\Rightarrow$ (b). Let $E \leq_e R_R$, and choose a complement A for $r_R(R/E)$ in E_R. Then, R/A is faithful, whence by (a) there exist $a_1, \ldots, a_n \in R$ such that $\bigcap_{i=1}^n (A : a_i) = 0$. Set $X = \bigcap_{i=1}^n (A \oplus r_R(R/E) : a_i)$. Observing that $a_i X A \leq A$ for all i, we have $XA = 0$, so that $A = 0$, because R is semiprime and $X \leq_e R_R$. Thus, $r_R(R/E)$ is essential in R_R, which shows that R is right bounded.

For the second condition of (b), let I be a two-sided ideal such that $(R/I)_R$ is nonsingular, and set $J = l_R(I)$. Since $I \oplus J \leq_e R_R$ and since $(R/I)_R$ is nonsingular, it follows that $I = l_R(J)$. Now, let $B \leq R_R$ for which $r_R(R/B) = I$. Then, $r_R(R/BJ) = 0$, whence by (a) there exist $b_1, \ldots, b_m \in R$ such that $\bigcap_{j=1}^m (BJ : b_j) = 0$, from which we obtain $\bigcap_{j=1}^m (B : b_j) = I$. Thus, the ring R/I satisfies the condition (C_1).

(b) $\Rightarrow$ (c). Obvious.

(c) $\Rightarrow$ (a). Note that R is right nonsingular, because R is a semiprime and right bounded ring. For (a), let C be a cyclic faithful right R-module. It then follows from [3, Lemma 2] that $C/Z(C)$ is also faithful. Thus, the second condition of (c) implies that $C/Z(C)$, and hence C, is co-faithful. □

A ring R is said to be *biregular* provided that for each $a \in R$, there exists $e \in B(R)$ such that $RaR = eR$. Also, recall that a two-sided ideal I of R has *bounded index (of nilpotence)* if there exists a positive integer n such that $a^n = 0$ for all nilpotent elements a of I; the least such positive integer is said to be the *index (of nilpotence)* of I. If there exist no such positive integers, then I has index ∞, or the index of I is ∞. In particular, the regular rings of index 1 are said to be *abelian*. They are precisely the regular rings in which all idempotents are central (see [1, Theorem 3.2]).

Next we shall show that any regular ring with (C_1) is a biregular ring of bounded index, for which we need the following lemmas.

Lemma 2 (c.f. [1, Proposition 2.10]). *Let R be a regular ring and n a positive integer. Let P be a projective right R-module and A a finitely generated submodule of $P^{(n)}$. Then, there exist submodules $P_1, \dots, P_k$ of P and nonnegative integers $n_1, \dots, n_k$ such that $P = P_1 \oplus \cdots \oplus P_k$ and $A \cong P_1^{(n_1)} \oplus \cdots \oplus P_k^{(n_k)}$.*

Proof. By virtue of [1, Lemma 2.7], there exist decompositions $P = U_i \oplus V_i$ $(i = 1, \dots, n)$ such that $A \cong U_1 \oplus \cdots \oplus U_n$. Applying [1, Theorem 2.8] $n-1$ times in succession, we actually obtain a desired decomposition $P = P_1 \oplus \cdots \oplus P_k$, where $k = 2^n$ (see the proof of [1, Proposition 2.10]). □

Recall that an R-module M is *directly finite* if M is not isomorphic to any proper direct summand of itself, or equivalently, for $x, y \in \mathrm{End}_R(M)$, $xy = 1$ implies $yx = 1$ (see [1, Lemma 5.1]).

Lemma 3. *For a regular ring R, the following conditions are equivalent:*

(a) *R is a biregular ring of bounded index;*

(b) *R is right bounded, and there exists a positive integer n such that $R/r_R(eR) \lesssim (eR)^{(n)}$ for every idempotent e of R;*

(c) *R is right bounded, and for idempotents $e_1, e_2, \dots$ of R such that $\{Re_nR \mid n = 1, 2, \dots\}$ is independent, there exists $f \in B(R^{\aleph_0})$ such that $R^{\aleph_0}(e_n)R^{\aleph_0} = fR^{\aleph_0}$, where $R^{\aleph_0}$ is the direct product of $\aleph_0$ copies of R.*

Proof. (a) $\Rightarrow$ (b). Assume that R is a biregular ring of bounded index n. Then, by [1, Lemma 6.20 and Corollary 7.10], R is right bounded.

For the second condition of (b), let e be an idempotent of R. Then, biregularity of R implies that there exists $f \in B(R)$ such that $ReR = fR$, so that $fR \lesssim (eR)^{(m)}$ for some positive integer m. Thus, according to Lemma 2, there exist $P_1, \dots, P_k \leq eR_R$ and nonnegative integers $n_1, \dots, n_k$ such that $eR = P_1 \oplus \cdots \oplus P_k$ and $fR \cong P_1^{(n_1)} \oplus \cdots \oplus P_k^{(n_k)}$. Since R has bounded index n, it follows from [1, Theorem 7.2] that each $n_i \leq n$. Therefore, we obtain $R/r_R(eR) \cong fR \lesssim (eR)^{(n)}$, as desired.

(b) $\Rightarrow$ (c). Assuming that (b) holds, we may generally show that any direct product R^Λ of Λ copies of R is biregular, which will obviously imply the second condition of (c). So, let $(e_\lambda)_{\lambda \in \Lambda}$ be an idempotent of R^Λ. Then, by the second condition of (b), $R/r_R(e_\lambda R) \lesssim (e_\lambda R)^{(n)}$ for all $\lambda \in \Lambda$; hence by [1, Theorem 1.11], for each $\lambda \in \Lambda$, there exists $f_\lambda \in B(R)$ such that $R = r_R(e_\lambda R) \oplus f_\lambda R$, and $f_\lambda R$ is isomorphic to a direct summand of $(e_\lambda R)^{(n)}$. The existence of an epimorphism from $(e_\lambda R)^{(n)}$ onto $f_\lambda R$ shows that for each $\lambda \in \Lambda$, there exist $a_{\lambda 1}, \dots, a_{\lambda n}, b_{\lambda 1}, \dots, b_{\lambda n} \in R$ such that $f_\lambda = \sum_{i=1}^n a_{\lambda i} e_\lambda b_{\lambda i}$, and $f_\lambda R = Re_\lambda R$. Thus, in R^Λ, we have $(f_\lambda) =$

$\sum_{i=1}^{n}(a_{\lambda i})(e_\lambda)(b_{\lambda i})$, so that $R^\Lambda(e_\lambda)R^\Lambda=(f_\lambda)R^\Lambda$. Therefore, R^Λ is biregular.

(c) $\Rightarrow$ (a). Assume that (c) holds. Then, the second condition of (c) obviously implies that R is biregular.

To prove that R has bounded index, we first claim the following.

CLAIM. R has no infinite independent set $\{I_n \mid n=1,2,\ldots\}$ of nonzero two-sided ideals such that the index of each I_n is at least n.

Suppose, to the contrary, that R does have such an infinite independent set $\{I_n \mid n=1,2,\ldots\}$. It then follows from [1, Theorem 7.2] that each I_n contains a nonzero idempotent e_n such that $(e_nR)^{(n)}\lesssim I_n$. Obviously, $\{Re_nR \mid n=1,2,\ldots\}$ is independent, whence according to (c), we obtain $(f_n)\in B(R^{\aleph_0})$ such that $R^{\aleph_0}(e_n)R^{\aleph_0}=(f_n)R^{\aleph_0}$. Thus, there exists a positive integer k such that $(f_n)R^{\aleph_0}\lesssim((e_n)R^{\aleph_0})^{(k)}_{R^{\aleph_0}}$, and so $f_nR\lesssim(e_nR)^{(k)}_R$ for all n. In particular, by [1, Theorem 1.11] the embedding $f_{k+1}R\lesssim(e_{k+1}R)^{(k)}$ implies that $(e_{k+1}R)^{(k)}\cong f_{k+1}R\oplus X$ for some right R-module X. On the other hand, by the choice of e_n's, we have $(e_{k+1}R)^{(k+1)}\lesssim Re_{k+1}R=f_{k+1}R$, and so $f_{k+1}R\cong(e_{k+1}R)^{(k+1)}\oplus Y$ for some right R-module Y. Thus, we obtain $f_{k+1}R\cong f_{k+1}R\oplus e_{k+1}R\oplus X\oplus Y$, that is, $f_{k+1}R$, and hence R, is not directly finite. But, this will imply a contradiction (by modifying the proof of [5, Proposition 7]) as follows. Since R is not directly finite, it follows from [1, Proposition 5.5] that R contains an infinite set $\{g_n \mid n=1,2,\ldots\}$ of nonzero pairwise orthogonal idempotents such that $g_mR\cong g_nR$ for all m,n. Now, let A be a complement for $\bigoplus_{n=1}^{\infty}g_nR$ in R_R. Since R is right bounded, the essential right ideal $A\oplus(\bigoplus_{n=1}^{\infty}g_nR)$ contains a two-sided ideal I which is essential in R_R. If we take a to be an arbitrary element of I, then by biregularity of R there exists $e\in B(R)$ such that $RaR=eR$. Note that $e\in A\oplus g_1R\oplus\cdots\oplus g_lR$ for some l, and so $eg_{l+1}=0$. But then, $g_nR\cong g_{l+1}R$, so that $eg_n=0$ for all n, from which e, and hence a, belongs to A. Since $a(\in I)$ is arbitrary, it follows that $I\leq A$, whence A is essential in R_R. This shows that $\bigoplus_{n=1}^{\infty}g_nR=0$, which is a contradiction. Therefore, the claim must hold.

Now, set $I=\sum\{I' \mid I' \text{ is a two-sided ideal of } R \text{ with bounded index }\}$. We shall show by using this claim first that I is essential in R_R. To this end, set $J/I=Z((R/I)_R)$, and note that J is a two-sided ideal of R. Then, the ring R/J has no infinite independent set of nonzero two-sided ideal of R. Indeed, suppose not, and take an infinite independent set $\{J_n/J \mid n=1,2,\ldots\}$ of nonzero two-sided ideals of R/J. Then, observing that $r_Rl_R(J)=J$ because R is a semiprime ring and $(R/J)_R$ is nonsingular, we see that $\{l_R(J)\cap J_n \mid n=1,2,\ldots\}$ is an infinite independent set of nonzero two-sided ideals of R. In particular, since each $l_R(J)\cap J_n$ is not contained in I, the choice of I implies that each $l_R(J)\cap J_n$ has index ∞. But, this contradicts the claim above. Thus, the ring R/J has no infinite independent set of nonzero two-sided ideals, which means that R/J has a finite independent set $\{H_1/J, \ldots, H_n/J\}$ of nonzero two-sided ideals such that as two-sided ideals, $\bigoplus_{k=1}^{n} (H_k/J)$ is essential in R/J, and each H_k/J is uniform. Furthermore, note by biregularity of the ring R/J that each H_k/J must be simple as a two-sided ideal and $R/J = (H_1/J) \oplus \cdots \oplus (H_n/J)$. In addition, by right boundedness of R and by nonsingularity of $(R/J)_R$ it is easy to see that the ring R/J, and hence each the ring H_k/J, is also right bounded. Since any right bounded and simple ring is artinian, it follows that R/J is a semisimple artinian ring, so that R/J, and hence $l_R(J)(\lesssim R/J)$, obviously has bounded index. Consequently, $l_R(J) \leq I \leq J$, that is, $l_R(J) = 0$, while $I \leq_e J_R$ and $J \oplus l_R(J) \leq_e R_R$. Therefore, I is indeed essential in R_R, as desired.

To conclude, we shall show that the ideal I has bounded index so that by virtue of [1, Corollary 7.5] the ring R may have bounded index, which will complete the proof of the lemma. If I does not have bounded index, then by [1, Corollary 7.8] there exists an infinite set $\{K_n \mid n = 1, 2, \ldots\}$ of nonzero two-sided ideals of R such that for each n, $K_n \subsetneqq K_{n+1}$ and the index of K_{n+1} is greater than that of K_n. Observing that $(l_R(K_n) \cap K_{n+1}) \oplus K_n \leq_e (K_{n+1})_R$ for all n, we see by [1, Corollary 7.5 and Proposition 7.7] that $\{l_R(K_n) \cap K_{n+1} \mid n = 1, 2, \ldots\}$ is an infinite independent set of nonzero two-sided ideals of R such that the index of each $l_R(K_{n+1}) \cap K_{n+2}$ is greater than that of $l_R(K_n) \cap K_{n+1}$, which contradicts again the claim above. Therefore, the ideal I has bounded index, as desired. This completes the proof of the lemma. $\square$

Note that both the classes of right bounded rings and of directly finite rings are closed under direct summands, and direct products, and also that in any regular ring R of index ∞, for each $n = 1, 2, \ldots$, there exists nonzero idempotent $e_n \in R$ such that $(e_nR)^{(n)} \lesssim R$. Then, observing the proofs of (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) in the lemma above, we see the following.

REMARK 2. For a regular ring R, the conditions (a), (b), (c) in Lemma 3 are also equivalent to the following conditions:

(d) *$R^{\aleph_0}$ is a biregular and right bounded ring;*
(e) *Any direct product of copies of R is a biregular and right bounded ring;*
(f) *$R^{\aleph_0}$ is a biregular and directly finite ring;*
(g) *Any direct product of copies of R is a biregular and directly finite ring.*

Here we shall show that any regular ring satisfying (C_1) is a biregular ring of bounded index.

Corollary 4. *Let R be a regular ring. If R is right bounded, and if for idempotents $e_1, e_2, \ldots$ of R such that $\{Re_nR \mid n = 1, 2, \ldots\}$ is independent, the R-module $R/(\bigcap_{n=1}^{\infty} r_R(e_n))$ is co-faithful modulo its annihilator, then R is a biregular ring of bounded index.*

In particular, if R satisfies (C_1), then R is a biregular ring of bounded index.

Proof. By the second hypothesis we see, as in the proof of (b) $\Rightarrow$ (c) in Lemma 3, that R is biregular.

Let $e_1, e_2, \ldots$ be idempotents of R such that $\{Re_nR \mid n = 1, 2, \ldots\}$ is independent. To prove the corollary, it suffices by Lemma 3 to obtain $f \in B(R^{\aleph_0})$ such that $R^{\aleph_0}(e_n)R^{\aleph_0} = fR^{\aleph_0}$. By hypothesis, there exists a positive integer k and a monomorphism $\varphi : R/(\bigcap_{n=1}^{\infty} r_R(e_nR)) \to (R/(\bigcap_{n=1}^{\infty} r_R(e_n)))^{(k)}$. For each $m = 1, 2, \ldots$, let $\pi_m : (R/(\bigcap_{n=1}^{\infty} r_R(e_n)))^{(k)} \to (R/r_R(e_m))^{(k)}$ be the natural epimorphism. Since R is biregular, for each m there exists $f_m \in B(R)$ such that $f_mR = Re_mR$ and $(1 - f_m)R = r_R(e_mR)$. Noting that $\{Re_nR \mid n = 1, 2, \ldots\}$ is independent, we obtain $\operatorname{Ker} \pi_m\varphi = (1 - f_m)R/(\bigcap_{n=1}^{\infty} r_R(e_nR))$; hence $f_mR \lesssim (e_mR)^{(k)}$ for all m. Consequently, it follows from the same argument as in the proof of (b) $\Rightarrow$ (c) in Lemma 3 that $R^{\aleph_0}(e_n)R^{\aleph_0} = (f_n)R^{\aleph_0}$, as desired.

The second assertion now follows from Lemma 1. □

Concerning Lemma 3, we observe the following well known examples.

EXAMPLE 1. (1) There exists a regular ring R which is right bounded and biregular, but R does not have bounded index.

For each $n = 1, 2, \ldots$, choose a division ring D_n, and set $Q = \prod_{n=1}^{\infty} M_n(D_n)$. Let R be the subring of Q consisting of all elements $(x_n) \in Q$ such that for all but finitely many n, the matrix x_n is of the form $\begin{pmatrix} a_n & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & a_n \end{pmatrix}$ $(\in M_n(D_n))$ for some $a_n \in D_n$. Then, R is a regular ring with Q the maximal quotient ring. Also, it is easy to see that R is as desired.

(2) There exists a regular ring R satisfying the second condition of (b) (and hence, of (c)) in Lemma 3, but R does not have bounded index.

Let V_D be an infinite dimensional vector space over a division ring D, and set $Q = \operatorname{End}_D(V)$ and $I = \{x \in Q \mid \dim_D xV < \dim_D V\}$. Then, I is the unique maximal two-sided ideal of Q. We consider $R = Q/I$. Let $\overline{x}$ $(= x + I)$ be an arbitrary nonzero element of R where $x \in Q$. Then, $xV \cong V_D$, and so there exists a Q-isomorphism $\varphi : xQ \to Q$. Since Q is right self-injective, there exist $y, z \in Q$ such that $yxz = 1$, from which we obtain $R \lesssim \overline{x}R$. Thus, R satisfies the second condition of (b).

But, since R is a simple non-artinian ring, it is not right bounded, and hence does not have bounded index.

According to [1, Theorem 3.4], a regular ring R is abelian if and only if for right ideals A, B of R such that $A \cap B = 0$, there exist no nonzero homomorphisms from one to the other. We thus call an R-module M *abelian* if M has the same property for its submodules. Obviously, any submodule of an abelian module is also abelian.

Sublemma. *Let M be a right R-module over a right nonsingular ring R.*

(1) *If M is abelian, then so is $M/Z(M)$.*

(2) *If M is nonsingular and abelian, then so is $E(M)$, the injective hull of M_R.*

Proof. Observe that if X and Y are right R-modules with $X' \leq_e X_R$ and $Y' \leq_e Y_R$ such that Y is nonsingular and such that $\mathrm{Hom}_R(X'', Y') = 0$ for all $X'' \leq X'_R$, then $\mathrm{Hom}_R(X, Y) = 0$. This immediately implies the assertion (2).

For (1), set $Z = Z(M)$, and let W be a complement for Z in M. Let $N_1, N_2 \leq M_R$ such that $N_1 \cap N_2 = Z$. Noting that $N_i \cap W \cong ((N_i \cap W) \oplus Z)/Z \leq_e N_i/Z$ for $i = 1, 2$, we see by the observation above that $\mathrm{Hom}_R(N_1/Z, N_2/Z) = 0$. Thus, M/Z is abelian. $\square$

To decompose regular rings with (C) into finite direct products of full matrix rings over abelian regular rings, we need the following lemma (c.f. [4, Lemma 2]).

Lemma 5. *For a regular ring R, the following conditions are equivalent:*

(a) *R has an abelian right R-module which is a generator for* Mod-R;

(b) *R is isomorphic to a finite direct product of full matrix rings over abelian regular rings.*

Proof. (b) $\Rightarrow$ (a). Assume that $R = \prod_{i=1}^{k} M_{n(i)}(S_i)$, where each S_i is an abelian regular ring. For each $i = 1, \ldots, k$, let e_i be the matrix unit in $M_{n(i)}(S_i)$ which has a 1_{S_i} in $(1,1)$ position as its only nonzero entry, and set $e = e_1 + \cdots + e_k$. Then, eR is actually an abelian R-module which is a generator for Mod-R.

(a) $\Rightarrow$ (b). The condition (a), a matter of fact, means that R has an abelian right R-module which is a finitely generated projective generator for Mod-R, as shown in the following claim, which will be often used in the next section as well.

CLAIM. Every abelian right R-module which is a generator for Mod-R is finitely generated projective.

To show this, let M be an abelian right R-module which is a generator for Mod-R. Then, $M_R^{(n)} \cong R \oplus X$ for some integer n and for some module X_R; hence $(M/Z(M))_R^{(n)} \cong R \oplus (X/Z(X))$, i.e., $M/Z(M)$ is also a generator. If $M/Z(M)$

is finitely generated projective, then $M = Z(M) \oplus Y$ for some module Y_R, whence $Z(M) = 0$ (and hence, actually, M is finitely generated projective), because $\mathrm{Hom}_R(Y, Z(M)) = 0$ and $Y \cong M/Z(M)$ generates $Z(M)$. Thus, to prove that M is finitely generated projective, we may assume by Sublemma that M is nonsingular.

Since M is a generator, there exist homomorphisms $\varphi_1, \dots, \varphi_n$ from M to R and $x_1, \dots, x_n \in M$ such that $\sum_{i=1}^n \varphi_i(x_i) = 1$. Now, consider a homomorphism $\varphi : M \to R^{(n)}$ defined by $x \mapsto (\varphi_1(x), \dots, \varphi_n(x))$. Then, φ is monic. To see this, set $K = \mathrm{Ker}\,\varphi$, let N be a complement for K in M, and set $A = \bigcap_{i=1}^n (K \oplus N : x_i)$. Indeed, let x be an arbitrary element of K, and let B be a complement for $r_R(x)$ in R_R. If $NB \neq 0$, then we have a nonzero homomorphism $B \to N$, which induces a nonzero homomorphism $xR\ (\leq K) \to E(N)$, the injective hull of N_R. But, this contradicts the assumption that M, and hence $E(M)$, is abelian (by Sublemma). Thus, $NB = 0$, from which we have $(A \cap B)^2 \leq \sum_{i=1}^n \varphi_i(x_i(A \cap B))(A \cap B) \leq \sum_{i=1}^n \varphi_i(N)(A \cap B) = 0$, so that $A \cap B = 0$. The essentiality of A in R_R then implies that $B = 0$, and so $x = 0$. Consequently, φ is monic. Thus, M can be embedded in $R_R^{(n)}$, whence by [1, Theorem 1.11] every finitely generated submodule of M is projective and a direct summand of M. As a result, if F is a finitely generated submodule of $M^{(l)}$ for some positive integer l, then induction on l shows that F is isomorphic to a finite direct sum of l submodules of M. Since $R_R \lesssim M^{(n)}$, it then follows that $R \cong M_1 \oplus \cdots \oplus M_n$ for some submodules $M_1, \dots, M_n$ of M. Set $P = \sum_{i=1}^n M_i$. Then, by the observation above, P is finitely generated projective and a direct summand of M, while $R \lesssim P_R^{(n)}$ and hence by [1, Theorem 1.11], P is a generator for Mod-R. Now, noting that M is abelian, we obtain $M = P$, whence M must be finitely generated projective, which completes the proof of Claim.

Thus, we have an abelian finitely generated projective generator P for Mod-R. Since R_R can be embedded in a finite direct sum of copies of P, it follows from Lemma 2 that there exist submodules $P_1, \dots, P_k$ of P and nonnegative integers $n_1, \dots, n_k$ such that $P = P_1 \oplus \cdots \oplus P_k$ and $R \cong P_1^{(n_1)} \oplus \cdots \oplus P_k^{(n_k)}$. We then see by [1, Theorem 3.4] that each ring $\mathrm{End}_R(P_i)$ is abelian, and $\mathrm{Hom}_R(P_i, P_j) = 0$ for $i \neq j$. Therefore, R has a desired decomposition $R \cong M_{n_1}(\mathrm{End}_R(P_1)) \times \cdots \times M_{n_k}(\mathrm{End}_R(P_k))$, which completes the proof of the lemma. □

We observe the following two examples concerning the condition (C_1).

EXAMPLE 2. (1) There exists a regular ring R such that every factor ring of R satisfies (C_1), but R is not isomorphic to a finite direct product of full matrix rings over abelian regular rings.

Choose a subfield F in the field of real numbers, and set $F_n = F$ for $n = 1, 2, \dots$, and $Q = \prod_{n=1}^\infty M_2(F_n)$, and let $a = (a_n)\ (\in Q)$, where each $a_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Now, set

$$R = \bigoplus_{n=1}^{\infty} M_2(F_n) + 1_Q F + aF,$$

i.e., R is the subring of Q consisting of all elements $(x_n) \in Q$ for which there exist $a, b \in F$ such that $x_n = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ for all but finitely many n. Then, R is a regular ring with Q the maximal quotient ring.

First we shall show that R satisfies (C_1). To this end, according to Lemma 1, let C be a cyclic faithful nonsingular right R-module. Then, there exists an idempotent $e = (e_n) \in Q$ such that $C \cong eR$. Noting that each $e_n \neq 0$ because eR_R is faithful, we can easily show that $r_R(\{e, ea\}) = 0$, so that eR is co-faithful. Thus, R satisfies (C_1). Now, let I be a two-sided ideal of R and consider the ring $\overline{R} = R/I$. Set $J = l_R(I)$, and $S = \bigoplus_{n=1}^{\infty} M_2(F_n)$. Since $I \oplus J \leq_e R_R$ and since S is the socle of R_R which is also a maximal two-sided ideal of R, it follows that either $I \oplus J = R$, or $I \oplus J = S$. If $I \oplus J = R$, then Lemma 1 implies that $\overline{R}$ satisfies (C_1). So, assume that $I \oplus J = S$. Then, there exists a subset N_1 of $\mathbf{N}$, the set of positive integers, such that $I = \bigoplus_{n \in N_1} M_2(F_n)$. If $\mathbf{N} - N_1$ is infinite, then $I = r_R(\bigoplus_{n \in \mathbf{N} - N_1} M_2(F_n))$; hence by Lemma 1 again, $\overline{R}$ satisfies (C_1). On the other hand, if otherwise, then $\overline{R}$ is isomorphic to a semisimple artinian ring $\prod_{n \in \mathbf{N} - N_1} M_2(F_n) \times \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in F \right\}$, which obviously satisfies (C_1). Therefore, in any case, $\overline{R}$ does satisfy (C_1).

Next, suppose that R is isomorphic to a finite direct product of full matrix rings over abelian regular rings. Then, R contains an idempotent $f = (f_n)$ such that fR_R is faithful and the ring fRf is abelian. Since each $f_n = f_n^2 \neq 0$, there exists $k \geq 1$ such that $f_n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ for all $n \geq k$. If we take $c = (c_n) \in R$ such that $c_k = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $c_n = 0$ otherwise, then $c = fcf$ is a nonzero nilpotent element of fRf, which is a contradiction. Therefore, R is as desired.

(2) There exists a regular ring R which is a biregular ring of bounded index, but R does not satisfy (C_1).

For each $n = 1, 2, \ldots$, choose a division ring D_n, and an integer $k \geq 2$, and set $Q = \prod_{n=1}^{\infty} M_k(D_n)$. Let R be the subring of Q consisting of all elements $(x_n) \in Q$ such that for all but finitely many n, the matrix x_n is of the form $\begin{pmatrix} a_n & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \cdot \\ & & & & a_n \end{pmatrix}$ ($\in M_k(D_n)$) for some $a_n \in D_n$. Then, R is a biregular regular ring of bounded index with Q the maximal quotient ring.

Set $A_R = \bigoplus_{n=1}^{\infty} \begin{pmatrix} D_n & \cdot & \cdot & \cdot & D_n \\ 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix}$, and consider the cyclic right R-module $C = R/A$. Then, it is easy to see that C is faithful, and also that for any finitely many elements $c_1, \dots, c_n \in C$, there exists a nonzero element of $\bigcap_{i=1}^{n} r_R(c_i)$. Therefore, R does not satisfy (C_1).

Here we digress from our subject and consider the conditions (C) and (C_1) on regular rings R along with two conditions below.

(C_0) R is isomorphic to a finite direct product of full matrix rings over abelian regular rings.

(C_2) R is a biregular ring of bounded index.

By Remark 1, Corollary 4 and Example 2 combined with Theorem A and Example 3(1) which will be shown in the next section, we see the following.

REMARK 3. For regular rings, the following proper implications hold:

$$(C) \implies (C_0) \implies (C_1) \implies (C_2).$$

We need a few more lemmas below.

Lemma 6. (1) *Let $R = \prod_{i=1}^{n} R_i$ be a ring decomposition. Then, R satisfies the condition that every cyclic faithful right R-module contains a (cyclic projective) submodule which is a generator if and only if so does each R_i.*

Furthermore, even if we replace "cyclic" by "finitely generated" in the condition above, the assertion also holds.

(2) *Let R be a semiprime ring with (C). If I is a two-sided ideal such that $(R/I)_R$ is nonsingular, then R/I is also a ring with (C).*

Proof. (1) Immediate.

(2) Set $J = l_R(I)$, and let $A \leq R_R$ such that $r_R(R/A) = I$. Then, R/AJ is faithful, whence it contains a generator B/AJ for Mod-R. Since $I = l_R(J)$, it is easy to see that $(B + A)/A$ $(\leq R/A)$ is a generator for Mod-R/I. Thus, the ring R/I satisfies (C). $\square$

Lemma 7 (c.f. [6, Lemma 2.5]). *Let R be a right nonsingular and semiprime ring with Q the maximal right quotient ring. For every two-sided ideal I of R, there exists $e \in B(Q)$ such that $I \leq_e eQ_R$.*

Lemma 8. *For a right nonsingular and semiprime ring R with Q the maximal* *right quotient ring, the following conditions are equivalent:*

(a) *Q is directly finite, and every right R-submodule M of Q generated by at most two elements contains an element x such that $r_R(x) = r_R(M)$;*

(b) *Every finitely generated faithful right R-submodule of Q contains a unit in Q.*

Proof. (a) $\Rightarrow$ (b). Let M be a faithful right R-submodule of Q generated by $x_1, \ldots, x_n$. For the module $\sum_{i=1}^{2} x_i R$, the second condition of (a) implies that $r_R(\sum_{i=1}^{2} x_i R) = r_R(y_2)$ for some $y_2 \in \sum_{i=1}^{2} x_i R$. Next, apply the condition again for the module $y_2 R + x_3 R$ to obtain $y_3 \in y_2 R + x_3 R$ such that $r_R(\sum_{i=1}^{3} x_i R) = r_R(y_2 R + x_3 R) = r_R(y_3)$. Continuing in this manner, we obtain $y_n \in \sum_{i=1}^{n} x_i R = M$ such that $r_R(y_n) = r_R(M) = 0$. Since Q is directly finite, the element y_n must be a unit in Q.

(b) $\Rightarrow$ (a). To prove that Q is directly finite, let e be an idempotent of Q such that $eQ \cong Q$. Since eQ_Q is faithful, we see by using Lemma 7 that eR_R is faithful, and hence by (b) that $eQ = Q$. Thus, Q is directly finite.

For the second condition of (a), let M be a right R-submodule of Q generated by at most two elements. By Lemma 7, there exists $f \in B(Q)$ such that $r_R(M) = fQ \cap R$. Noting that $M \oplus fR$ is a finitely generated faithful R-submodule of Q, by (b) we obtain $x \in M$ and $y \in fR$ such that $r_R(x + y) = 0$, which implies that $r_R(x) = r_R(M)$, as desired. □

3. Results

By using the results in the preceding section, we shall prove the following our main theorem.

Theorem A. *For a regular ring R, the following conditions are equivalent:*

(a) *Every cyclic faithful right R-module contains a submodule which is a generator for* Mod-R;

(b) *Every cyclic faithful right R-module contains a cyclic projective submodule which is a generator for* Mod-R;

(c) *For every right ideal A of R such that R/A is faithful, there exists $a \in R$ such that $aR \cap A = 0$ and $RaR = R$;*

(d) *$R \cong \prod_{i=1}^{k} M_{n(i)}(S_i)$, where $n(1) = 1$, and $n(i) \geq 2$ for $i = 2, 3, \ldots, k$, and where each S_i is an abelian regular ring such that for $i = 2, 3, \ldots, k$, every finitely generated faithful right S_i-submodule of $Q(S_i)$ contains a unit in $Q(S_i)$.*

Proof. (b) $\Leftrightarrow$ (c) and (b) $\Rightarrow$ (a). Immediate.

(a) $\Rightarrow$ (d). According to Corollary 4 and [1, Corollary 7.4], the ring R, and hence $Q(R)$, has bounded index, whence $Q(R)$ contains an idempotent e such that $eQ(R)_{Q(R)}$ is faithful and abelian. Observe that eR_R is an abelian module which is also faithful by Lemma 7. It then follows from the condition (a) and Lemma 5 that R has a decomposition $R = S_1 \times \prod_{i=2}^{k} M_{n(i)}(S_i)$, where each $n(i) \geq 2$, and where each S_i is an abelian regular ring. To show that for each $i = 2, \dots, k$, the ring S_i has the desired property in (d), it suffices by Lemmas 6 and 8 to show that in case $R = M_n(S)$ satisfies (a) where $n \geq 2$ and where S is an abelian regular ring, every right S-submodule X of $Q(S)$ generated by at most two elements contains an element whose annihilator coincides with that of X.

If X is cyclic, then it obviously contains such an element, because S is abelian. So, assume that $X = xS + yS$, where $x, y \in Q(S)$. Take a (central) idempotent e of $Q(S)$ to satisfy $r_S(X) = (1-e)Q(S) \cap S$. Let A and B be right $M_n(eS)$-submodules of $M_n(eQ(S))$ generated by $\begin{pmatrix} ex & 0 & \cdot & \cdot & \cdot & 0 & ey \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}$ and $\begin{pmatrix} e & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}$, respectively. Observing that $r_{eS}(exS + eyS) = 0$, we see that A is a cyclic faithful abelian $M_n(eS)$-submodule of $M_n(eQ(S))$. Since by Lemma 6 the ring $M_n(eS)$ also satisfies the condition (a), it follows from Claim in the proof of Lemma 5 that A contains a finitely generated projective generator P for Mod-$M_n(eS)$. In particular, $B \lesssim P^{(l)} \leq A^{(l)}$ for some integer l, while B is an abelian $M_n(eS)$-module, whence by Lemma 2 we may take $l = 1$, and so $eS \lesssim exS + eyS$. Thus, there exist $s, t \in S$ such that $r_{eS}(exs + eyt) = 0$, which implies that $r_S(xs + yt) = r_S(X)$, as desired.

(d) $\Rightarrow$ (b). Assume that (d) holds. Since any abelian regular ring obviously satisfies the condition (b), it suffices by Lemma 6 to show that in case $R = M_n(S)$ where $n \geq 2$ and where S is an abelian regular ring such that every finitely generated faithful right S-submodule of $Q(S)$ contains a unit in $Q(S)$, the ring R actually satisfies the condition (b).

Indeed, let C be a cyclic faithful right R-module, and set $Q = Q(R) = M_n(Q(S))$. Then, there exists an idempotent $e \in Q$ such that $C/Z(C) \cong eR$. Note from Lemma 1 and [3, Lemma 2] that eR_R, and hence eQ_Q, is faithful. Also, let f be the matrix unit in R which has a 1_S in $(1,1)$ position as its only nonzero entry. Since Lemma 3 implies that $fQ \leq Q \lesssim (eQ)^{(k)}$ for some integer k and since fQ_Q is abelian, it follows from Lemma 2 that $fQ \lesssim eQ$. Thus, by virtue of [1, Corollary 7.11 and Theorem 4.1], there exist two decompositions $Q_Q = A_1 \oplus A_2 \oplus A_3 = B_1 \oplus B_2 \oplus B_3$ such that $A_1 \oplus A_2 = eQ, A_3 = (1-e)Q, B_1 = fQ$, and $B_2 \oplus B_3 = (1-f)Q$ along with Q-isomorphisms $\varphi_i : B_i \to A_i$ for $i = 1, 2, 3$. Set $\varphi = \bigoplus_{i=1}^{3} \varphi_i : Q_Q \to Q_Q$, the direct sum of φ_i's, and set $v = \varphi(1)$ and $\varphi(u) = 1$ (for some $u \in Q$). Then, $vu = 1$, and hence $uv = 1$, i.e., $v = u^{-1}$, because Q is directly finite. Expressing $1-f = x+y$, where $x \in B_2, y \in B_3$, and noting that $e + (1-e) = 1 = u^{-1}fu + u^{-1}xu + u^{-1}yu = (\varphi_1(f)u + \varphi_2(x)u) + \varphi_3(y)u$, we obtain $e = u^{-1}fu + u^{-1}xu$, and so $ue = (f+x)u$. By the choices of f and x and by the unity of u, the element ue may be expressed as $ue = \begin{pmatrix} u_1 & \cdot & \cdot & \cdot & u_n \\ & & & & \\ & & * & & \\ & & & & \end{pmatrix} \in M_n(Q(S))$, where $\sum_{i=1}^n u_i Q(S) = Q(S)$, and hence, in particular, $\sum_{i=1}^n u_i S_S$ is faithful. It then follows from the hypothesis of S that there exist $s_1, \dots, s_n \in S$ such that $\sum_{i=1}^n u_i s_i$ is a unit in $Q(S)$. This induces a monomorphism $fR \to ueR$ defined by

$$\begin{pmatrix} a_1 & \cdot & \cdot & \cdot & a_n \\ 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \mapsto ue \begin{pmatrix} s_1 & 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s_n & 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix} \begin{pmatrix} a_1 & \cdot & \cdot & \cdot & a_n \\ 0 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & 0 \end{pmatrix},$$

which implies that $fR \lesssim ueR \cong eR \cong C/Z(C)$. Since fR is obviously a cyclic projective generator for Mod-R, we conclude that $C/Z(C)$, and hence C, actually contains a cyclic projective generator, which completes the proof of the theorem. $\square$

We may replace "cyclic" by "finitely generated" in the equivalent conditions of the theorem above, as shown in the following theorem.

Theorem B. *For a regular ring R, the following conditions are equivalent:*

(a) *Every finitely generated faithful right R-module contains a submodule which is a generator for* Mod-R;

(b) *Every finitely generated faithful right R-module contains a finitely generated projective submodule which is a generator for* Mod-R;

(c) *For every positive integer n and for every right ideal X of $M_n(R)$ such that $M_n(R)/X$ is faithful, there exists $\theta \in M_n(R)$ such that $\theta M_n(R) \cap X = 0$ and $M_n(R)\theta M_n(R) = M_n(R)$;*

(d) *$R \cong \prod_{i=1}^k M_{n(i)}(S_i)$, where each S_i is an abelian regular ring such that every finitely generated faithful right S_i-submodule of $Q(S_i)$ contains a unit in $Q(S_i)$.*

To prove Theorem B, we provide the following lemma by using Theorem A.

Lemma 9. *For an abelian regular ring S, the following conditions are equivalent:*

(a) *Every finitely generated faithful right S-module contains a submodule which is a generator for* Mod-S;

(b) *Every finitely generated faithful right S-module contains a finitely generated projective submodule which is a generator for* Mod-S;

(c) *Every finitely generated faithful right S-submodule of $Q(S)$ contains a* *unit in* $Q(S)$.

Proof. (b) $\Rightarrow$ (a). Obvious.

(a) $\Rightarrow$ (c). Let M be a finitely generated faithful right S-submodule of $Q(S)$. Then, the condition (a) implies that M contains a generator G_S. Since $Q(S)$, and hence G, is abelian, it follows from Claim in the proof of Lemma 5 that G must be finitely generated projective. Thus, there exist $x_1, \ldots, x_n \in G$ such that $G = x_1S \oplus \cdots \oplus x_nS$, and then $r_S(\sum_{i=1}^n x_i) = 0$. Therefore, $\sum_{i=1}^n x_i$ $(\in M)$ is a unit in $Q(S)$.

(c) $\Rightarrow$ (b). Let M be a finitely generated faithful right S-module. Then, there exists a positive integer n and an epimorphism $\varphi : S^{(n)} \to M$. Set $P = S^{(n)}$, and $T = M_n(S)$. Also, let F denote the functor $\mathrm{Hom}_S({}_TP_S, \_) : \text{Mod-}S \to \text{Mod-}T$, and note that the functor F is a category equivalence. Then, we obtain an exact sequence in Mod-T:

$$0 \to F(\mathrm{Ker}\,\varphi) \to F(P) \to F(M) \to 0$$

and $F(P) \cong T_T$. Thus, $F(M)$ is a cyclic faithful right T-module. Since by the condition (c) and Theorem A, every cyclic faithful right T-module contains a cyclic projective generator, we conclude that $F(M)$, and hence M, contains a finitely generated projective generator, as desired. □

Proof of Theorem B. As in the proof of (c) $\Rightarrow$ (b) in the lemma above, we see that the conditions (b) and (c) are equivalent. In addition, note that the conditions (a) and (b) on rings are Morita-invariant. Then, the theorem is immediate from the lemma above, Theorem A and Lemma 6. □

REMARK 4. By Theorems A, B and their proofs, we see that for a regular ring R, the following conditions are equivalent:

(a) *R satisfies the equivalent conditions of Theorem* B;

(b) *Every faithful right R-module generated by at most two elements contains a submodule which is a generator for* Mod-R;

(c) *For every positive integer n, the matrix ring $M_n(R)$ satisfies the equivalent conditions of Theorem* A;

(d) *The matrix ring $M_2(R)$ satisfies the equivalent conditions of Theorem* A.

Recall that a regular ring is (*right*) *continuous* if it contains all the idempotents of the maximal (right) quotient ring (see [1, Theorem 13.13]).

REMARK 5. The matrix rings over any continuous abelian regular rings satisfy the equivalent conditions of Theorem B.

Indeed, let S be an abelian regular ring which is continuous. We must show that every finitely generated faithful S-submodule of $Q(S)$ contains a unit in $Q(S)$. So, let $X_S = x_1S + \cdots + x_nS$ be a finitely generated faithful right S-submodule of $Q(S)$. For each $i = 1, \ldots, n$, take an idempotent e_i (of S) to satisfy $x_1Q(S) + \cdots + x_iQ(S) = e_iQ(S)$. Then, we have $Q(S) = \sum_{i=1}^n x_iQ(S) = \bigoplus_{i=1}^n (x_i(1 - e_{i-1})Q(S)) = (\sum_{i=1}^n x_i(1 - e_{i-1}))Q(S)$ (where $e_0 = 0$), which shows that $\sum_{i=1}^n x_i(1 - e_{i-1})$ ($\in X$) is a unit in $Q(S)$, as desired.

Let S be an abelian regular ring, and k (≥ 2) an integer, and set $R = M_k(S)$. Let x be an arbitrary element of $Q(S)$, and consider the right R-submodule C of $Q(R)$ generated by $\begin{pmatrix} 1 & 0 & \cdots & 0 & x \\ 0 & \cdots & \cdots & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix}$. Then, C is a cyclic faithful abelian R-module. Now, assume that C is a generator for Mod-R. It then follows from Claim in the proof of Lemma 5 that C_R is projective, so that $S + xS$ is a projective S-module, from which we have $x \in S$. As a result, if every cyclic faithful right R-module is a generator for Mod-R, then $S = Q(S)$, i.e., S is self-injective.

Therefore, Theorems A and B combined with the argument above immediately imply the following two corollaries, respectively.

Corollary A ([6, Theorem 4.3]). *For a regular ring R, the following conditions are equivalent:*

(a) *R is right GFC, i.e., every cyclic faithful right R-module is a generator for* Mod-R;

(b) *R is isomorphic to a finite direct product of an abelian regular ring and full matrix rings over self-injective abelian regular rings.*

Corollary B ([5, Theorem 9]). *For a regular ring R, the following conditions are equivalent:*

(a) *R is right FPF, i.e., every finitely generated faithful right R-module is a generator for* Mod-R;

(b) *R is isomorphic to a finite direct product of full matrix rings over self-injective abelian regular rings.*

We conclude with two examples to illustrate Theorems A and B.

EXAMPLE 3. (1) There exists a regular ring R which is a full matrix ring over an abelian regular ring, but R does not satisfy the equivalent conditions of Theorem A.

Choose an at most countable division ring D with $D - \{0\} = \{a_n \mid n = 1, 2, \ldots\}$ and $a_0 = 0$. For each $n = 1, 2, \ldots$, set $D_n = D$, and set $Q = \prod_{n=1}^{\infty} D_n$, and

$S = \bigoplus_{n=1}^{\infty} D_n + 1_Q D (\subset Q)$. Then, S is an abelian regular ring with Q the maximal quotient ring. Let k (≥ 2) be an integer, and set $R = M_k(S)$.

Now, partition $\mathbf{N}$, the set of positive integers, into countably many pairwise disjoint countable sets $N_i = \{n_i(0), n_i(1), n_i(2), \ldots\}$ $(i = 1, 2, \ldots)$, and take $x = (x_n), y = (y_n) \in Q$ as follows:

$$x_n = a_j \text{ if } n = n_i(j) \text{ for some } i,\ j,$$

$$y_n = \begin{cases} 1 & \text{if } n = n_i(j) \text{ for some } i,\ j \text{ with } j \leq i, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $xS + yS$ is a faithful S-submodule of Q. But, it is easy to see that for every $s, t \in S$, some entry of $xs + yt$ must be zero; hence $xS + yS$ contains no units in Q. Thus, the cyclic faithful right R-module generated by $\begin{pmatrix} x & 0 & \cdot & \cdot & \cdot & 0 & y \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}$ can not contain a generator for Mod-R.

(2) There exists a regular ring R which satisfies the equivalent conditions of Theorem B, but not all cyclic faithful right R-modules are generators for Mod-R.

In fact, as can be seen from Remark 5 combined with the argument following the remark, the matrix rings $M_n(S)$ $(n \geq 2)$ over any continuous abelian non-self-injective regular ring S (e.g. [1, Example 13.8]) is one example with which the above may be illustrated. The following is such "another" example.

For each $n = 1, 2, \ldots$, choose a field F_n which contains $\mathbf{R}$, the field of real numbers, and set $Q = \prod_{n=1}^{\infty} F_n$, and $S = \bigoplus_{n=1}^{\infty} F_n + 1_Q \mathbf{R}$ $(\subset Q)$. Then, S is an abelian regular ring with Q the maximal quotient ring. Let k (≥ 2) be an integer, and set $R = M_k(S)$.

Then, R has the desired property. To this end, according to Lemma 8, we must first show that every right S-submodule X of Q generated by at most two elements contains an element whose annihilator coincides with that of X. So, let $X = xS + yS$, where $x = (x_n)$, $y = (y_n) \in Q$, and let N_1 denote the set $\{n \in \mathbf{N} \mid x_n \neq 0 \text{ or } y_n \neq 0\}$. Since for each $n \in N_1$, the set $H_n = \{(a, b) \in \mathbf{R} \times \mathbf{R} \mid x_n a + y_n b = 0,\ a^2 + b^2 = 1\}$ is finite, we have $\bigcup_{n \in N_1} H_n \subsetneqq \{(a, b) \in \mathbf{R} \times \mathbf{R} \mid a^2 + b^2 = 1\}$; hence there exist $a, b \in \mathbf{R}$ such that $x_n a + y_n b \neq 0$ for all $n \in N_1$. Now, taking $s = (s_n), t = (t_n) \in S$ such that $s_n = a$, $t_n = b$ for all $n = 1, 2, \ldots$, we see that $r_S(X) = r_S(xs + yt)$, as desired. Thus, every finitely generated faithful right R-module contains a finitely generated projective generator for Mod-R.

Now, choose $z \in Q - S$. Then, as seen in the argument following Remark 5, the cyclic faithful right R-module generated by $\begin{pmatrix} 1 & 0 & \cdot & \cdot & \cdot & 0 & z \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{pmatrix}$ can not be a generator for Mod-R.

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